PHASE ANGLE FLUCTUATIONS IN THE HYSTERETIC JOSEPHSON JUNCTION: OLD THEORY? NEW PHENOMENON?

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The Kramers equation governing the fluctuations with time (t) of the phase difference $\theta(k)$ across the hysteretic Josephson junction is solved in the limit $G \to 0$, where G is the dissipation coefficient. The results are expressed, for clarity, in term of the power spectrum corresponding to the autocorrelation function $\langle [\mathrm{d} \cos \theta(t)/\mathrm{d}t] [\mathrm{d} \cos \theta(t)/\mathrm{d}t]_0 \rangle$ computed at thermodynamic equilibrium. In the limit $G \to 0$ the spectrum is structured with resonance peaks which may indicate the presence of several new characteristic frequencies of the hysteretic Josephson effect in the limit $G \to 0$.

Introduction. The Kramers equation [1] is now known to govern several different rate processes of importance in physics and other disciplines where account is taken of the interaction between non-linear and random processes. An example is molecular motion in the liquid state, which may be described [2] in terms of the theory of brownian motion in potential wells. The RSJ model [3] of the hysteretic Josephson junction envisages a rate process in terms of the fluctuations of the phase difference, θ, across the junction. These fluctuations are governed by the Kramers equation in the space of θ and $\dot{\theta}$. It is the purpose of this letter to use the theoretical methods developed in the discipline of molecular dynamics [4] to solve this Kramers equation in the limit $G \to 0$, where G is the dissipation coefficient, given by

$$G = 1/\omega_J RC\,, \tag{1}$$

where ω_J is the Josephson plasma frequency, R the resistance, and C the capacitance of the junction.

Theory. The Kramers equation for the conditional probability density function $\rho(\theta(t), \dot{\theta}(t), t | \theta(0), \dot{\theta}(0), 0)$ is given in the RSJ model of the hysteretic Josephson junction by

$$\partial\rho/\partial t + \dot{\theta}\partial\rho/\partial\theta + (I_0 - \sin\theta)\partial\rho/\partial\dot{\theta} = G(\partial/\partial\dot{\theta})(\dot{\theta}\rho + T\partial\rho/\partial\dot{\theta})\,. \tag{2}$$

Here I_0 is the externally driven current in units of the critical current I_J. The time t is expressed in units of ω_J^{-1}, the reciprocal of the Josephson plasma frequency and the temperature T in units of $\hbar\omega_J/k_B$.

It is the purpose of this letter to provide a solution for eq. (2) in the limit $G \to 0$, where it is known from recent work in liquid state molecular dynamics that the equilibrium power spectrum from an equation closely analogous to eq. (2) becomes highly structured with non-harmonic resonance peaks. Before describing the solution of eq. (2) it is useful to note that it corresponds to the Langevin equation

$$\ddot{\theta}(t) + G\dot{\theta}(t) - [I_0 - \sin\theta(t)] = \dot{W}(t)\,, \tag{3}$$

where $W(t)$ is a Wiener process, representing the influence of random noise. In the absence of noise ($W(t) = 0$ for all t), eq. (3) reduces to

$$\ddot{\theta} + G\dot{\theta} + \sin\theta = I\,. \tag{4}$$

The general solution of eq. (2) is

$$\rho(\dot\theta, \theta, t) = \exp(-\dot\theta^2/4T) \sum_{n=0}^{\infty} D_n(\dot\theta/T^{1/2})\phi_n(\theta, t) , \quad (5)$$

where D_n are the Hermite polynomials. The ϕ_n functions are periodic and can be expanded in terms of a Fourier series:

$$\phi_n(\theta, t) = \sum_{p=-\infty}^{\infty} A_p^n(t) \exp(\mathrm{i}p\theta) . \quad (6)$$

Standard differential–difference algebra leads to the linear recurrence relation:

$$\dot A_r^m(t) + (\mathrm{i}rT^{1/2} - I_0)A_r^{m-1}(t) + \mathrm{i}rT^{1/2}(m+1)A_r^{m+1}(t) - \mathrm{i}[A_{r-1}^{m-1}(t) - A_{r+1}^{m-1}(t)] + mA_r^m(t) = 0 . \quad (7)$$

Laplace transformation of eq. (7) leads to the simple matrix equation

$$\mathbf{MA}(s) = \mathbf{A}(0) \quad (8)$$

for the coefficients $A_r^m(s)$. The complex matrix **M** is a sparse, banded, tridiagonal, which is truncated at a given r and m for convergence..

It is convenient to express the behaviour of the phase angle θ from eq. (2) in terms of its self-correlation function $\langle\cos\theta(t)\cos\theta(0)\rangle$. This is not the usual course taken by the theory of the hysteretic Josephson effect, but is standard in the theory of molecular dynamics [5] because the Fourier transform of the orientational self-correlation function is related to the fequency dependent dielectric loss. In the present context $\langle\cos\theta(t)\cos\theta(0)\rangle$ measures the self-correlation of the cosine of the phase-difference from the Kramers equation (2). The orientational s.c.f. $\langle\cos\theta(t)\cos\theta(0)\rangle$ is related to the coefficients $A_1^0(t)$ and $A_{-1}^0(t)$ by:

$$\langle\cos\theta(t)\cos\theta(0)\rangle = \tfrac{1}{2}(2\pi)^{3/2}T^{1/2} \times [\langle\cos\theta(0)A_1^0(t)\rangle + \langle\cos\theta(0)A_{-1}^0(t)\rangle] \quad (9)$$

and is obtained from eq. (8) subject to thermal equilibrium conditions:

$$\langle\cos\theta(0)A_p^0(0)\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos\theta(0)\,\mathrm{e}^{-\mathrm{i}p\theta(0)} \times \exp[I_0\theta(0) + \cos\theta(0)]\,\mathrm{d}\theta(0) . \quad (10)$$

The size of the matrix **M** used in this letter is 100 by 100. Eq. (8) is solved for the power spectrum $\alpha(\omega)$ corresponding not to $\langle\cos\theta(t)\cos\theta(0)\rangle$ itself but to its second derivative:

$$\alpha(\omega) \propto \omega^2 f(s) + \mathrm{i}\omega f(0) , \quad (11)$$

$$f(s) = \tfrac{1}{2}[A_1^0(s) + A_{-1}^0(s)] , \quad (12)$$

which is much more sensitive to the resonance peaks which appear from eq. (2) as $G \to 0$.

Results and discussion. In the special case $I_0 = 0$ the power spectrum $\alpha(\omega)$ is structured as illustrated in fig. 1. The peaks become more prominent and more widely spaced as the temperature parameter T is increased, with the other parameters kept constant. In the limit $G \to 0$ therefore the Josephson junction hysteresis process starts to show characteristic frequencies in addition to the plasma frequencies ω_J. These are characterised by the Fourier transform of the self-correlation function of $\cos\theta(t)$, where $\theta(t)$ is the time-dependent phase-angle across the junction. These peaks appear when the dissipation coefficient G becomes very small, or when the Josephson plasma fre-

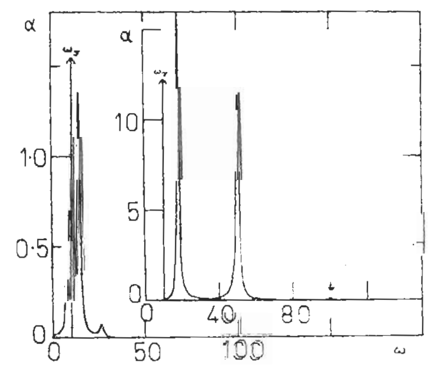

Fig. 1. Appearance of peaks in the power spectrum $\alpha(\omega)$ with parameters: $G = 0.1$; $T = 1$ and $I_0 = 0$. Inset: $T = 4$. Frequency: reduced units. Note that for $RC = 1$ the Josephson plasma frequency occurs at $\omega_J = 10.0$ for $G = 1.0$ in reduced units. This is marked on the diagrams.

quency becomes very high, or when the resistance or capacitance of the junction becomes high. This self-correlation pattern will, in turn, affect the hysteresis properties of the Josephson junction itself, and might be useful in the design of circuitry based on the Josephson effect. It may be useful to regard the appearance of these peaks as the development of structure in the Josephson plasma frequency ω_J, normally a delta function.

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References

[1] H.A. Kramers, Physica 7 (1940) 284.
[2] C.J. Reid, Mol. Phys. 49 (1983) 331.
[3] D.E. McCumber, J. Appl. Phys. 39 (1968) 3113.
[4] M.W. Evans, P. Grigolini and G. Pastori-Parravicini, eds., Memory function approaches to stochastic problems in condensed matter, two-volume special issue of: Advances in chemical physics, gen. eds. Prigogine and Rice (Wiley/Interscience, New York, 1984).
[5] W.T. Coffey, M.W. Evans and P. Grigolini, Molecular diffusion and spectra (Wiley/Interscience, New York, 1984).
[6] M.W. Evans, G.J. Evans, W.T. Coffey and P. Grigolini, Molecular dynamics (Wiley/Interscience, New York, 1982).