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SOME PROPERTIES OF LONGITUDINAL FIELDS AND PHOTONS

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CONTENTS

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Modern Nonlinear Optics, Part 2, Edited by Myron Evans and Stanisław Kielich. Advances in Chemical Physics Series, Vol. LXXXV.
ISBN 0-471-57546-1

THE MAGNETOSTATIC FLUX DENSITY $\mathbf{B}_\Pi$ *OF THE ELECTROMAGNETIC FIELD: DEVELOPMENT AND CLASSICAL INTERPRETATION*

I. INTRODUCTION

It has been demonstrated[1–7] recently that the electromagnetic plane wave generates the novel quantity $\mathbf{B}_\Pi$ which is in units of tesla, and the symmetry of uniform, divergentless, magnetostatic flux density, i.e., positive to parity inversion $\hat{P}$ and negative to motion reversal $\hat{T}$. In quantum field theory $\mathbf{B}_\Pi$ becomes[2] the novel magnetostatic flux density quantum $\hat{B}_\Pi$, a boson operator defined by $\hat{B}_\Pi = B_0\hat{J}/\hbar$, where $\hat{J}$ is the angular momentum boson operator of the photon, B_0 is the magnetic flux density amplitude associated with a single photon, and $\hbar$ is the reduced Planck constant. The boson operator $\hat{B}_\Pi$ is generated by each photon of the beam as it propagates linearly at the speed of light c. For this reason, and from considerations of symmetry,[8] there can be no electrostatic field $\mathbf{E}_\Pi$ associated with a plane wave, and $\mathbf{E}_\Pi$ cannot be generated from $\mathbf{B}_\Pi$ by Faraday induction in free space. However, $\mathbf{B}_\Pi$ interacts with matter through an interaction Hamiltonian operator $-\hat{m}\cdot\mathbf{B}_\Pi$, where $\hat{m}$ is a magnetic dipole moment operator, and $\mathbf{B}_\Pi$ can be used either in classical or quantum forms to describe observable phenomena such as the inverse Faraday effect,[9] an optical Faraday effect,[3] optical Zeeman effect,[4,5] optical Cotton-Mouton effect,[6] and other related phenomena[10,11] in which light magnetizes matter. Well-known phenomena of optics can also be reinterpreted in terms of $\mathbf{B}_\Pi$, for example, antisymmetric light scattering.[12] The classical vector $\mathbf{B}_\Pi$ is proportional[1] to the vector cross product $\mathbf{E}\times\mathbf{E}^*$, where $\mathbf{E}$ is the usual electric field strength of an electromagnetic plane wave, and $\mathbf{E}^*$ is its complex conjugate.[13–15] Accordingly, in free space,

$$\mathbf{B}_\Pi = \frac{\mathbf{E}\times\mathbf{E}^*}{2E_0ci} = B_0\mathbf{k} = \left(\frac{I_0}{\varepsilon_0c^3}\right)^{1/2}\mathbf{k} = \left(\frac{|\mathbf{N}|}{2\varepsilon_0c^3}\right)^{1/2}\mathbf{k} \tag{1}$$

in which $\mathbf{k}$ is a unit axial vector in Z, the propagation axis, and E_0 is the scalar amplitude of $\mathbf{E}$; $B_0 = E_0/c$ is the scalar magnetic flux density amplitude, and ε_0 is the free space permittivity. Here I_0 is the scalar intensity in Wm^{-2} of the plane wave and $|\mathbf{N}|$ denotes the scalar magnitude of the well-known Poynting vector:

$$\mathbf{N} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}^* \tag{2}$$

This paper is concerned with the development and classical interpretation of the novel vector $\mathbf{B}_\Pi$ in analogy with its well-known relative $\mathbf{N}$. The latter is a flux of energy denisty of the plane wave, and takes meaning[15] only when the wave interacts with matter, at the simplest level the electronic charge. It is well known that $\mathbf{N} = 2I_0\mathbf{n}$, where $\mathbf{n}$ in free space is the unit propagation vector in the axis Z of propagation of the plane wave. The vectors $\mathbf{N}$ and $\mathbf{n}$ are negative to both $\hat{P}$ and $\hat{T}$, and are polar vectors, whereas $\mathbf{B}_\Pi$ is $\hat{P}$-positive, $\hat{T}$-negative, and is an axial vector. $\mathbf{N}$ is proportional to the scalar part of the free space intensity tensor

$$I_{ij} = \varepsilon_0 cE_iE_j^* \tag{3}$$

and $\mathbf{B}_\Pi$ is proportional to its antisymmetric part, which in vector notation is $\varepsilon_0c\mathbf{E}\times\mathbf{E}^*$. Therefore, $\mathbf{N}$ and $\mathbf{B}_\Pi$ are different parts of the same tensor property. It follows that as for $\mathbf{N}$ (Ref. 15), its relative $\mathbf{B}_\Pi$ takes meaning only when the wave interacts with matter.

Section II develops a novel continuity equation for $\mathbf{B}_\Pi$ which links it to a novel magnetic density $U_\Pi = -B_0/c$ in the same way as $\mathbf{N}$ is linked by a continuity equation of electromagnetic field theory to the energy density U (Ref. 15). Thus, $\mathbf{N}$ and $\mathbf{B}_\Pi$ are vector fields and U and U_Π are scalar fields. The scalar field U can also be interpreted as electromagnetic power per unit volume, and $\mathbf{N}$ can be interpreted as power per unit area. In the same way $\mathbf{B}_\Pi$ becomes interpretable in Section II as magnetostatic power per unit area, and U_Π as magnetostatic power per unit volume generated by a completely circularly polarized electromagnetic plane wave.

Section III defines the vector potential $\mathbf{A}_\Pi$ associated with $\mathbf{B}_\Pi$ starting directly from a consideration of the cross product $\mathbf{E}\times\mathbf{E}^*$, suitably scaled by the denominator $2E_0ci$. It is shown that a vector triple product of the type $\mathbf{r}\times(\mathbf{E}\times\mathbf{E}^*)$, where $\mathbf{r}$ is a position vector in (X, Y, Z), has all the characteristics of the vector potential normally associated with a uniform, divergentless, magnetostatic field. The latter is identified therefore as $\mathbf{B}_\Pi$, and is related to $\mathbf{A}_{\Pi}$ through

$$\mathbf{B}_{\Pi} = \nabla \times \mathbf{A}_{\Pi} \tag{4}$$

Section IV interprets the cross product $\mathbf{E} \times \mathbf{E}^*$ with the antisymmetric part of Maxwell's electromagnetic stress tensor,[14] which is part of the electromagnetic energy/momentum four tensor. Thus, $\mathbf{B}_{\Pi}$ is proportional to a vorticity in the classical electromagnetic field. This is illustrated by determining the equations of motion of an electron in $\mathbf{B}_{\Pi}$ by solving the novel Lorentz equation

$$\mathbf{p} + 2e\mathbf{A}_{\Pi} = \text{constant} \tag{5}$$

where $\mathbf{p}$ is the electron's momentum and e is its charge. The field $\mathbf{B}_{\Pi}$ drives the electron forward in a helical trajectory, with constant linear velocity in z. It is shown finally that this is the same trajectory as that of the electron in a circularly polarized plane wave, obtained by solving the Lorentz equation for this case. In other words, the solutions of the Lorentz equation (5) are also solutions of the Lorentz equation of motion of the electron in a circularly polarized plane wave, showing that a plane wave can generate the characteristics of a magnetostatic flux density $\mathbf{B}_{\Pi}$, which takes meaning as it interacts with the electron, driving the latter in a helical trajectory.

The paper ends with a short discussion of the interpretation of $\mathbf{B}_{\Pi}$ in the required relativistic context.

II. THE CONTINUITY EQUATION FOR $\mathbf{B}_{\Pi}$

Maxwell's phenomenological equations lead to the following well-known[15] continuity equation when electromagnetic radiation interacts with matter:

$$\nabla \cdot \mathbf{N} = -\frac{\partial U}{\partial t} - \mathbf{E} \cdot \mathbf{J}^* = 0 \tag{6}$$

where $\mathbf{J}^*$, the current density, is zero in free space. The energy density U is defined in free space as the time-averaged quantity:

$$U = \tfrac{1}{2}(\mathbf{H}^* \cdot \mathbf{B} + \mathbf{E}^* \cdot \mathbf{D}) \tag{7}$$

with $\mathbf{B} = \mu_0 \mathbf{H}$ and $\mathbf{D} = \varepsilon_0 \mathbf{E}$. Here μ_0 is the permeability of free space.

Clearly, in free space, the wave does no work on matter, such as electronic charge, because the density $\mathbf{E} \cdot \mathbf{J}^*$ of power lost from the fields $\mathbf{E}$ and $\mathbf{B}^*$ is zero. U is therefore a field energy density, which takes meaning only if there is field–matter interaction.

By considering the product $\mathbf{E} \times \mathbf{E}^*$, which is proportional, as we have seen, to the antisymmetric vector part of light intensity, it can be demonstrated as follows that there exists a novel continuity equation linking $\mathbf{B}_{\Pi}$ and U_{Π}. Using the vector relation,

$$\nabla \cdot (\mathbf{E} \times \mathbf{E}^*) = \mathbf{E}^* \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{E}^*) \tag{8}$$

and the Maxwell equations in free space,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{E}^* = -\frac{\partial \mathbf{B}^*}{\partial t} \tag{9}$$

implies

$$\nabla \cdot (\mathbf{E} \times \mathbf{E}^*) = \nabla \cdot (\mathbf{B} \times \mathbf{B}^*) = 0 \tag{10}$$

that is,

$$\nabla \cdot \mathbf{B}_{\Pi} = 0 \tag{11}$$

which shows that $\mathbf{B}_{\Pi}$ is uniform and divergentless. From Eqs. (9) and (10), we can write

$$\nabla \cdot (\mathbf{E} \times \mathbf{E}^*) = 2E_0 c i \nabla \cdot \mathbf{B}_{\Pi} = -\mathbf{E}^* \cdot \frac{\partial \mathbf{B}}{\partial t} + \mathbf{E} \cdot \frac{\partial \mathbf{B}^*}{\partial t} \tag{12}$$

Integration by parts of the right side of Eq. (12) gives the result

$$\int \mathbf{E}^* \cdot \frac{\partial \mathbf{B}}{\partial t}\, dt - \int \mathbf{E} \cdot \frac{\partial \mathbf{B}^*}{\partial t}\, dt = \frac{1}{2}(\mathbf{E}^* \cdot \mathbf{B} - \mathbf{E} \cdot \mathbf{B}^*) = -2iE_0 B_0 \tag{13}$$

Defining the quantity

$$\begin{aligned} U_{\Pi} &= \frac{1}{2E_0 c i}\left(\int \mathbf{E}^* \cdot \frac{\partial \mathbf{B}}{\partial t}\, dt - \int \mathbf{E} \cdot \frac{\partial \mathbf{B}^*}{\partial t}\, dt\right) \\ &= -\frac{B_0}{c} \end{aligned} \tag{14}$$

implies

$$\nabla \cdot \mathbf{B}_\Pi = -\frac{\partial U_\Pi}{\partial t} = 0 \tag{15}$$

which is a free space continuity equation for $\mathbf{B}_\Pi$. Equation (15) is the precise counterpart of the free space continuity equation (6) for **N**:

$$\nabla \cdot \mathbf{N} = -\frac{\partial U}{\partial t} = 0 \tag{16}$$

Equation (15) is novel to this work, whereas Eq. (16) is standard in classical electrodynamics.[15]

The continuity equations (15) and (16) have the same structure and must be interpreted in the same way. Thus, in classical electrodynamics, the Poynting vector **N** is the flux of the density (U) of the electromagnetic energy. Similarly, the novel $\mathbf{B}_\Pi$ is the flux of the magnetostatic density of the electromagnetic plane wave, a flux that is uniform and does not vary with time. U is the electromagnetic field energy density in free space (i.e., electromagnetic power per unit volume[15]), and therefore U_Π is the magnetostatic density in free space generated by an electromagnetic plane wave, or the magnetostatic power of the wave per unit volume occupied by that wave in free space.

Both Eqs. (15) and (16) are continuity equations relating[15] the time rate of change of a density (the scalar fields U or U_Π) to the divergence of a flux (the vector fields **N** or $\mathbf{B}_\Pi$). It is well established[15] that the notion of electromagnetic field energy density U takes meaning only when there is interaction between the field and matter. Similarly, U_Π takes meaning only when there is wave–matter interaction. Equations (15) and (16) are both statements based on the existence in free space of the light intensity tensor I_{ij}, which is Hermitian.[16] Equation (16) is concerned with the scalar part of I_{ij}, and Eq. (15) with its vector part, which, as we have seen, is the quantity $\varepsilon_0 c\mathbf{E} \times \mathbf{E}^*$ proportional to $\mathbf{B}_\Pi$. The existence of $\mathbf{B}_\Pi$, and of Eq. (15), is an inevitable consequence of the fact that I_{ij} is a tensor,[16] with a vector (i.e., antisymmetric) component that is purely imaginary as a consequence of the Hermitian nature of I_{ij} (Ref. 16). It follows that the classical free space electromagnetic plane wave generates $\mathbf{B}_\Pi$ and its associated scalar field U_Π.

III. THE VECTOR POTENTIAL $\mathbf{A}_\Pi$ IN FREE SPACE

Since $\mathbf{B}_\Pi$ is a magnetostatic flux density, it is assumed that there exists a vector potential $\mathbf{A}_\Pi$ such that

$$\mathbf{B}_\Pi = \nabla \times \mathbf{A}_\Pi \tag{17}$$

Since $\mathbf{B}_\Pi$ and U_Π are well defined in free space, it follows from the assumption (17) that $\mathbf{A}_\Pi$ would also be well defined in free space. If Eq. (17) were true, it follows that $\mathbf{A}_\Pi$ would be a function[17] of the type

$$\mathbf{A}_\Pi = -\tfrac{1}{2}\mathbf{r} \times \mathbf{B}_\Pi \tag{18}$$

where r is a positive coordinate vector in frame (X, Y, Z). From Eq. (1),

$$\mathbf{A}_\Pi = -\tfrac{1}{2}\mathbf{r} \times (\mathbf{E} \times \mathbf{E}^*)/(2E_0 c i) \tag{19}$$

Defining

$$\mathbf{r} \equiv X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k} \tag{20}$$

and using the vector relation

$$\mathbf{r} \times (\mathbf{E} \times \mathbf{E}^*) = \mathbf{E}(\mathbf{r} \cdot \mathbf{E}^*) - \mathbf{E}^*(\mathbf{r} \cdot \mathbf{E}) \tag{21}$$

with[17–19]

$$\begin{aligned} \mathbf{E} &= E_0(\mathbf{i} - i\mathbf{j})e^{i\phi} \\ \mathbf{E}^* &= E_0(\mathbf{i} + i\mathbf{j})e^{-i\phi} \end{aligned} \tag{22}$$

where ϕ is the phase[17] of the plane wave, we have

$$\mathbf{A}_\Pi = \frac{B_0}{2}(X\mathbf{j} - Y\mathbf{i}) \tag{23}$$

Since

$$|\mathbf{B}_\Pi| = B_0 \tag{24}$$

it is clear that $\mathbf{B}_\Pi$ and $\mathbf{A}_\Pi$ of Eq. (24) are related by Eq. (17).

In other words there exists a vector potential $\mathbf{A}_\Pi$ in free space whose curl is $\mathbf{B}_\Pi$ that is defined by the vector field $(B_0/2)(X\mathbf{j} - Y\mathbf{i})$, a field that is generated in free space by the classical electromagnetic plane waves

(22), both solutions of Maxwell's equations. Note that $\mathbf{B}_{\Pi}$ is also a solution of Maxwell's equations in vacuo. In the same way, $\mathbf{N}$ is not in itself a solution of Maxwell's equations, but is generated therefrom through a cross product $\mathbf{E} \times \mathbf{B}^*/\mu_0$.

It has been demonstrated that $\mathbf{B}_{\Pi}$ is related to a well-defined $\mathbf{A}_{\Pi}$ in free space, and to a well-defined scalar field U_{Π}. The next section considers the interaction of $\mathbf{B}_{\Pi}$ with an electron, i.e., wave–matter interaction.

IV. INTERACTION OF $\mathbf{B}_{\Pi}$ WITH AN ELECTRON

From Eq. (18) it is clear that

$$\dot{\mathbf{A}}_{\Pi} = -\tfrac{1}{2}\mathbf{v} \times \mathbf{B}_{\Pi} \tag{25}$$

where $\mathbf{v} = \partial \mathbf{r}/\partial t$ is a velocity in frame (X, Y, Z). Consider the interaction of $\mathbf{B}_{\Pi}$ with an electron. Since action and reaction are equal and opposite, the action of $\mathbf{B}_{\Pi}$ on the electronic charge e is balanced by a reaction of e upon $\mathbf{B}_{\Pi}$. However, as usual in classical electrodynamics, we assume that $\mathbf{B}_{\Pi}$ is not changed greatly by the action of e upon it,[13–15] so that the Lorentz equation of motion applies. Since there is no $\mathbf{E}_{\Pi}$ present,

$$\dot{\mathbf{p}} = e\mathbf{v} \times \mathbf{B}_{\Pi} \tag{26}$$

where $\mathbf{p}$ is the momentum of the electron, and $\mathbf{v}$ is its velocity $(\mathbf{p}/m)$ in the field $\mathbf{B}_{\Pi}$. With Eq. (25) the Lorentz equation can be rewritten directly in terms of the novel vector potential $\mathbf{A}_{\Pi}$:

$$\dot{\mathbf{p}} + 2\dot{\mathbf{A}}_{\Pi} e = 0 \tag{27}$$

The equations of motion of e in $\mathbf{B}_{\Pi}$ thus become[15] those of Coriolis acceleration:

$$\left.\begin{aligned} \dot{v}_X &= \Omega v_Y \\ \dot{v}_Y &= -\Omega v_X \end{aligned}\right\} \qquad \dot{v}_Z = 0 \tag{28}$$

where the parameter Ω is defined as

$$\Omega = \left(1 - \frac{v^2}{c^2}\right)^{1/2} \left(\frac{e}{m}\right) B_{\Pi Z} \tag{29}$$

Equations (28) can be rewritten[15] as

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(v_X + \mathrm{i}v_Y) &= -\mathrm{i}\Omega(v_X + \mathrm{i}v_Y) \\ v_X + \mathrm{i}v_Y &= a \exp(-\mathrm{i}\Omega t) \end{aligned} \tag{30}$$

where a is complex and defined by

$$a = v_{0t} \exp(-\mathrm{i}\alpha) \tag{31}$$

where v_{0t} and α are real. Here

$$v_{0t} = \left(v_X^2 + v_Y^2\right)^{1/2} \tag{32}$$

is the velocity of the electron in the XY plane. The trajectory of the electron in the field $\mathbf{B}_{\Pi}$ is therefore that of a helix

$$\begin{aligned} X &= X_0 + r \sin(\Omega t + \alpha) \\ Y &= Y_0 + r \cos(\Omega t + \alpha) \\ Z &= Z_0 + v_{0Z} t \end{aligned} \tag{33}$$

where the radius of the helix $\mathbf{r}$ is defined by

$$r = \frac{p_t}{eB_{\Pi Z}} = \frac{m v_{0t}}{eB_{\Pi Z}} \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \tag{34}$$

where p_t is the projection of the momentum on to the XY plane. Note that the same result can be obtained directly from the equation

$$\dot{\mathbf{p}} = -2\mathbf{A}_{\Pi} e \tag{35}$$

to give

$$\dot{v}_X = -\Omega v_Y \text{ etc.} \tag{36}$$

We now demonstrate that the motion of an electron in a circularly polarized plane wave is precisely the same as the motion of an electron in the novel magnetic field $\mathbf{B}_{\Pi}$. Assume that the reaction of the electron upon the circularly polarized electromagnetic field is negligible, so that the

Lorentz equation again applies:

$$\dot{\mathbf{p}} = e(\mathbf{E} + v \times \mathbf{B}) \tag{37}$$

Here **E** and **B** are plane wave solutions of Maxwell's equations in free space:

$$\mathbf{E} = E_0(\mathbf{i} + \mathrm{i}j)e^{\mathrm{i}\phi} \qquad \mathbf{B} = B_0(\mathbf{j} - \mathbf{ii})e^{-\mathrm{i}\phi} \tag{38}$$

i.e., they are the usual oscillating, phase-dependent, electric and magnetic field vectors of the wave. Note carefully that **B** is quite different in nature from $\mathbf{B}_\Pi$. The velocity of the electron in frame (X, Y, Z) is

$$\mathbf{v} = v_X\mathbf{i} + v_Y\mathbf{j} + v_Z\mathbf{k} \tag{39}$$

Substituting Eqs. (38) and (39) into the Lorentz equation (37) and using $|\mathbf{B}_\Pi| = B_0 = E_0/c$ gives the equation of motion:

$$\dot{v}_X = \Omega(c - v_Z)e^{-\mathrm{i}\phi} \tag{40}$$

$$\dot{v}_Y = \mathrm{i}\Omega(c + v_Z)e^{-\mathrm{i}\phi} \tag{41}$$

$$\dot{v}_Z = \Omega(v_X + \mathrm{i}v_Y)e^{-\mathrm{i}\phi} \tag{42}$$

Multiplying Eq. (41) by i and adding to Eq. (40) gives

$$\frac{d}{dt}(v_X + \mathrm{i}v_Y) = -2v_Z\Omega e^{-\mathrm{i}\phi} \tag{43}$$

If we assume a solution of the type

$$v_Z = \frac{\mathrm{i}}{2}(v_X + \mathrm{i}v_Y)e^{\mathrm{i}\phi} \tag{44}$$

then separation of variables occurs

$$\frac{d}{dt}(v_X + \mathrm{i}v_Y) = -\mathrm{i}\Omega(v_X + \mathrm{i}v_Y) \tag{45}$$

$$\frac{d}{dt}v_Z = -2\mathrm{i}\Omega v_Z e^{-2\mathrm{i}\phi} \tag{46}$$

Equation (44) is a consistent solution of Eqs. (40)–(42), a solution that implies that the time average of v_Z must be zero

$$\langle v_Z \rangle_t = 0 \tag{47}$$

because the time average of the oscillating function

$$\langle e^{\mathrm{i}\phi} \rangle_t = \langle \cos\phi \rangle_t + \mathrm{i}\langle \sin\phi \rangle_t \tag{48}$$

is zero.[15] This is consistent with the fact that the unaveraged v_Z itself is in general nonzero and complex. Equations (42) and (46) both imply that $\langle \dot{v}_Z \rangle_t$ is also zero. Equation (44) is therefore a solution of Eqs. (40)–(42) when both $\langle v_Z \rangle_t$ and $\langle \dot{v}_Z \rangle_t$ vanish.

Note that Eq. (45) is the same as Eq. (30), derived when considering the motion of one electron in $\mathbf{B}_\Pi$, and can be solved to give the trajectory of the electron in a circularly polarized plane wave. This is, from Eq. (45), a circle:

$$\begin{aligned} X &= X_0 + r\sin(\Omega t + \alpha) \\ Y &= Y_0 + r\cos(\Omega t + \alpha) \\ Z &= Z_0 \end{aligned} \tag{49}$$

a conclusion that is consistent with that of Landau and Lifshitz[15] from the relativistic Hamilton-Jacobi equations of the electromagnetic field, but derived in a different way by solving the Lorentz equation assuming Eqs. (38).

For an initially stationary electron, i.e., for $v_{0Z} = 0$, Eqs. (49) are identical with Eqs. (33), describing the trajectory of the electron in the novel field $\mathbf{B}_\Pi$. The trajectories (49) are consistent with the assumption (44), which implies that the time averages of v_Z and $\dot{v}_Z$ both vanish.

In summary, the trajectory of an initially stationary electron in the field $\mathbf{B}_\Pi$ is the same as that in the fields **E** and **B**: An initially stationary electron moves in a circle under the influence either of $\mathbf{B}_\Pi$ or of **E** and **B**. In the former, the velocity in Z vanishes identically; in the latter this component is zero on the average.

An observer noting this trajectory would not be able to define unambiguously the influence that causes the electron to move as it does in a circle, be this the wave or the field $\mathbf{B}_\Pi$. The influence upon an initially stationary electron of a circularly polarized electromagnetic plane wave is identical in all respects in the plane perpendicular to the propagation axis with the influence of $\mathbf{B}_\Pi$ upon that electron. Therefore, the motion in this plane of an electron in a circularly polarized electromagnetic plane wave can be represented exactly by a magnetostatic field $\mathbf{B}_{\Pi}$, which influences an electron to move in the same trajectory. $\mathbf{B}_{\Pi}$ is therefore a real, physically meaningful, influence.

Furthermore, we have shown that $\mathbf{B}_{\Pi}$ is in units of tesla, is $\hat{T}$-negative and $\hat{P}$-positive, and is accompanied by a well-defined scalar field U_{Π} and a well-defined vector potential $\mathbf{A}_{\Pi}$. It follows that $\mathbf{B}_{\Pi}$ has several characteristics of a magnetostatic field. However, $\mathbf{B}_{\Pi}$ is clearly not identical with an ordinary magnetostatic field, because it is a property of light. Apparently there is no $\mathbf{E}_{\Pi}$, and $\mathbf{E}_{\Pi}$ cannot be generated from $\mathbf{B}_{\Pi}$ by Faraday induction. If there were an $\mathbf{E}_{\Pi}$ an electron's trajectory in that $\mathbf{E}_{\Pi}$ would be a catenary[15]; clearly, from Eqs. (33) and (49), this is not the case. An electric field $\mathbf{E}_{\Pi}$ cannot be generated from products such as $\mathbf{E} \times \mathbf{E}^*$, $\mathbf{B} \times \mathbf{B}^*$, $\mathbf{E} \times \mathbf{B}^*$, or $\mathbf{B} \times \mathbf{E}^*$ on the grounds of fundamental $\hat{P}$ and $\hat{T}$ symmetry.

Finally, further physical interpretation can be placed upon $\mathbf{E} \times \mathbf{E}^*$ by considering the Maxwell stress tensor[15]:

$$\sigma_{\alpha\beta} = -\varepsilon_0 E_\alpha E_\beta^* - \mu_0 H_\alpha H_\beta^* + \tfrac{1}{2}\delta_{\alpha\beta}(\varepsilon_0 E_\alpha E_\beta^* + \mu_0 H_\alpha H_\beta^*) \qquad (50)$$

which is the momentum flux density of electromagnetic radiation, and part of its energy momentum four tensor.[15] It is clear that the antisymmetric component of $\sigma_{\alpha\beta}$ is proportional to $\mathbf{E} \times \mathbf{E}^*$ in vector notation, so that the field $\mathbf{B}_{\Pi}$ is also proportional to the antisymmetric part of $\sigma_{\alpha\beta}$. In this context the antisymmetric part of stress in mechanics is a vorticity, with the same symmetry as angular momentum, so we deduce that $\mathbf{B}_{\Pi}$ is proportional to the angular momentum of classical radiation, as in our operator equation $\hat{B}_{\Pi} = B_0 \hat{J}/\hbar$ (Ref. 2) of quantum field theory. Clearly, angular momentum takes meaning in the energy momentum four tensor of the electromagnetic field through the antisymmetric part of the Maxwell stress tensor, a vorticity, i.e., the antisymmetric part of the momentum tensor per unit volume.

V. DISCUSSION

It has been argued that the notion of $\mathbf{B}_{\Pi}$ is consistent with several properties of a magnetostatic field, but it must be borne in mind that $\mathbf{B}_{\Pi}$ is generated by a photon that travels at the speed of light. The classical theory with which we have been concerned must come to terms with the fact that $\mathbf{B}_{\Pi}$ is relativistic in nature. In so doing[8,15] it becomes clear that the only relativistically invariant component of $\mathbf{B}_{\Pi}$ is that in the propagation axis, which is, of course, consistent with Eq. (1). Furthermore, it can be shown[8] that there is no Faraday induction by a time-modulated $\mathbf{B}_{\Pi}$;

i.e., the hypothetical $\mathbf{E}_{\Pi}$ cannot be generated from $\partial \mathbf{B}_{\Pi}/\partial t$ through Faraday's law. This is a consequence of the Lorentz transformations. Furthermore, the existence of a $\mathbf{B}_{\Pi}$ is consistent with the fact that a valid solution of the Maxwell equations in free space is

$$\mathbf{B} = B_0(\mathbf{j} - \mathbf{i}\mathrm{i})e^{-i\phi} + B_Z\mathbf{k} \qquad (51)$$

where $B_Z k$ is in general a magnetostatic field such as $\mathbf{B}_{\Pi}$. It has been shown that there exists a vector potential $\mathbf{A}_{\Pi}$ such that $\mathbf{B}_{\Pi} = \nabla \times \mathbf{A}_{\Pi}$, and there exists a scalar field U_{Π} linked to $\mathbf{B}_{\Pi}$ by a continuity equation. Since $\mathbf{A}_{\Pi}$ is defined in terms of X and Y coordinates it is relativistically invariant, i.e., does not change under Lorentz transformation. This is consistent with the fact that $\mathbf{B}_{\Pi}$ is defined in Z, and is a magnetic flux density, and so is also invariant to Lorentz transformation.

Finally, it is straightforward to deduce that

$$\mathbf{N} = \pm c U_{\mathbf{n}} \qquad \mathbf{B}_{\Pi} = \pm c U_{\Pi}\mathbf{k} \qquad (52)$$

where $\mathbf{n}$ is a propagation vector whose magnitude is refractive index, and $\mathbf{k}$ is a unit axial vector. The first of these equations shows that the relation between energy and momentum in an electromagnetic field is the same as that in a particle moving at the speed of light,[15] i.e., the photon of quantum field theory. The second equation shows that there exists a similar proportionality between the magnetic flux density $\mathbf{B}_{\Pi}$ and U_{Π}.

VI. CONCLUSION

The classical theory of fields has been used to develop and interpret the concepts $\mathbf{E} \times \mathbf{E}^*$ and $\mathbf{B}_{\Pi}$, and it has been shown that these concepts are self-consistent and physically meaningful. For example, the motion of an electron in $\mathbf{B}_{\Pi}$ is the same as that in $\mathbf{E}$ and $\mathbf{B}$ of a plane wave. The latter can therefore be thought of as generating a magnetostatic field.

Acknowledgments

The Materials Research Laboratory of Penn State University and the Cornell Theory Center are thanked for research support. Faramarz Farahi and Yasin Raja are thanked for many interesting communications.

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THE ELEMENTARY STATIC MAGNETIC FIELD OF THE PHOTON*

I. INTRODUCTION

It is well known that the photon has an intrinsic, unremovable spin, which can be expressed as its quantized angular momentum operator $\hat{J}$ (Refs. 1–4). This is the essential explanation in quantum-field theory for the existence of classical left and right circular polarization in electromagnetic plane waves. In this paper it is argued that the photon also generates an intrinsic and unremovable static magnetic field (flux density in tesla) which can be described through the operator equation

$$\hat{B}_{\Pi} = B_0 \frac{\hat{J}}{\hbar} \tag{1}$$

where B_0 is the scalar magnetic flux density amplitude of a beam of N photons (for example, a circularly polarized laser beam). The expectation value of the component of the operator $\hat{B}_{\Pi}$ in the propagation axis of the laser is $B_0 M_J$, where M_J is the azimuthal equantum number associated with the operator $\hat{J}$. Classically, this expectation value is $\pm \mathbf{B}_{\Pi} \cdot \mathbf{k}$, where $\mathbf{k}$

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is a unit axial vector in the propagation axis, axis Z of the laboratory frame of reference (X, Y, Z).

The derivation of the fundamental operator equation (1) is given in Section II, followed in Section III by a suggestion for a key experiment to test the theory, which consists of reflecting at right angles a circularly polarized laser beam from a beam of electrons, and of measuring the frequency shift in the reflected laser due to the interaction

$$\Delta H_{\Pi} = -\hat{m} \cdot \hat{B}_{\Pi} \tag{2}$$

between $\hat{B}_{\Pi}$ and the electron's magnetic dipole moment operator $\hat{m}$. Section IV develops some consequences in spectroscopy of the existence of $\hat{B}_{\Pi}$, specifically a quantum field theory of the optical Zeeman effect, splitting due to $\hat{B}_{\Pi}$ in spectra at visible frequencies, and optical NMR and ESR,[5] in which $\hat{B}_{\Pi}$ shifts and splits conventional resonance features in liquids and condensed matter. Finally, a discussion is given of some other immediately interesting consequencies of the existence of $\hat{B}_{\Pi}$, for example, an optical Stern-Gerlach effect.

II. DERIVATION OF THE OPERATOR EQUATION FOR $\hat{\mathbf{B}}_{\Pi}$

It is seen immediately that the operator equation (1) can be derived on the basis of symmetry and dimensions alone, and in this section the rigorous quantum field theoretical derivation is given using the recent results of Tanaś and Kielich[6] and of the present author.[7–10] Before embarking on this it is instructive to note the role of fundamental symmetries, namely the motion reversal operator $\hat{T}$, and the parity inversion operator $\hat{P}$. The operators $\hat{B}_{\Pi}$ and $\hat{J}$ have the same $\hat{P}$ and $\hat{T}$ symmetries, respectively, positive and negative, so one is proportional to the other through a T- and P-positive scalar quantity. Furthermore, the unit of the operator $\hat{J}$ in quantum mechanics is the reduced Planck constant $\hbar$, and therefore the scalar proportionality constant must be a scalar magnetic flux density amplitude, B_0, in tesla. In a laser beam of N photons, the constant is the laser's scalar flux density amplitude in tesla. When $N = 1$ (one photon), B_0 remains finite, and it follows that the single photon generates a quantum of magnetostatic flux density, described in quantum field theory by $\hat{B}_{\Pi}$ of Eq. (1).

To derive this result rigorously it is convenient to consider first the classical equivalent of $\hat{B}_{\Pi}$, which is a vector quantity in the propagation axis of the laser:

$$\mathbf{B}_{\Pi} = B_0 \mathbf{k} \tag{3}$$

where **k** is a unit axial vector. The classical $\mathbf{B}_\Pi$ is proportional[7-10] to the conjugate product[11-15]

$$\mathbf{\Pi}^{(A)} = \mathbf{E} \times \mathbf{E}^* \tag{4}$$

a vector cross product of the electric field strength **E** of a circularly polarized laser with its complex conjugate $\mathbf{E}^*$:

$$\begin{aligned} \mathbf{E} &= E_0(\mathbf{i} + i\mathbf{j})\exp(i\phi) \\ \mathbf{E}^* &= E_0(\mathbf{i} - i\mathbf{j})\exp(-i\phi) \end{aligned} \tag{5}$$

Here, as usual, E_0 is the scalar electric field strength amplitude in volts per meter of the laser, **i** and **j** are unit polar vectors in X and Y of the laboratory frame (X, Y, Z), mutually orthogonal to the propagation axis Z, and ϕ is the phase. From Eq. (5)

$$\mathbf{\Pi}^{(A)} = -2E_0^2\mathbf{k}i \tag{6}$$

where **k** is a unit axial vector in Z. The product $\mathbf{\Pi}^{(A)}$ is an axial vector which is also negative to T and positive to P. Equation (6) can be rewritten using the fundamental in vacuo relation

$$E_0 = cB_0 \tag{7}$$

as

$$\mathbf{\Pi}^{(A)} = -2E_0c(B_0\mathbf{k})i \equiv -2E_0c\mathbf{B}_\Pi i \tag{8}$$

with the definition

$$\mathbf{B}_\Pi \equiv B_0\mathbf{k} \tag{9}$$

where c is the (scalar) speed of light.

Clearly, $\mathbf{\Pi}^{(A)}$ and $\mathbf{B}_\Pi$ must have the same T and P symmetries, and so does the unit axial vector **k**. The classical quantity $\mathbf{B}_\Pi$ has been defined[7-10] as equivalent magnetostatic flux density in vacuo of a circularly polarized plane wave. It has no dependence on the phase ϕ of the wave, and therefore none on the angular frequency ω and propagation vector **k**. Using the relation between intensity (watts per square meter) and electric field strength

$$I_0 = \tfrac{1}{2}\varepsilon_0 c E_0^2 \tag{10}$$

where ε_0 is the permittivity in vacuo,[1]

$$\varepsilon_0 = 8.854 \times 10^{-12}C^2J^{-1}m^{-1} \tag{11}$$

we arrive at

$$\mathbf{B}_\Pi = i\left(\frac{\varepsilon_0}{8I_0c}\right)^{1/2}(\mathbf{E} \times \mathbf{E}^*) \tag{12}$$

which shows clearly that $\mathbf{B}_\Pi$ is directly proportional to the conjugate product $\mathbf{E} \times \mathbf{E}^*$, and that $\mathbf{B}_\Pi$ is a real quantity.

The classical third Stokes parameter is the real scalar

$$S_3 = -i(E_XE_Y^* - E_YE_X^*) \tag{13}$$

so that

$$\mathbf{B}_\Pi = -\left(\frac{\varepsilon_0}{8I_0c}\right)^{1/2}S_3\mathbf{k} \tag{14}$$

Equation (14) defines $\mathbf{B}_\Pi$ in (8). Equation (10) defines $\mathbf{B}_\Pi$ in terms of S_3, and shows that the former is a real axial vector that changes sign with the laser circular polarization (left to right). The transition from classical to quantized field theory is made through the third Stokes operator $\hat{S}_3$, recently introduced by Tanaś and Kielich[6]:

$$\hat{S}_3 \equiv -\left[\frac{2_\pi\hbar\omega}{n^2(\omega)V}\right]i\left(\hat{a}_X^+\hat{a}_Y - \hat{a}_Y^+\hat{a}_X\right) \tag{15}$$

Here $n(\omega)$ is the refractive index, V is the quantization volume, and $\hat{a}^+$ and $\hat{a}$ denote, respectively, the creation and annihilation operator. Defining a coherent state of a laser beam of N photons by the Schrödinger equation[6]

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \tag{16}$$

provides the expectation value

$$\langle\alpha|\hat{S}_3|\alpha\rangle = |\alpha_+|^2 - |\alpha_-|^2 \tag{17}$$

with

$$\alpha_{\pm} = \frac{1}{\sqrt{2}}(\alpha_X \mp i\alpha_Y) \tag{18}$$

We define the operator $\hat{\mathbf{B}}_{\Pi}$ using Eqs. (14) and (15):

$$\hat{B}_{\Pi} \equiv \left(\frac{\varepsilon_0}{8I_0c}\right)^{1/2} \hat{S}_3 \tag{19}$$

and rewrite Eq. (19) as

$$\hat{B}_{\Pi} = \left(\frac{\varepsilon_0}{8I_0c}\right)^{1/2} \left[\frac{2\pi\omega}{n^2(\omega)V}\right]\left[\hbar\left(\hat{a}_+^+\hat{a}_+ - \hat{a}_-^+\hat{a}_-\right)\right] \tag{20}$$

with

$$\hat{a}_{\pm} = \frac{1}{\sqrt{2}}(\hat{a}_X \mp i\hat{a}_Y) \tag{21}$$

The quantity

$$\hbar\left(\hat{a}_+^+\hat{a}_+ - \hat{a}_-\hat{a}_-\right) \equiv \hbar\left(\hat{n}_+ - \hat{n}_-\right) \tag{22}$$

has the units of quantized angular momentum because $(\hat{n}_+ - \hat{n}_-)$ is dimensionless. Here

$$\begin{aligned} \hat{n}_+ &\equiv \hat{a}_+^+\hat{a}_+ \\ \hat{n}_- &\equiv \hat{a}_-^+\hat{a}_- \end{aligned} \tag{23}$$

are the number of photons operators.[6]

The total angular momentum of a beam of N photons propagating in Z is known independently[17] to be $NM_J\hbar$, where M_J is the azimuthal quantum number associated with the photon's angular momentum operator $\hat{J}$. Defining the angular momentum eigenfunction of a single photon by $|JM_J\rangle$ we arrive at the Schrödinger equation:

$$\hbar\left(\hat{n}_+ - \hat{n}_-\right)|JM_J\rangle = \hbar M_J N|JM_J\rangle \tag{24}$$

From Eq. (20) and (24)

$$\frac{n^2(\omega)V}{2\pi\omega}\hat{S}_3|JM_J\rangle = \hbar NM_J|JM_J\rangle \tag{25}$$

and with the identity

$$\hat{J} = \left[\frac{n^2(\omega)V}{2\pi\omega N}\right]\hat{S}_3 \tag{26}$$

Eq. (25) becomes the standard Schrödinger equation describing the angular momentum of one photon:

$$\hat{J}|JM_J\rangle = \hbar M_J|JM_J\rangle \tag{27}$$

From (19) and (26)

$$\hat{B}_{\Pi} = \left(\frac{\varepsilon_0}{8I_0c}\right)^{1/2}\left[\frac{2\pi\omega N}{n^2(\omega)V}\right]\hat{J} \equiv \zeta\hat{J} \tag{28}$$

showing that $\hat{B}_{\Pi}$ is directly proportional to $\hat{J}$. Considerable insight to the nature of the constant ζ is obtained with the results of Tanaś and Kielich[6]

$$S_0 = \frac{2\pi\omega\hbar}{n^2(\omega)V}\langle\alpha|\hat{S}_0|\alpha\rangle \tag{29}$$

$$N = \langle\alpha|\hat{S}_0|\alpha\rangle \tag{30}$$

for the zeroth Stokes operator $\hat{S}_0$ and its classical equivalent, the Stokes parameter S_0. Furthermore, we make use of the classical result[18]

$$S_0 = 2E_0^2 \tag{31}$$

which follows from our definitions (5) of E and E^*. From Eqs. (28)–(31):

$$E_0^2 = \frac{\pi\omega\hbar N}{n^2(\omega)V} \tag{32}$$

Using Eqs. (10) and (32) in (28) gives, finally, the fundamental and simple operator equation we seek to prove:

$$\hat{B}_{\Pi} = B_0\left(\frac{\hat{J}}{\hbar}\right) \tag{33}$$

III. A KEY EXPERIMENT FOR $\hat{B}_{\Pi}$

The theoretical existence of $\hat{B}_{\Pi}$ implies many different things experimentally, because a circularly polarized laser acts as a simple magnet, and delivers equivalent (or "latent" or "potential") static magnetic flux density through a vacuum, flux density which is able to form a scalar interaction Hamiltonian with a dipole moment operator $\hat{m}$:

$$\Delta H = -\hat{m} \cdot \hat{B}_{\Pi}$$

The electron carries an elementary $\hat{m}$:

$$\hat{m} = g_e \gamma_e \hat{I} \tag{34}$$

where g_e is the electron's g factor[1] (2.002), and $\hat{I}$ is the electron's angular momentum operator. The key experiment devised in this section isolates the effect of the Hamiltonian (2) on a circularly polarized visible frequency laser reflected at right angles from an electron beam. There is a frequency shift

$$\Delta f = \frac{\langle \hat{m} \cdot \hat{B}_{\Pi} \rangle}{h} \tag{35}$$

in the reflected beam, which provides a method of measuring $\hat{B}_{\Pi}$ spectrally with a high-resolution spectrometer. The derivation of Eq. (35) is based on conservation of momentum and kinetic energy when a circularly polarized laser beam of N photons is reflected at right angles from the electron beam. Consider the collision of one photon of the beam with one electron, whereby the former is reflected by an angle θ and the latter by an angle θ'. Initially, the electron is at rest with relativistic energy $m_e c^2$, where m_e is its mass.[1] After the collision, the electron's linear momentum magnitude is p and it translational kinetic energy is $(p^2c^2 + m_e^2c^4)^{1/2}$. The initial linear momentum of the photon is h/λ_i where λ_i is its wavelength, and its initial energy is hc/λ_i. The photon strikes the electron, considered stationary,[1] is deflected through an angle θ, and emerges with linear momentum h/λ_f and translational kinetic energy hc/λ_f. The electron after collision moves off at an angle θ' to the incident photon's trajectory. Conserving linear momentum and translational kinetic energy gives three equations

$$p\cos\theta' + \frac{h}{\lambda_f}\cos\theta = \frac{h}{\lambda_i} \tag{36}$$

$$p\sin\theta' + \frac{h}{\lambda_f}\sin\theta = 0 \tag{37}$$

$$m_e c^2 + \frac{hc}{\lambda_i} = \left(p^2c^2 + m_e^2c^4\right)^{1/2} + \frac{hc}{\lambda_f} \tag{38}$$

which can be solved simultaneously to give the standard Compton equation for the wavelength shift

$$(\lambda_f - \lambda_i) = \frac{2h}{m_e c}\sin^2\frac{\theta}{2} + \frac{h}{2m_e c}\frac{\lambda_f}{\lambda_i}\cos^2\theta \tag{39}$$

At $\theta = 90°$

$$\lambda_f - \lambda_i = \frac{h}{m_e c}, \tag{40}$$

a result that has no classical counterpart.[1] The wavelength shift is in the X-ray region of the spectrum.

The theory of Compton's effect, embodied in Eqs. (36)–(38), takes no account of the interaction energy

$$\Delta E_{\Pi} = -\langle \hat{m} \cdot \hat{B}_{\Pi} \rangle \tag{41}$$

In consequence, Eq. (38) for the conservation of kinetic energy must be modified to

$$m_e c^2 + \frac{hc}{\lambda_i} = \left(p^2c^2 + m_e^2c^4\right)^{1/2} + \frac{hc}{\lambda_f} + \Delta E_{\Pi} \tag{42}$$

i.e., ΔE_{Π} contributes to the total energy after collision. Also the theory

(36) to (38) takes no account of the intrinsic angular momenta of either photon or electron, and there must also be conservation of rotational kinetic energy and angular momentum. However, equations (36), (37), and (42) suffice to solve for $\lambda_f - \lambda_i$ in the presence of ΔE_Π. For an electromagnetic beam of N photons (a circularly polarized laser) reflected at 90° off the electron beam, solving (36), (37), and (42) gives

$$\lambda_f - \lambda_i = \frac{h/m_e c + \lambda_i \lambda_f (\Delta E_\Pi / hc)(1 - \Delta E_\Pi / 2 m_e c^2)}{1 - \Delta E_\Pi / m_e c^2} \tag{43}$$

We consider the order of magnitude of ΔE_Π compared with $m_e c^2$. The magnitude of the observable associated with the operator $\hat{m}$ is the Bohr magneton multiplied by the electron's g factor (2.002), and is about 10^{-23} J T^{-1}. The magnitude of the observable with the elementary photon operator $\hat{B}_\Pi$ is[7]

$$|\mathbf{B}_\Pi| \sim 10^{-7} I_0^{1/2} \tag{44}$$

and for I_0 of 1.0 W cm^{-2} (10 000 W m^{-2}) is of the order 10^{-5} T. Therefore ΔE_Π is of the order 10^{-28} J for this intensity. However, $m_e c^2$ is of the order 10^{-13} J per electron, so that to an excellent approximation

$$\frac{\Delta E_\Pi}{m_e c^2} \ll 1 \tag{45}$$

and Eq. (43) reduces to

$$\lambda_f - \lambda_i = \frac{h}{m_e c} + \lambda_i \lambda_f \frac{\Delta E_\Pi}{hc} \tag{46}$$

which is consistent with Eq. (40) for $\Delta E_\Pi = 0$. It is useful to express Eq. (46) in terms of wave numbers:

$$\bar{\nu}_f = \frac{m_e c \bar{\nu}_i}{m_e c + h \bar{\nu}_i} - \frac{m_e \Delta E_\Pi}{m_e hc + h^2 \bar{\nu}_i} \tag{47}$$

We now compare the order of magnitude of $m_e hc$ (about 10^{-55} kg J m) with $h^2 \bar{\nu}_i$, which is about $4.4 \times 10^{-67} \bar{\nu}_i$ kg J m; and find that for $\bar{\nu}_i \le 10^9$

m^{-1} (10^7 cm^{-1})

$$\bar{\nu}_f - \bar{\nu}_i = -\frac{\Delta E_\Pi}{hc} \tag{48}$$

to a very good approximation. In terms of frequency in hertz

$$\Delta f = -\frac{\Delta E_\Pi}{h} = \frac{\langle \hat{m} \cdot \hat{B}_\Pi \rangle}{h} \tag{49}$$

which is Eq. (35).

This equation shows that at electromagnetic frequencies well below 10^7 cm^{-1} (wave numbers), for example, at visible frequencies, the change in frequency in hertz in a circularly polarized laser reflected at right angles off an electron beam is given by Eq. (49). This is based on the interaction energy $\langle \hat{m} \cdot \hat{B}_\Pi \rangle$ between $\hat{m}$ of the electron and $\hat{B}_\Pi$ of the photon, two elementary properties of quantized matter. In general, the frequency shift is proportional to the square root of the incident laser intensity I_0. The interaction energy is quantized according to the quantum theory[19–21] of operator products, and can be expressed as the expectation value

$$\begin{aligned} \Delta E_\Pi &= -\langle IJFM_F | \hat{m} \cdot \hat{B}_\Pi | I'J'F'M'_F \rangle \\ \hat{m} &= -2.002 \gamma_e \hat{I} \equiv -g_e \gamma_e \hat{I} \end{aligned} \tag{50}$$

where I is the angular momentum quantum number of the electron ($I = \frac{1}{2}$), and J is the angular momentum quantum number of the photon (a positive integral quantity greater than zero[1]). Here γ_e is the gyromagnetic ratio and g_e the electronic g factor.[1] The quantum number F is given by the Clebsch-Gordan series

$$F = J + I, \ldots, |J - I| \tag{51}$$

and the expectation value of the Z component of the resultant angular momentum operator $\hat{F}$ is given by $M_{F\hbar}$, with the selection rule

$$\Delta M_F = \pm 1 \tag{52}$$

and M_F having $(2F + 1)$ values from $-F$ to F as usual. Therefore, depending on the value of J, there are several different values possible of the frequency shift Δf; i.e., an analysis of the reflected laser beam will reveal their presence as a spectrum. The diagonal matrix elements of (50) can be worked out analytically,[19-21] giving the frequency shift

$$\Delta f = -\frac{10^{-7} I_0^{1/2} g_e g \gamma_e}{(2\pi)}$$

$$g = \left[3(2F+1)I(I+1)(2I+1)J(J+1)(2J+1)\right]^{1/2} \times \begin{bmatrix} I & I & 1 \\ J & J & 1 \\ F & F & 0 \end{bmatrix} \quad (53)$$

Here the quantity in braces is the well-known 9-j symbol. For $I = \frac{1}{2}$, $J = 1$, and $F = \frac{3}{2}$, this is -0.05, and for an incident circularly polarized laser intensity of 1.0 W cm^{-2} the frequency shift in hertz from Eq. (19) is $\sim 20\,000 M_F$ Hz. For $I = \frac{1}{2}$, $J = 1$, $F = \frac{1}{2}$, the shift is $\sim -10\,000 M_F$ Hz. For modest incident laser intensity the shift is already in the kilohertz range, easily measurable, and has the following characteristics:

1. It should change sign with respect to the incident laser frequency if the incident laser's circular polarity is switched from left to right (i.e., if the azimuthal expectation value of the photon's static magnetic flux density operator is changed from $B_0|M_J|$ to $-B_0|M_J|$).
2. There should be no frequency shift or spectral detail if the incident laser is linearly polarized, because the net static flux density delivered by the photon beam is zero, being 50% $B_0|M_J|$ and 50% $-B_0|M_J|$.
3. The shifts with respect to the incident laser frequency should be proportional to the square root of the laser's incident intensity I_0.

These features should provide adequately for the measurement of the novel elementary property $\hat{B}_\Pi$, and the method can be extended to other elementary particles by using, for example, a neutron beam in place of the laser beam. This would allow an experimental determination of whether neutrons also have elementary magnetic flux density similar to $\hat{B}_\Pi$ of the photon (the elementary "magneton" of electromagnetic radiation).

IV. DISCUSSION

In addition to the frequency shift phenomenon introduced in this paper, it is possible to predict novel phenomena due to $\hat{B}_\Pi$ wherever the photon interacts with matter, one of these being the optical Zeeman effect and another the optical Faraday effect. Both effects have been suggested in a semiclassical context recently[22-26] and their existence is reinforced by that

of $\hat{B}_\Pi$ on a fundamental level. In the optical Zeeman effect the magneton $\hat{B}_\Pi$ plays the role taken by a static magnetic flux density in the conventional Zeeman effect and its relatives, the anomalous Zeeman effect, and Paschen-Back effect. In the optical Faraday effect the magneton rotates the plane of polarization of a linearly polarized problem. Clearly, the interaction energy in these magneton-based effects must be constructed from the quantum theory of operator products, as in Eq. (53), and off-diagonal components of the matrix elements so obtained are important in general, as well as diagonal elements. In molecules, these off-diagonal elements must probably be worked out numerically, but for this purpose there are many standard ab initio packages available.

Clearly, the magneton $\hat{B}_\Pi$ is capable also of generating optically induced resonance spectra (optical NMR and ESR), and evidence for this has been obtained recently,[27] although a full theoretical description is not yet available, and must probably be generated ab initio, using software packages such as HONDO or GAUSSIAN 90 by taking into account the interaction of the magneton $\hat{B}_\Pi$ with the large and complicated chiral test molecule used by Warren et al.[27] to show interesting, site-specific effects of a low-power, circularly polarized laser on a conventional one- and two-dimensional NMR spectrum. This technique appears to have considerable promise, especially if the laser intensity could be increased by pulsing. In principle, considerable increase in resolution of conventional resonance spectra is obtainable.[7-10,27]

Finally, the optical equivalent of the Stern-Gerlach experiment is possible in principle by using an expanding or focused laser beam to generate the optical equivalent of a magnetic field gradient in the axis of propagation of a circularly polarized laser beam coaxial with a beam of atoms, such as silver atoms. The magneton theory of this will be the subject of future work.

APPENDIX

The neutron has a magnetic moment and quantum number $I = \frac{1}{2}$, but is approximately 10 000 times heavier than the electron. The theory of spectral detail in a circularly polarized laser beam reflected off a beam of neutrons can be set up in the same way as for an electron beam, but the expected splitting in the reflected laser beam is much smaller and much more difficult to detect with a spectrometer. However, the presence of such detail would be further evidence for the existence of the magneton $\hat{B}_\Pi$. It is also possible to replace the electron beam by a beam of atoms with net electronic dipole moment, for example, and for any material with net electronic or nuclear dipole moment the interaction with the magneton produces spectral detail in the reflected laser beam, which can be analyzed spectroscopically. The experiment is not confined, furthermore, to beams, but can also proceed, in principle, by reflecting the circularly polarized laser beam from a material of interest with a net magnetic dipole moment. In this context, the behavior of superconducting surfaces is particularly interesting[28] and the magneton $\hat{B}_{\Pi}$ could well provide an entirely novel way of analyzing type I and II superconductors by reflecting a laser beam from the surface of the material at right angles, and looking for the specific magnetic effects due to $\hat{B}_{\Pi}$. Type II superconductors are of particular interest[28] because they remain superconducting in the presence of magnetic flux density, which is known to propagate in such material in the form of quantized flux lines, each carrying one quantum[28] of magnetic flux. In this case it might be expected that the magneton $\hat{B}_{\Pi}$ would be converted in the type II superconductor to the quantum of magnetic flux hc/m_e. Furthermore, Bitter imaging techniques[28] can be utilized in superconductors to detect the presence of magnetization due to the magneton $\hat{B}_{\Pi}$ of the circularly polarized laser, which can also be used to scan the surface of the sample and induce individual vortices of magnetization in the superconducting sample.

More generally and fundamentally it is interesting to speculate on the possibility that elementary particles with spin are also capable of generating magnetons of flux akin to $\hat{B}_{\Pi}$ of the photon. The latter is massless and travels at c in vacuo, whereas the neutron, for example, has mass and does not travel at c. The electron also has mass and does not travel at c. Nevertheless, the electron and neutron both have intrinsic, irremovable spin, essentially in the same way as the photon, and in terms of symmetry and dimensionality, both electron and neutron can generate magnetons of flux through equations identical in structure to Eq. (1) of the text. However, neither electron nor neutron are electromagnetic plane waves, but different types of wave, and the question comes down to whether a beam of electrons or neutrons carries a finite, scalar flux density amplitude akin to $\hat{B}_{\Pi}$ of the photon. It is known that the neutron, for example, obeys the Planck relation between energy and frequency, but there appears to be no evidence that the Maxwell equations can be written for neutrons or electrons, and solved to generate plane waves akin to electromagnetic waves. It appears at present that the electron and neutron generate elementary magnetic dipole moments and that the photon generates the elementary magnetic field $\hat{B}_{\Pi}$.

These speculations can be extended to other elementary particles with intrinsic spin (i.e., angular momentum operators) and experiments can be devised to test the speculations. For example, if the electron does indeed generate its own magneton, a quantized magnetic flux density operator, $\hat{B}_e$, a beam of electrons reflected off a beam of neutrons will generate the interaction Hamiltonian

$$\Delta H_1 = -\hat{m}_n \cdot \hat{B}_e \tag{A.1}$$

where $\hat{m}_n$ is the magnetic dipole moment of the neutron. This is quantized as in Eq. (53), and consequently the energy of the emerging electron beam must record in some way the presence of ΔH_1. If the electron beam has wave properties, it should be analyzable spectrally, and the spectral pattern due to the interaction ΔH_1 should be measurable experimentally. Electron diffraction is evidence that electrons can behave as waves as well as particles, which is a result of the de Broglie principle. Proceeding with the speculative logic in this way, it becomes clear that reflecting a beam of any particle with intrinsic elementary spin from any other particle beam with intrinsic elementary magnetic dipole moment could, in principle, result in an interaction energy of type (A.1). In other words, we speculate on the possibility that elementary particles in general can each generate its own magneton.

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THE PHOTON'S MAGNETOSTATIC FLUX QUANTUM: SYMMETRY AND WAVE PARTICLE DUALITY—FUNDAMENTAL CONSEQUENCES IN PHYSICAL OPTICS

I. INTRODUCTION

It has recently been demonstrated theoretically that there exists an operator $\hat{B}_{\Pi}$ of the quantized electromagnetic field that describes the photon's magnetostatic flux density:

$$\hat{B}_{\Pi} = B_0 \frac{\hat{J}}{\hbar} \tag{1}$$

Here B_0 has been interpreted[2–5] as a scalar magnetic flux density amplitude of a beam of circularly polarized light consisting of one photon, and $\hat{J}$ is the boson operator[6,7] describing that photon's quantized angular momentum. The classical equivalent of $\hat{B}_{\Pi}$ is a novel axial vector $\mathbf{B}_{\Pi}$, which is directed in the propagation axis of the beam. In this paper it is demonstrated using elementary tensor algebra, and from inspection of the Maxwell equations of the classical field, that there is another possible interpretation of the scalar amplitude B_0, designated henceforth by $(B_0)_+$,

where the + subscript is to be interpreted as "positive to parity inversion." It turns out that B_0 can be interpreted both as a scalar and as a pseudoscalar quantity, designated $(B_0)_-$, where the minus subscript means "negative to parity inversion." This is designated "symmetry duality," and is shown in this work to imply that $\hat{B}_{\Pi}$ can be defined simultaneously in terms of the photon's angular momentum operator $\hat{J}$ and linear momentum operator $\hat{p}$, a result that is a generalization of a keystone of wave mechanics, the de Broglie wave particle duality.[6] The latter is linked through $\hat{B}_{\Pi}$ to a symmetry duality in Maxwell's classical equations.

It has already been shown theoretically[2–5] and experimentally[8,9] that circularly polarized light can magnetize, leading, for example, to the inverse Faraday effect[10–13] and novel, potentially very useful, light-induced shifts in NMR spectroscopy[8,9] in one and more dimensions. The existence of the operator $\hat{B}_{\Pi}$ and its classical equivalent $\mathbf{B}_{\Pi}$ makes it much easier to interpret these magnetization effects by treating circularly polarized light as a "magnet" generating this novel flux quantum per photon. The $\hat{B}_{\Pi}$ concept also makes it relatively straightforward to forecast the existence of novel spectral phenomena, such as optical Zeeman, anomalous Zeeman, and Paschen-Back effects,[3] an optical Faraday effect and optically induced magnetic circular dichroism,[4] and optical Stern-Gerlach effect, using a focused laser beam to produce a light-induced magnetic field gradient, optical ESR effects, optically induced effects in interacting beams, such as a beam of circularly polarized photons reflected[5] from a beam of polarized electrons, and so on. All these effects can be thought of as arising from the replacement (or augmentation) of an ordinary magnet by or with a circularly polarized laser. These theories allow scope for the development of several novel analytically useful methods.

In this paper it is shown that $\mathbf{B}_{\Pi}$ is related directly to the ubiquitous,[14] pseudoscalar, third Stokes parameter S_3 of the classical electromagnetic plane wave, which becomes in quantum-field theory the third Stokes operator of Tanaś and Kielich.[1] Therefore, it follows immediately that several well-known phenomena of physical optics can be reinterpreted fundamentally in terms of the operator $\hat{B}_{\Pi}$, or its classical equivalent $\mathbf{B}_{\Pi}$. Examples include ellipticity in the plane wave, ellipticity developed in the measuring beam of the electrical Kerr effect, and circular dichroism, which are shown in this work to be magneto-optic phenomena. Therefore, not only does $\hat{B}_{\Pi}$ allow this reinterpretation, in both classical and quantum field theory, it also allows a link to be made between de Broglie wave particle duality and symmetry duality in the classical Maxwell equations. It appears, therefore, to go to the root of physical optics and field theory.

In Section II we develop the mathematical basis of symmetry duality with elementary vector and tensor algebra, before embarking in Section

III on a discussion of symmetry duality in the link between $\mathbf{B}_\Pi$ and S_3. In Section IV we develop the link between wave particle duality and the symmetry duality in Maxwell's equations demonstrated in Section III, and discuss qualitatively the implications for elementary particle theory. In Section V we develop the link between $\mathbf{B}_\Pi$ and S_3 into a novel explanation for ellipticity and circular dichroism in physical optics.

II. SYMMETRY DUALITY IN THE VECTOR PRODUCT OF TWO POLAR VECTORS

It is well known that the components of a vector that can be written as the cross product of two polar vectors do not change sign under parity inversion ($\hat{P}$) and that the vector so formed is an axial vector,[15] or pseudovector. The conjugate product of the classical electromagnetic field[2–5]

$$\mathbf{\Pi}^{(\mathrm{A})} = \mathbf{E} \times \mathbf{E}^* = 2\left(E_0^2\right)_+ \mathrm{i}\mathbf{e}_+ \tag{2}$$

where $\mathbf{E}^*$ is the polar complex conjugate of the polar electric field strength vector $\mathbf{E}$, is an axial vector, therefore. Here, $\mathbf{e}_+$ is an axial unit vector, positive to $\hat{P}$, and the quantity $(E_0^2)_+$ is a scalar, also positive to $\hat{P}$. The overall motion reversal ($\hat{T}$) symmetry of $\mathbf{\Pi}^{(\mathrm{A})}$ is negative, and it is natural to define $\mathbf{e}_+$ as a $\hat{T}$-negative unit vector, so that $(E_0^2)_+$ is a $\hat{T}$ positive scalar.

It appears at first sight that these definitions are both necessary and sufficient for the complete definition of the axial vector $\mathbf{\Pi}^{(\mathrm{A})}$; but mathematically, there is an alternative, which is revealed through writing any arbitrary axial vector as

$$\mathbf{C} = C_+\mathbf{e}_+ = C_-\mathbf{e}_- \tag{3}$$

where C_+ and $\mathbf{e}_+$ are respectively $\hat{P}$-positive scalar and unit axial vector quantities, and where C_- and $\mathbf{e}_-$ are respectively $\hat{P}$-negative pseudoscalar and $\hat{P}$-negative polar unit vector quantities. The overall $\hat{P}$ symmetry of the complete axial vector $\mathbf{C}$ is positive in both cases.

This seemingly mundane observation in elementary vector analysis has far-reaching consequences in the theory of the classical and quantized electromagnetic fields. In tensor algebra, the general vector cross product $\mathbf{C} = \mathbf{A} \times \mathbf{B}$ is written with the third rank antisymmetric (or alternating) unit tensor, $\varepsilon_{\alpha\beta\gamma}$, known as the Levi-Civita symbol[7,15]:

$$C_\alpha = \tfrac{1}{2}\varepsilon_{\alpha\beta\gamma}(A_\beta B_\gamma - A_\gamma B_\beta) \equiv \tfrac{1}{2}\varepsilon_{\alpha\beta\gamma}C_{\beta\gamma} \tag{4}$$

where the $\hat{P}$ symmetry of $\varepsilon_{\alpha\beta\gamma}$ is negative, so that $C_{\beta\gamma}$ is a $\hat{P}$-negative antisymmetric polar tensor of rank two. Evidently, C_α must be $\hat{P}$-positive, and is the rank one axial tensor (i.e., an axial vector). However, $C_{\beta\gamma}$ can also be written[15] as

$$C_{\beta\gamma} = -\mathrm{i}\varepsilon_{\beta\gamma}C_- \tag{5}$$

where $\varepsilon_{\beta\gamma}$ is the $\hat{P}$-positive, axial, unit antisymmetric tensor of rank two, and C_- is the pseudoscalar of Eq. (3). Equation (5) shows that the polar antisymmetric tensor of rank two can be reduced, quite generally, to a pseudoscalar, a particular result of a generalization in the relativistic theory of the classical electromagnetic field.[15] Note that $C_{\beta\gamma}$ is purely imaginary from the Hermitian properties of the general second-rank tensor, which can always be written as a sum of real symmetric and imaginary antisymmetric parts.[15]

Therefore,

$$C_\alpha = -\frac{\mathrm{i}}{2}\varepsilon_{\alpha\beta\gamma}\varepsilon_{\beta\gamma}C_- \tag{6}$$

or

$$C_\alpha \equiv -\mathrm{i}C_+\varepsilon_{\alpha+} = -\frac{\mathrm{i}}{2}\varepsilon_{\alpha\beta\gamma}\varepsilon_{\beta\gamma}C_- \tag{7}$$

where $\varepsilon_{\alpha+}$ is the rank one axial unit tensor, positive to $\hat{P}$, and C_+ is a $\hat{P}$-positive scalar. Recall that C_- is a $\hat{P}$-negative pseudoscalar. Equations (3) and (7), using vector and tensor notation, respectively, are expressions of symmetry duality, a purely mathematical result that shows that a scalar and pseudoscalar may both be used to define an axial vector. Clearly, if we take the magnitude ($|\mathbf{C}|$) of the axial vector $\mathbf{C}$ in Eq. (3), we obtain

$$\begin{aligned} C^2 &\equiv \mathbf{C}\cdot\mathbf{C} = C_+^2 = C_-^2 \\ |\mathbf{C}| &= \left|(C^2)^{1/2}\right| = |C_+| = |C_-| \end{aligned} \tag{8}$$

so that the positive parts of the scalar C_+ and pseudoscalar C_- are equal in absolute magnitude. This same result can be obtained from the tensor

Equation (7) by taking a particular Z component:

$$C_Z = -iC_+\varepsilon_{Z+} = -\frac{i}{2}C_-(\varepsilon_{ZXY}\varepsilon_{XY} + \varepsilon_{ZYX}\varepsilon_{YX}) \tag{9}$$

where the Einstein convention of summation over repeated indices has been used on the right side. With the component definitions $\varepsilon_{ZXY} = 1$, $\varepsilon_{XY} = 1$, $\varepsilon_{ZYX} = -1$, and $\varepsilon_{YX} = -1$, we obtain

$$C_+\varepsilon_{Z+} = C_-\varepsilon_{Z-} \tag{10}$$

where

$$\varepsilon_{Z-} \equiv \varepsilon_{ZXY}\varepsilon_{XY} + \varepsilon_{ZYX}\varepsilon_{YX} \tag{11}$$

is the Z component of the $\hat{P}$-negative polar unit tensor of rank one, $\varepsilon_{\alpha-}$. Note that Eqs. (3) and (10) are identical in symmetry character for the considered Z components of C. Equation (10), which is a direct and fundamental consequence of elementary tensor algebra, again shows the symmetry duality between scalar and pseudoscalar in the definition of the axial, or pseudo, vector. It is now possible to apply the purely mathematical principle of symmetry duality to the classical, nonrelativistic (or relativistic) field to obtain novel information of fundamental importance in physical optics, particularly in respect of a $\hat{P}$-positive, $\hat{T}$-negative axial vector, a novel magnetostatic field, $\mathbf{B}_\Pi$,[2-5] associated with the electromagnetic plane wave or in the quantized field, the magnetostatic flux density operator $\hat{B}_\Pi$ of the photon.

III. AN EXAMPLE OF SYMMETRY DUALITY: THE RELATION BETWEEN $\mathbf{B}_\Pi$ AND THE STOKES PARAMETER S_3

Consider the classical electromagnetic wave in free space, so that the real scalar refractive index is unity. It follows from Maxwell's equations for a plane wave that

$$E_0 = cB_0 \tag{12}$$

where E_0 and B_0 are $\hat{P}$- and $\hat{T}$-positive scalars, amplitudes, respectively, of the electric field strength and magnetic flux density. The intensity of the wave is defined in free space by

$$I_0 = \varepsilon_0 c E_0^2 \tag{13}$$

where ε_0 is the free space permittivity[6] in S.I. units, and c is the speed of light in vacuo. With Eqs. (12) and (13), Eq. (2) can be rewritten as

$$\mathbf{\Pi}^{(A)} = 2(E_0)_+ ci\mathbf{B}_\Pi \tag{14}$$

where we have defined the magnetostatic flux density vector $\mathbf{B}_\Pi$[2-5] of the classical electromagnetic plane wave in free space:

$$\mathbf{B}_\Pi \equiv (B_0)_+\mathbf{e}_+ \tag{15}$$

where $\mathbf{e}_+$ is a $\hat{P}$-positive unit axial vector. The overall $\hat{T}$ symmetry of $\mathbf{B}_\Pi$ is negative, and the overall $\hat{P}$ symmetry is positive. In the introduction we have given an account of the role of $\mathbf{B}_\Pi$ in the reinterpretation of well-known effects, such as circular dichroism and ellipticity, and its mediating role in new effects such as optical NMR and ESR,[9] optical Faraday[4] and Zeeman[3] effects, optical Stern-Gerlach effects, optical Compton scattering,[5] and so on.[2-5] Its quantized equivalent is the magnetostatic flux density operator of Eq. (1), in which $(B_0)_+$ is defined as the $\hat{P}$- and $\hat{T}$-positive scalar magnetic flux density amplitude of one photon.

Again, as in Section II, it would appear at first sight as if the definition of the seemingly mundane quantity B_0 as a $\hat{P}$- and $\hat{T}$-positive scalar is sufficient. Remarkably, however, this is not the case, there is an alternative definition possible of the novel classical vector $\mathbf{B}_\Pi$ which uses B_0 as a pseudoscalar. Not only does this emerge naturally from the Maxwell equations for the plane wave, it also provides a natural link between $\mathbf{B}_\Pi$ and the third Stokes parameter S_3.[1,7,14,15]

These conclusions emerge straightforwardly from the equations linking the $\mathbf{E}$ and $\mathbf{B}$ vectors of the classical electromagnetic plane wave in a medium of refractive index n, defined through the classical wave vector $\boldsymbol{\kappa}$, a $\hat{T}$- and $\hat{P}$-negative polar vector directed in the propagation axis Z of the plane wave[7]:

$$\boldsymbol{\kappa} = \frac{\omega}{c}\mathbf{n} \qquad n = \frac{c}{v} \tag{16}$$

Here ω is the angular frequency in radians per second of the plane wave, as usual. Maxwell's equations give[7]

$$\mathbf{B} = \frac{1}{c}\mathbf{n} \times \mathbf{E} \qquad \mathbf{E} = -\frac{c}{n^2}\mathbf{n} \times \mathbf{B} \tag{17}$$

In free space, the positive absolute magnitude of the $\hat{P}$- and $\hat{T}$-positive scalar n is unity. Using Equation (17) yields the conjugate product

$$\mathbf{E} \times \mathbf{E}^* = -\frac{c}{n^2}\mathbf{E} \times (\mathbf{n} \times \mathbf{B}^*) = -\frac{\mathbf{n}}{n}\frac{c(\mathbf{E} \cdot \mathbf{B}^*)}{n} \tag{18}$$

We note that the vector $\mathbf{n}$ is a $\hat{P}$-negative, $\hat{T}$-negative polar vector, defined as usual[7] as a propagation vector whose scalar magnitude is equal to the $\hat{T}$-positive scalar refractive index n; the dot product $\mathbf{E} \cdot \mathbf{B}^*$ is a $\hat{T}$-positive pseudoscalar. Equation (18) reduces to

$$\underset{\text{Scalar}}{\left(\frac{E_0 B_0 c}{n}\right)_+} \underset{\text{Axial unit vector}}{\mathbf{e}_+} = \underset{\text{Pseudoscalar}}{\left(\frac{E_0 B_0 c}{n}\right)_-} \underset{\text{Polar unit vector}}{\frac{\mathbf{n}}{|\mathbf{n}|}} \tag{19}$$

in which we have designated the various symmetries. It follows algebraically that

$$(B_0)_+ \mathbf{e}_+ = (B_0)_- \frac{\mathbf{n}}{n} \tag{20}$$

which can be rewritten in the notation of Section II as an example of symmetry duality in the Maxwell equations:

$$(B_0)_+ \mathbf{e}_+ = (B_0)_- \mathbf{e}_- \qquad \mathbf{e}_- \equiv \frac{\mathbf{n}}{n} \tag{21}$$

This shows that classical vector $\mathbf{B}_\Pi$ can be defined simultaneously in terms of the unit axial vector $\mathbf{e}_+$ and the unit polar vector $\mathbf{e}_-$, which is related to the propagation vector $\boldsymbol{\kappa}$, the photon linear momentum. In free space, with $n = 1$,

$$\mathbf{B}_\Pi = (B_0)_+ \mathbf{e}_+ = (B_0)_- \mathbf{e}_- = (B_0)_- \frac{c}{\omega}\boldsymbol{\kappa} \tag{22}$$

demonstrating a duality between the classical angular and linear momentum of the plane wave. We shall see that this is none other than the classical equivalent of the de Broglie wave particle duality for the photon in the quantized field. Before making the transition to the quantized field, however, another fundamentally new result emerges when we consider the definition[15] of the Stokes parameter S_3:

$$E_\alpha E_\beta^* - E_\beta E_\alpha^* = -\mathrm{i}\varepsilon_{\alpha\beta}(S_3)_- \tag{23}$$

so that

$$(S_3)_- \equiv (E_0^2)_- \tag{24}$$

is a pseudoscalar quantity, implying inter alia the symmetry duality

$$\mathbf{\Pi}^{(A)} = 2(E_0^2)_+ \mathrm{i}\mathbf{e}_+ = 2(S_3)_- \mathrm{i}\mathbf{e}_- \tag{25}$$

It follows directly that the magnetostatic vector $\mathbf{B}_\Pi$ can be defined in free space ($n = 1$) in terms of $(S_3)_-$ as follows:

$$\mathbf{B}_\Pi = \frac{(S_3)_-}{2E_0 c}\mathbf{e}_- \equiv (B_0)_- \mathbf{e}_- \tag{26}$$

and we find that the role of B_0 as pseudoscalar is none other than the Stokes parameter S_3 scaled by an appropriate $\hat{P}$- and $\hat{T}$-positive scalar quantity. Thus, $\mathbf{B}_\Pi$ can be defined in free space through the symmetry duality

$$\mathbf{B}_\Pi = (B_0)_+ \mathbf{e}_+ = \frac{(S_3)_-}{2E_0 c}\mathbf{n} = \frac{(S_3)_-}{2E_0 c}\frac{c}{\omega}\boldsymbol{\kappa} \tag{27}$$

where the unit polar vector $\mathbf{n}$ can be identified with the unit vector $\mathbf{e}_-$ of this section. We thus forge a novel and fundamental link between the pseudoscalar magnitude of $\mathbf{B}_\Pi$ and the pseudoscalar S_3.

VI. SYMMETRY DUALITY AND WAVE PARTICLE DUALITY FOR THE PHOTON

Equation (1) shows that the photon's novel magnetic field operator $\hat{B}_\Pi$ is directly proportional to its well-defined[6] angular momentum boson operator $\hat{J}$ through B_0 in its scalar representation $(B_0)_+$, interpreted as the magnetic flux density amplitude of a single photon. The latter is a massless lepton that propagates at the speed of light and is not localized in space[16] unlike a massive lepton such as the electron or proton. These well-known properties are contained in Eq. (1), in that B_0 varies with intensity I_0 for a beam of circularly polarized light containing one photon, and therefore B_0 for one photon depends on the beam cross section, a finite area. The eigenvalues of the operator $\hat{J}$ are known to be $M_J\hbar$, $M_J = \pm 1$; there is no $M_J = 0$ component from relativistic considerations.[6,7] Therefore, the eigenvalues of $\hat{B}_\Pi$ are $\pm(B_0)_+$, where $(B_0)_+$ is a scalar, the positive eigenvalue corresponds to one particular circular polarization, and vice versa,[7] as in the convention for the operator $\hat{J}$.

We now use the result[6,7] that the eigenvalue of the linear momentum operator $\hat{p}$ of the photon is

$$\mathbf{p} \equiv \langle \hat{p} \rangle = \hbar\boldsymbol{\kappa} \tag{28}$$

where $\boldsymbol{\kappa}$ is the wave vector as defined classically in the preceding section. It follows straightforwardly from Eqs. (22) and (28) that in free space ($n = 1$)

$$\hat{B}_\Pi = (B_0)_+ \frac{\hat{J}}{\hbar} = \frac{(B_0)_-}{n} \frac{c}{\omega} \frac{\hat{p}}{\hbar} \tag{29}$$

which expresses the duality of Eq. (22) in terms of quantum field theory, and shows that the $\hat{B}_\Pi$ operator of the photon is simultaneously proportional to both its angular and linear momentum operators. Equation (29) summarizes a duality in symmetry, linear/angular momentum, and wave–particle character with the results

$$\begin{aligned} \hat{T}[(B_0)_+] &= +\hat{T}[(B_0)_-] \\ \hat{P}[(B_0)_+] &= -\hat{P}[(B_0)_-] \end{aligned} \tag{30}$$

Equation (29) implies the free space relation

$$\hat{p} = n\frac{\omega}{c}\hat{J} \qquad n = 1 \tag{31}$$

The expectation value of $\hat{p}$ is therefore given by the expectation value of $\hat{J}$, which is $\pm\hbar$. Taking without loss of generality the positive eigenvalue $\hbar$, we have, with $n = 1$,

$$p = \frac{\omega}{c}\hbar \tag{32}$$

which is the de Broglie wave particle duality for the photon.

We have therefore succeeded in relating directly the de Broglie wave-particle duality of quantum mechanics to the novel symmetry duality (22) of classical electromagnetic field theory. It has also been shown that the novel flux quantum $\hat{B}_\Pi$ is definable simultaneously in terms of $\hat{J}$ and $\hat{p}$, one operator being directly proportional to the other, implying that both must be quantized in the same way. In a sense, therefore, $\hat{B}_\Pi$ is the keystone of de Broglie's concept of duality for the photon.

Furthermore, contemporary elementary particle theory argues that the photon is a chiral entity, a massless lepton that travels in any frame of reference at c, and whose chirality, in consequence,[17] is well defined in terms of the eigenvalues of Dirac's $\hat{\gamma}_5$ operator. The chirality of a lepton with mass (i.e., a massive lepton) such as the electron is not well defined, leading to the idea[17] that mass itself is ill-defined chirality. Well-defined chirality in the photon can be thought of as a consequence of superimposed linear and angular momentum, and Eq. (29) shows that there is a duality between these two fundamental quantities. It appears therefore that the novel $\hat{B}_\Pi$ operator of the photon is a true chiral influence as defined by Barron,[17] and is therefore fundamentally different in nature from a magnetostatic flux density, such as a magnetic field generated in an electromagnet. The latter is now known to be an example of a false chiral influence,[17] and cannot, for example, be a cause of enantioselective synthesis. This is in contrast to the circularly polarized electromagnetic field, which le Bel[18] in 1874 conjectured to be a truly chiral influence, and which is now known to influence enantioselectivity in chemical reactions. The definition of the $\hat{B}_\Pi$ operator in Eq. (29) also allows insight to the symmetry of natural optical activity, i.e., circular dichroism and optical rotatory dispersion, as developed in the next section.

It may be conjectured that a magnetostatic flux quantum $\hat{B}_\Pi$ is always carried by a massless lepton whose chirality can be precisely defined as the eigenvalues of the Dirac $\hat{\gamma}_5$ operator; and, conversely, that the massive lepton does not support $\hat{B}_\Pi$ and does not have precisely defined eigenvalues of $\hat{\gamma}_5$. This conjecture would imply that fundamentally, $\hat{B}_\Pi$ is always a consequence of the absence of mass. It would therefore follow that the neutrino (and antineutrino) carries a $\hat{B}_\Pi$ field, but that the electron, neutron, and proton do not. However, it is not clear whether the neutrino has a classical counterpart such as the classical electromagnetic plane wave, the counterpart of the photon. If the parallel between photon and neutrino can be carried further, it would appear that the neutrino must also be thought of as unlocalized in space. This would imply inter alia that localization in space implies the presence of mass and the absence of well-defined chirality (or well-defined eigenvalues of $\hat{\gamma}_5$), and that the absence of mass implies the absence of space localization. Carrying the argument further, wave particle duality in a massive lepton such as the electron has been observed, because an electron beam can be diffracted, for example, but since the electron is localized and does not have well-defined chirality, its wave nature must be fundamentally different from that of the photon, and in consequence, no $\hat{B}_{\Pi}$ can be constructed or defined for the electron. Wave particle duality in the electron is therefore fundamentally different in nature from duality in the photon. The electron has a magnetic dipole moment that is proportional to the electron's spin angular momentum operator through the gyromagnetic ratio. We therefore conjecture that a massless lepton cannot support a magnetic dipole moment, because its effective gyromagnetic ratio would be infinite, but can support a magnetostatic flux quantum. The opposite is true for a massive lepton. With these assumptions, the $\hat{B}_{\Pi}$ operator of a massless lepton would always be able to form an interaction Hamiltonian operator to first order with the magnetic dipole moment operator of a massive lepton, an example being a photon beam interacting with an electron beam,[5] or a neutrino beam with a neutron beam and so on, giving rise to measurable effects in principle. The inference overall, therefore, is that a beam of massless leptons, for example, photons or neutrinos, can magnetize but cannot be magnetized, whereas a beam of massive leptons cannot magnetize but can be magnetized.

The charge conjugation symmetry operator can be defined as $\hat{C}$ (which operates to reverse the sign of charge), and with this definition we recall the fundamental Luders-Pauli-Villiers theorem[17]:

$$\hat{C}\hat{P}\hat{T} = \hat{1} \tag{33}$$

The violation of $\hat{P}$ has been observed[17] in a number of different ways, whereas the violation of $\hat{T}$ has been observed in only one critical experiment.[17] The violation of $\hat{P}$ leads to the result that the space-inverted enantiomers of a truly chiral entity such as the photon or neutrino are not degenerate, or exactly the same in energy, because of the existence of the $\hat{P}$ violating electroweak force.[17] In contrast, the space-inverted "enantiomers" of a falsely or pseudo chiral entity, such as an ordinary magnetic field, are precisely the same in energy.[17] Thus, it is important to note that the true enantiomer of the photon, or neutrino, is not generated by space inversion or by application of the $\hat{P}$ operator, i.e., by reversing the linear momentum and keeping the angular momentum the same. Assuming that the photon is uncharged, so that $\hat{C}$ has no effect, its true or exact enantiomer must be generated by simultaneous $\hat{P}$ and $\hat{T}$ violation in order to conserve the validity of the Luders-Pauli-Villiers theorem. (33). The true enantiomer of the left-handed photon is presumably, therefore, an object that must be designated the right-handed "antiphoton," and there is a very small, but nonzero, energy difference between the left-handed photon and the right-handed photon. If the right-handed photon is to be regarded as having a different energy from the left-handed photon, then either $\hat{P}$ has been violated and $\hat{T}$ and $\hat{C}$ have been conserved, or $\hat{T}$ has been violated and $\hat{P}$ and $\hat{C}$ have been conserved. Assuming that $\hat{C}$ has no effect on the photon, because it is uncharged, the combined operation $\hat{P}\hat{T}$ must be used to generate the right antiphoton from its true enantiomer, the left photon, and vice versa. The photon is an object whose chirality is generated only as a result of its simultaneous translational and rotational motion, and the novel $\hat{B}_{\Pi}$ operator is a fundamental manifestation of this chirality. The latter is conserved, furthermore, in the photon and antiphoton, because both travel at the speed of light and both are massless. It is also known[17] that neutrinos conserve chirality, in that only left-handed neutrinos and right-handed antineutrinos exist. The $\hat{P}$ violating weak force is known to play a critical part in the interaction of left-handed neutrinos with left, but not with right, spin-polarized relativistic electrons, and of right-handed antineutrinos with right, but not left, polarized relativistic electrons.[17]

These arguments lead to the interesting possibility that a beam of, say, left photons, each carrying a flux quantum $\hat{B}_{\Pi}$, may interact differently with a beam of left and right polarized relativistic electrons, each carrying the magnetic dipole moment $\hat{m}$, through the interaction Hamiltonian operator

$$\Delta\hat{H} = -\hat{m} \cdot \hat{B}_{\Pi}$$

This difference may be picked up by observation of the Zeeman splitting caused by $\Delta\hat{H}$ in, for example, a circularly polarized visible laser beam reflected from a polarized, relativistic, electron beam. Such an experiment has been proposed recently to evaluate the effect of $\hat{B}_{\Pi}$ (Ref. 5).

V. THE ROLE OF $\mathbf{B}_{\Pi}$ IN ELLIPTICITY AND ASSOCIATED EFFECTS IN PHYSICAL OPTICS, FOR EXAMPLE, CIRCULAR DICHROISM

The link between $|\mathbf{B}_{\Pi}|$ and the third Stokes parameter $(S_3)_-$ can be expressed through the intensity I_0 as

$$|\mathbf{B}_{\Pi}| = (B_0)_- = \left(\frac{\varepsilon_0}{4I_0c}\right)^{1/2}(S_3)_- \tag{34}$$

so that it follows that whenever $(S_3)_-$ occurs in physical optics, it can be replaced by the pseudoscalar quantity B_0, multiplied by the scalar $(4I_0 c/\varepsilon_0)^{1/2}$. This is a key link between the photon's magnetostatic field operator $\hat{B}_\Pi$, in its classical limit, and the ubiquitous $(S_3)_-$, revealing immediately the root cause of several well-known phenomena in physical optics.

As an example, the ellipticity η of an elliptically polarized beam of light is related to $(S_3)_-$ by

$$(S_3)_- = 2E_0^2 \sin 2\eta \tag{35}$$

with

$$\eta = \tan^{-1}\frac{b}{a} \tag{36}$$

where a and b are respectively the major and minor axes of the polarization ellipse.[7] This shows that there is a direct link between ellipticity and the vector $\mathbf{B}_\Pi$, which is the classical equivalent of the operator $\hat{B}_\Pi$. In the theory of the electrically induced Kerr effect,[7] for example, ellipticity is developed in an initially circularly polarized measuring beam after it has passed through a material to which a static, uniform, electric field has been applied perpendicular to the propagation direction of the beam and at 45° to the aximuth of an incident linearly polarized beam. For the emerging beam in the electric Kerr effect it can be shown that

$$|\mathbf{B}_\Pi| = (B_0)_- = \left(\frac{I_0}{\varepsilon_0 c^3}\right)^{1/2} \sin(2\eta) \tag{37}$$

showing that the root cause of ellipticity in the Kerr effect is the pseudoscalar magnitude $(B_0)_-$ of $\mathbf{B}_\Pi$. Note that for the incident, linearly polarized beam, $\mathbf{B}_\Pi$ is zero, but that in the transmitted, elliptically polarized beam it is nonzero.

Another example of the fundamental role of the pseudoscalar $(B_0)_-$ is the phenomenon of circular dichroism, which is a manifestation of optical activity, whereby the intensity of initially linearly polarized electromagnetic radiation transmitted by a structurally chiral material contains an excess of left over right circularly polarized components, or vice versa. In this context[7]

$$\frac{(S_3)_-}{(S_0)_+} = \frac{I_L - I_R}{I_L + I_R} \tag{38}$$

where $(S_0)_+$ is the zeroth Stokes parameter, a scalar quantity defined by

$$(S_0)_+ = 2E_0^2 \tag{39}$$

Therefore, the root cause of circular dichroism is the pseudoscalar magnitude $(B_0)_-$ of the vector $\mathbf{B}_\Pi$:

$$|\mathbf{B}_\Pi| = (B_0)_- = \left(\frac{1}{\varepsilon_0 c^3 I_0}\right)^{1/2} (I_R - I_L) \tag{40}$$

an equation that is valid at all electromagnetic frequencies.

The origin of circular dichroism therefore resides in the photon's magnetostatic flux quantum $\hat{B}_\Pi$. In other words, circular dichroism is magneto-optic in origin, and the observable $(I_R - I_L)$ is a spectral consequence of the interaction of $\hat{B}_\Pi$ with structurally chiral material. From Eq. (40), $I_R - I_L$ is proportional to the real pseudoscalar quantity $(B_0)_-$ after each photon of the beam emerges from the chiral material through which the beam has passed, i.e., after interaction has occurred between the incident flux quantum $\hat{B}_\Pi$ per photon and the appropriate molecular property tensor of the material.[7] This leads to a new way of describing the fundamental mechanism of natural optical activity by considering the mechanism of interaction of $\hat{B}_\Pi$ with a structurally chiral molecule, or center of optical activity. A quantum $\hat{B}_\Pi$ per photon is evidently absorbed and reemitted with different characteristics imparted by the chiral structure.

For a beam consisting of one photon, the observable $I_R - I_L$ provides an experimental measure of the transmitted elementary $\hat{B}_\Pi$ at each frequency of that beam. Although $\mathbf{B}_\Pi$ is itself independent of the phase of the beam, the interacting molecular property tensor depends on the beam frequency through semiclassical perturbation theory,[7] which gives

$$|\mathbf{B}_\Pi| = (B_0)_- = \left(\frac{I_0}{\varepsilon_0 c^3}\right)^{1/2} \tanh[\omega\mu_0 c l N \zeta''_{XYZ}(g)] \tag{41}$$

where μ_0 is the permeability in vacuo, ω the angular frequency of the beam, l the length of sample through which the beam has passed, and ζ''_{XYZ} an appropriately averaged molecular property tensor component, a pseudoscalar.[7] Equation (41) shows that all circularly dichroic spectra are signatures of the reemitted $\hat{B}_\Pi$ property of the photon.

More generally, any property in physical optics that involves $(S_3)_-$, in classical or quantized[1] form, necessarily involves $\mathbf{B}_\Pi$ or the quantized $\hat{B}_\Pi$

per photon. There are several of these phenomena, each of whose origin can be traced to the novel elementary flux quantum $\hat{B}_{\Pi}$ of the photon. Rayleigh refringent scattering theory, for example,[7] shows that $(S_3)_-$ is associated with a change $d\eta/dz$ in ellipticity in a beam passing through a sample of thickness z. It is immediately possible to say, therefore, that $d\eta/dz$ measures changes in the flux quantum $\hat{B}_{\Pi}$ per photon as it passes through the sample, i.e., as $\hat{B}_{\Pi}$ is absorbed and reemitted, a process from Eq. (29) that must involve changes in the incident photon's angular and linear momentum. Ellipticity is therefore magneto-optic in fundamental origin.

VI. DISCUSSION

One of the interesting consequences of the development in the preceding sections is that the speed of light c must be regarded as a $\hat{T}$-positive scalar quantity. This is because c is a universal constant that is relativistically the same in any frame of reference, and cannot be reversed by the motion reversal operator $\hat{T}$, because c is independent of motion. However, a velocity v that is less than c is $\hat{T}$-negative, because it is reversed by motion reversal in a given reference frame. In consequence, the scalar refractive index n, defined by c/v in a material, must be a $\hat{T}$-positive quantity. The value of n in vacuo is numerically unity and is the mathematical limit of c/v as $v \to c$. It is proper to regard n as being $\hat{T}$-positive in this limit, and this is the point of view utilized in this paper.

It follows that the unit polar vector $\mathbf{n}/n$ must be $\hat{T}$-negative, because it is the quotient of $\hat{T}$-negative/$\hat{T}$-positive quantities. In Eq. (20), for example, $\hat{T}[(B_0)_-] = +$, $\hat{T}(\mathbf{e}_+) = -$, $\hat{T}[(B_0)_+] = +$, $\hat{T}(\mathbf{n}/n) = -$, so that there is a balance of net $\hat{T}$ symmetries on either side of the equation.

To interpret rigorously the generalization, Eq. (29), of the de Broglie equation (32) in its "textbook" form, it must be borne in mind that the de Broglie duality rigorously implies the symmetry duality summarized in Eq. (30) for $\hat{P}$. Equation (31), therefore, is more rigorously expressed as

$$\hat{P} = n\frac{(B_0)_+}{(B_0)_-}\frac{\omega}{c}\hat{J} \qquad n = 1 \tag{42}$$

and Eq. (32) as

$$p = n\frac{(B_0)_+}{(B_0)_-}\frac{\omega}{c}\hbar \qquad n = 1 \tag{43}$$

The quotient $(B_0)_+/(B_0)_-$, and the $\hat{T}$-positive free space refractive index ($n = 1$) are missing or implied in the usual textbook definition of the de Broglie wave particle duality.

In conclusion, it has been demonstrated that there is an inherent symmetry duality in the definition of the magnetostatic flux quantum $\hat{B}_{\Pi}$, which is the root of the de Broglie wave particle duality for the photon. The operator $\hat{B}_{\Pi}$ can be defined simultaneously in terms of the angular and linear momentum operators of the photon. This type of symmetry duality occurs throughout physical optics, and is inherent in the fact that $\hat{B}_{\Pi}$, or its classical equivalent $\mathbf{B}_{\Pi}$, is at the root of several well-known effects, such as circular dichroism and ellipsometry of various kinds. The operator $\hat{B}_{\Pi}$ can also be used straightforwardly to predict and describe novel and useful spectroscopic effects that depend on magnetization by circularly polarized light.

Acknowledgments

The Leverhulme Trust is thanked for a Fellowship, and the Cornell Theory Center and the Materials Research Laboratory of Penn State is thanked for additional support.

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EXPERIMENTAL DETECTION OF THE PHOTON'S FUNDAMENTAL STATIC MAGNETIC FIELD OPERATOR: THE ANOMALOUS OPTICAL ZEEMAN AND OPTICAL PASCHEN-BACK EFFECTS

I. INTRODUCTION

In this paper we continue a systematic theoretical search for a method of detecting and measuring unequivocally the photon's fundamental static magnetic field operator[1-5]

$$\hat{B}_{\Pi} = B_0 \frac{\hat{J}_{\Pi}}{\hbar} \tag{1}$$

where B_0 is the scalar magnetic flux density amplitude in tesla of a circularly polarized laser beam, made up of N photons, $\hat{J}_{\Pi}$ is the photon's angular momentum operator, and $\hbar$ is the reduced Planck constant $h/2\pi$.

Fragmentary experimental evidence for the existence of $\hat{B}_{\Pi}$ is available through the inverse Faraday effect (IFE),[6-9] and the recent emergence of optical NMR (ONMR), or laser-enhanced nuclear magnetic resonance spectroscopy (LENS).[10, 11] Both techniques measure the ability of circularly polarized laser radiation to magnetize. The IFE measures bulk magnetization and ONMR measures light-induced resonance shifts, which are different for each resonating site and are therefore useful for sample identification and spectral analysis. However, the theoretical existence of $\hat{B}_{\Pi}$ also implies other effects, such as an optical Zeeman effect, in which the magnetic effect of a circularly polarized laser splits electric dipole transitions in atoms occurring in the visible frequency range. This paper provides a fairly rigorous quantum theory of the anomalous optical Zeeman effect and the optical Paschen-Back effect, in which both spin and orbital electronic angular momenta are considered in various coupling schemes.

The existence of an optical Zeeman effect in atoms appears to have been implicit in the theory of the inverse Faraday effect proposed by Pershan et al.,[6-9] Kielich et al.,[12] and Atkins and Miller.[13] The present author independently arrived at the existence of an optical Zeeman effect in a series of papers[14-17] based on symmetry considerations and semiclassical theory, considerations that also led to ONMR.[18-21] Recently, he has proposed theoretically the existence of the photon's $\hat{B}_{\Pi}$ operator, a fundamental property of the photon itself, whose classical equivalent is a static magnetic field, $\hat{B}_{\Pi}$, produced by circularly polarized electromagnetic

radiation at all frequencies.[1-4] It is important to realize that the operator $\hat{B}_{\Pi}$ (or the classical $\hat{B}_{\Pi}$) is different fundamentally from the usual **B** vector of electromagnetic plane waves.[22] The **B** vector is frequency dependent, whereas the **B** vector is not. The **B** vector has components in X and Y directions mutually perpendicular to the propagation axis Z of the laser, and no component in the Z axis, whereas $\mathbf{B}_{\Pi}$ is directed in the Z axis only. The **B** vector depends on the photon's linear momentum vector $\boldsymbol{\kappa}$ (i.e., the propagation vector), whereas $\mathbf{B}_{\Pi}$ does not. Again, $\mathbf{B}_{\Pi}$ is positive to the parity inversion operator $\hat{P}$ and negative to the motion reversal operator $\hat{T}$, and is therefore fundamentally different in symmetry from the Poynting vector[23] and the propagation vector.[24] It appears that $\mathbf{B}_{\Pi}$ (and its quantum field equivalent $\hat{B}_{\Pi}$) is a fundamentally new concept in electromagnetic field theory.

The optical Zeeman effect appears to be a promising method of detecting the effects of $\hat{B}_{\Pi}$ experimentally. Electric dipole transitions in atoms are readily measured and identified spectrally.[24] The key to the optical Zeeman effect is to replace the magnet of the conventional Zeeman effect[24] by a circularly polarized laser. In the ordinary Zeeman effect, where there is no consideration given to the role of net electronic spin angular momentum,[22] a singlet $^1S \rightarrow {}^1P$ transition is split by a magnet into three lines. Its optical equivalent has recently been proposed theoretically[5] using the concept summarized in Eq. (1), and produces a splitting of the original $^1S \rightarrow {}^1P$ electric dipole transition, a splitting pattern whose details are different in quantum-field theory (in which the operator $\hat{B}_{\Pi}$ forms a Hamiltonian with the electronic magnetic dipole moment operator $\hat{m}$ of the atom) and in semiclassical theory (in which the Hamiltonian is formed from a product of $\hat{m}$ and the classical vector $\mathbf{B}_{\Pi}$). The pattern also depends on the type of angular momentum interaction used in the quantum-field theory, i.e., whether a coupled or uncoupled representation of $\hat{B}_{\Pi}$ and $\hat{m}$ is used.[22] The semiclassical result is recovered[5] only in the uncoupled representation. In the coupled representation, the quantum field theory produces three lines, but the central line is displaced in frequency. In the uncoupled representation of the quantum field theory of the optical Zeeman effect, and in the semiclassical representation of the same problem, the splitting pattern obtained[5] is the same as that in the conventional Zeeman effect, i.e., two lines each side of a central line situated at the original frequency of the electric dipole transition $^1S \rightarrow {}^1P$.

In Section II, these findings are augmented by the consideration of electronic spin angular momentum in quantum field and semiclassical approaches using in the former different coupling models for the angular momenta involved in the interaction Hamiltonians. In Section III, the details are given of the spectral splitting due to the circularly polarized laser for each theoretical approach of Section II. Finally, some experimental details are discussed and estimates of the splittings in hertz are given for each theoretical approach, namely the quantum field theory in the fully coupled, semicoupled, and uncoupled representations, and the semiclassical approach where $\mathbf{B}_{\Pi}$ is considered as a classical field vector. Conditions are discussed under which the anomalous optical Zeeman effect gives way to the optical Paschen-Back effect in the quantum field and semiclassical representations of the same phenomena.

II. QUANTUM FIELD AND SEMICLASSICAL INTERACTION HAMILTONIANS

The core of the description of the anomalous optical Zeeman effect and the optical Paschen-Back effect in atoms is the construction of the interaction Hamiltonians between the novel photon property $\hat{B}_{\Pi}$ (quantum field theory) or $\mathbf{B}_{\Pi}$ (semiclassical theory) and the atom's net electronic magnetic dipole moment operator $\hat{m}$. In this section, first-order interaction Hamiltonians are constructed in the framework of quantum field and semiclassical descriptions of the same problem.

A. Quantum Field Theory

The interaction Hamiltonian is the operator product

$$\Delta \hat{H}_1 = -\hat{m} \cdot \hat{B}_{\Pi} \tag{2}$$

from which the energy of interaction is calculated from an expectation value such as

$$\Delta E_1 = -\frac{\gamma_e B_0}{\hbar} \langle SLJJ_{\Pi}FM_F | \hat{L} + 2.002\hat{S} | S'L'J'J'_{\Pi}F'M'_F \rangle \tag{3}$$

In this expression, the magnetic dipole moment of the atom is developed as

$$\hat{m} = \gamma_e(\hat{L} + 2.002\hat{S}) \tag{4}$$

where γ_e is the electronic gyromagnetic ratio, $\hat{L}$ is the operator describing the net electronic orbital angular momentum, and $2.002\hat{S}$ is the operator description of the net electronic spin angular momentum.[22] The $\hat{B}_{\Pi}$ operator is developed in terms of the photon's angular momentum operator in Eq. (1). The angular momentum quantum numbers S, L, and J_{Π} are associated with the operators $\hat{S}$, $\hat{L}$, and $\hat{J}_{\Pi}$, respectively. The interaction energy (3) is one in which a coupled representation[22] is considered f the three angular momenta just introduced. In this case the couplin scheme is

$$\mathbf{J} = \mathbf{L} + \mathbf{S} \qquad \mathbf{F} = \mathbf{J} + \mathbf{J}_{\Pi} \tag{5}$$

so that the values of the quantum number J, associated with the operat $\hat{J}$ are given as usual by the Clebsch-Gordan series

$$J = L + S, \ldots, |L - S| \tag{6}$$

Similarly, the overall quantum number F is defined by

$$F = J + J_{\Pi}, \ldots, |J - J_{\Pi}| \tag{7}$$

i.e., from a coupled representation of the $\hat{J}$ operator of the atom and th novel[1-5] $\hat{J}_{\Pi}$ operator of the photon whose effects we are attempting t describe.

There are other ways of writing the interaction energy for the give Hamiltonian (2), these being the semicoupled and uncoupled representa tions. In the former, the operators $\hat{L}$ and $2.002\hat{S}$ of the atom ar combined in a coupled angular momentum representation[22] to give the operator, but the interaction energy is formed as follows:

$$\Delta E_2 = -\frac{\gamma_e B_0}{\hbar} \langle SLJM_J; J_{\Pi}M_{J_{\Pi}} | \hat{L} + 2.002\hat{S} | S'L'J'M'_J; J'_{\Pi}M'_{J_{\Pi}} \rangle \tag{8}$$

i.e., with $\hat{J}$ and $\hat{J}_{\Pi}$ considered in an uncoupled representation, in whic the projections $\hat{M}_J$ and $\hat{M}_{J_{\Pi}}$ onto the azimuthal axis (Z, the laser' propagation axis) are well defined, but in which the net angular momen tum operator $\hat{F}$ is not. In the latter, the interaction energy is written as

$$\Delta E_3 = -\frac{\gamma_e B_0}{\hbar} \langle SM_S; LM_L; J_{\Pi}M_{J_{\Pi}} | \hat{L} + 2.002\hat{S} | S'M'_S; L'M'_L; J'_{\Pi}M'_{J_{\Pi}} \rangle \tag{9}$$

in which all three angular momentum operators—$\hat{L}$ and $2.002\hat{S}$ of the atom, and $\hat{J}_{\Pi}$ of the photon—are considered in a fully uncoupled repre sentation. All three representations are possible theoretically, and which i the most appropriate can be determined only by independent considera tion of the physics of the problem.

1. *The Coupled Representation*

The first stage is the usual one. The Wigner-Eckart theorem is used to separate out the M_F dependence[22, 25]:

$$\Delta E_1 = -(-1)^{F-M_F}\begin{pmatrix} F & 0 & F \\ -M_F & 0 & M_F \end{pmatrix}\langle SLJJ_\Pi F\|\hat{m}\cdot\hat{B}_\Pi\|SLJJ_\Pi F\rangle \quad (10)$$

We have restricted our consideration to diagonal elements of the interaction energy, and in this case, the 3-*j* symbol is[25]

$$(-1)^{F-M_F}\begin{pmatrix} F & 0 & F \\ -M_F & 0 & M_F \end{pmatrix} = (2F+1)^{-1/2} \quad (11)$$

The quantum nature of the interaction energy is therefore contained within the reduced matrix element in Eq. (10). This is a problem of the type first considered by Curl and Kinsey[26] and which is summarized in Eq. (13.8) of Silver,[25] one in which there are three types of commuting (independent) angular momenta, described by operators $2.002\hat{S}$, $\hat{L}$, and $\hat{J}_\Pi$ in spaces 1, 2, and 3, in the fully coupled representation of angular momentum quantum theory.[22] It is helpful to write the interaction energy (10) as

$$\Delta E_1 = -\frac{\gamma_e B_0}{\hbar}(2F+1)^{-1/2}\Big(\langle SLJJ_\Pi F\Big\|\Big[\big[\hat{1}^0\otimes\hat{L}^1\big]^1_0\otimes\hat{J}^1_{\Pi 0}\Big]^0_0\Big\|SLJJ_\Pi F\rangle$$
$$+2.002\langle SLJJ_\Pi F\Big\|\Big[\big[\hat{S}^1\otimes\hat{1}^0\big]^1_0\otimes\hat{J}^1_{\Pi 0}\Big]^0_0\Big\|SLJJ_\Pi F\rangle\Big) \quad (12)$$

in terms of tensor products of the type illustrated in Eq. (13.7) of Silver,[25] to which we refer the reader for background and details of irreducible tensorial methods. These methods allow the reduced matrix element to be written in terms of the 9-*j* symbols of atomic quantum mechanics,[25] allowing the interaction energy to be expressed simply as

$$\Delta E_1 = -\gamma_e B_0 \hbar g_1 \quad (13)$$

where the g_1 factor is a complicated combination of terms defined

through the individual angular momentum quantum numbers:

$$g_1 = (2J+1)[3(2F+1)J_\Pi(J_\Pi+1)(2J_\Pi+1)]^{1/2}\begin{bmatrix} J & J & 1 \\ J_\Pi & J_\Pi & 1 \\ F & F & 0 \end{bmatrix}$$
$$\times[(2S+1)L(L+1)(2L+1)]^{1/2}\begin{bmatrix} S & S & 0 \\ L & L & 1 \\ J & J & 1 \end{bmatrix} \quad (14)$$
$$+\,2.002[(2L+1)S(S+1)(2S+1)]^{1/2}\begin{bmatrix} S & S & 1 \\ L & L & 0 \\ J & J & 1 \end{bmatrix}$$

Note at this stage that there are several energy levels, because there are several allowed combinations of quantum numbers through the appropriate Clebsch-Gordan series. For each energy level there will be an individual g_1 factor. The physical meaning of this coupled representation is discussed later, and in Section III the result (13) is used in the context of electric dipole transitions in atomic states split by the photon property $\hat{B}_\Pi$ generated by a circularly polarized laser.

2. *The Semicoupled Representation*

This is, perhaps, the most realistic representation of the problem in quantum field theory, because of the nature of the photon operator $\hat{B}_\Pi$. The photon propagates at the speed of light and is massless, so that the azimuthal components of the angular momentum operator $\hat{J}_\Pi$ are always well defined (i.e., specified)[22] in terms of the azimuthal quantum numbers

$$M_{J_\Pi} = \pm 1$$

(A sign change in this context denotes switching from left to right circular polarization.) Relativity theory forbids any component of the photon angular momentum perpendicular to the azimuthal (propagation) axis Z. It appears natural, therefore, to combine the angular momentum operators $\hat{J}$ and $\hat{J}_\Pi$ in the uncoupled representation of the quantum theory of angular momentum coupling,[22, 25] a representation in which the azimuthal components of the angular momenta are specified, but in which the resultant angular momentum is not.[22]

In the semicoupled representation, the interaction energy (8) can therefore be written

$$\Delta E_2 = -\frac{\gamma_e B_0}{\hbar}\Big(\langle SLJM_J|\big[\hat{1}^0 \otimes \hat{L}^1\big]_0^1|S'L'J'M_J'\rangle\Big) + 2.002 \times \langle SLJM_J|\big[\hat{S}^1 \otimes \hat{1}^0\big]_0^1|S'L'J'M_J'\rangle\Big)\langle J_\Pi M_{J_\Pi}|\hat{J}_{\Pi 0}^1|J_\Pi' M_{J_\Pi}'\rangle \tag{15}$$

an expression that can be reduced using tensorial methods (using Eqs. (14.10) ff. of Silver[25]) to the form

$$\Delta E_2 = -\left(\frac{\gamma_e B_0}{\hbar}\right)M_J\left[\frac{3J(J+1) - L(L+1) + S(S+1)}{2J(J+1)}\right](-1)^{J_\Pi - M_{J_\Pi}} \times \begin{pmatrix} J_\Pi & 1 & J_\Pi' \\ -M_{J_\Pi} & 0 & M_{J_\Pi}' \end{pmatrix}\langle J_\Pi\|\hat{J}_\Pi\|J_\Pi'\rangle \tag{16}$$

This can be reduced further to the simple result

$$\Delta E_2 = -\gamma_e B_0 g_L M_J M_{J_\Pi}\hbar \tag{17}$$

using the following results[25] and notation:

$$g_L = \frac{3J(J+1) - L(L+1) + S(S+1)}{2J(J+1)} \tag{18}$$

$$(-1)^{J_\Pi - M_{J_\Pi}}\begin{pmatrix} J_\Pi & 1 & J_\Pi \\ -M_{J_\Pi} & 0 & M_{J_\Pi} \end{pmatrix} = M_{J_\Pi}\big[J_\Pi(J_\Pi+1)(2J_\Pi+1)\big]^{-1/2} \tag{19}$$

$$\langle J_\Pi\|\hat{J}_\Pi\|J_\Pi\rangle = \hbar\big[J_\Pi(J_\Pi+1)(2J_\Pi+1)\big]^{1/2} \tag{20}$$

In this semicoupled representation, therefore, the interaction energy in quantum field theory becomes the product of a g_L factor which depends only on the quantum numbers J, L, and S, with the azimuthal quantum number product $M_J M_{J_\Pi}$. The g_L factor in this case is recognizable as the Landé factor of atomic theory.[22, 25]

3. *The Uncoupled Representation*

This is a possible representation of the same problem, in which the three operators $2.002\hat{S}$, $\hat{L}$, and $\hat{J}_\Pi$ are decoupled operators acting independently on decoupled states, each operator acting independently on states built from independent sets of coordinates in spaces 1, 2, and 3 (Ref. 22). The interaction energy (9) is therefore written as

$$\Delta E_3 = -\frac{\gamma_e B_0}{\hbar}\Big(\langle SM_S; LM_L; J_\Pi M_{J_\Pi}|\hat{1}_0^0\hat{L}_0^1\hat{J}_{\Pi 0}^1|S'M_S'; L'M_L'; J_\Pi' M_{J_\Pi}'\rangle + 2.002\langle SM_S; LM_L; J_\Pi M_{J_\Pi}|\hat{S}_0^1\hat{1}_0^0\hat{J}_{\Pi 0}^1|S'M_S'; L'M_L'; J_\Pi' M_{J_\Pi}'\rangle\Big) \tag{21}$$

and the Wigner-Eckart theorem applied three times to give a superficially complicated result:

$$\begin{aligned}\Delta E_3 = -\gamma_e B_0\hbar(-1)^{S-M_S+L-M_L+J_\Pi-M_{J_\Pi}} &\times\Bigg[\begin{pmatrix} S & 0 & S' \\ -M_S & 0 & M_S' \end{pmatrix}\begin{pmatrix} L & 1 & L' \\ -M_L & 0 & M_L' \end{pmatrix}\begin{pmatrix} J_\Pi & 1 & J_\Pi' \\ -M_{J_\Pi} & 0 & M_{J_\Pi}' \end{pmatrix} \\ &\times \langle S\|\hat{1}\|S'\rangle\langle L\|\hat{L}\|L'\rangle\langle J_\Pi\|\hat{J}_\Pi\|J_\Pi'\rangle \\ &+ 2.002\begin{pmatrix} S & 1 & S' \\ -M_S & 0 & M_S' \end{pmatrix}\begin{pmatrix} L & 0 & L' \\ -M_L & 0 & M_L' \end{pmatrix}\begin{pmatrix} J_\Pi & 1 & J_\Pi' \\ -M_{J_\Pi} & 0 & M_{J_\Pi} \end{pmatrix} \\ &\times\langle S\|\hat{S}\|S'\rangle\langle L\|1'\|L'\rangle\langle J_\Pi\|J_\Pi\|J_\Pi'\rangle\Bigg]\end{aligned} \tag{22}$$

However, with standard results,[22, 25] such as

$$\langle S\|\hat{S}\|S\rangle = \big[S(S+1)(2S+1)\big]^{1/2}\hbar \tag{23}$$

$$\langle S\|\hat{1}\|S\rangle = (2S+1)^{1/2} \tag{24}$$

$$\begin{pmatrix} S & 1 & S \\ -M_S & 0 & M_S \end{pmatrix} = (-1)^{S-M_S}M_S\big[S(S+1)(2S+1)\big]^{-1/2} \tag{25}$$

$$\begin{pmatrix} S & 0 & S \\ -M_S & 0 & M_S \end{pmatrix} = (-1)^{S-M_S}(2S+1)^{-1/2} \tag{26}$$

the interaction energy in the decoupled representation of quantum field theory collapses to

$$\Delta E_3 = -\gamma_e B_0 \hbar M_{J_\Pi}(M_L + 2.002 M_S) \tag{27}$$

in which there is no g factor at all, and which is a simple product of azimuthal quantum numbers of the atom and the photon's novel $\hat{B}_\Pi$ operator in which we are interested.

B. Semiclassical Theory

The semiclassical representation of the anomalous optical Zeeman and Paschen-Back effects depends on the interaction Hamiltonian

$$\Delta \hat{H}_2 = -\hat{m} \cdot \mathbf{B}_\Pi \tag{28}$$

where $\mathbf{B}_\Pi$ is now a classical field vector,[1-5] not a quantum-mechanical operator. The interaction energy in this case is

$$\Delta E_4 = -\gamma_e |\mathbf{B}_\Pi| \langle SLJM_J | \hat{L} + 2.002 \hat{S} | S'L'J'M_J' \rangle \tag{29}$$

which can be reduced to

$$\Delta E_4 = -\gamma_e |\mathbf{B}_\Pi| g_L \hbar M_J \tag{30}$$

where g_L is the same Landé factor as in Eq. (17).

III. APPLICATION TO ELECTRIC DIPOLE TRANSITIONS IN ATOMS

In this section the results of Section II are applied to predict the splitting of a visible frequency electric dipole transition in an atom by a circularly polarized laser generating the flux quantum $\hat{B}_\Pi$ of Eq. (1). The selection rules governing such a transition in the conventional theory of the anomalous Zeeman effect in atoms are well known.[22, 25, 26] They are determined by rules on the existence of the 3-j symbol in the Wigner-Eckart expansion of the matrix elements of the transition electric dipole moment operator $\hat{\mu}$. For the Z component

$$\langle SLJM_J | \hat{\mu}_0^1 | S'L'J'M_J' \rangle = (-1)^{J-M_J} \begin{pmatrix} J & 1 & J' \\ -M_J & 0 & M_J' \end{pmatrix} \langle SLJ \| \hat{\mu}_0^1 \| S'L'J' \rangle \tag{31}$$

and the selection rules are

$$\Delta J = 0, \pm 1 \qquad \Delta M_J = 0 \tag{32}$$

Similarly, for the X and Y components of $\hat{\mu}$,

$$\Delta J = 0, \pm 1 \qquad \Delta M_J = \pm 1 \tag{33}$$

However, in the anomalous optical Zeeman effect, the atomic terms between which the electric dipole transition takes place are each being considered in the presence of the operator $\hat{B}_\Pi$, in the various coupling schemes of Section II. Therefore, the electric dipole transition selection rules must also be derived in the appropriate coupling scheme.

We shall consider an atomic transition[22] between the atomic terms $^2P_{1/2}$ and $^2D_{3/2}$. In the former, $L = 1$, $S = \frac{1}{2}$, and $J = \frac{1}{2}$; and in the latter, $L = 2$, $S = \frac{1}{2}$, and $J = \frac{3}{2}$. The Laporte (or parity) selection rule is also obeyed in such a transition, i.e., $\Delta L = 1$ in this case. The transition occurs at a frequency that is determined from the appropriate electric dipole selection rules,[22] and the spectrum in the absence of $\hat{B}_\Pi$ consists of a single line which can be measured at visible frequencies in a spectrometer.

We are specifically interested in how this line is affected by the presence of an additional, circularly polarized laser, generating the flux quantum $\hat{B}_\Pi$ of Eq. (1), and substituting for the usual magnet of the Zeeman effect.[22] In examining the effect of $\hat{B}_\Pi$ we use the four results, Eqs. (13), (17), (27), and (30), in turn. In each case the effect of $\hat{B}_\Pi$ is first determined on the $^2P_{1/2}$ atomic term, and then on the $^2D_{3/2}$ term. Each of these two terms is split into nondegenerate energy levels by the addition of quantized energy such as ΔE_1, described by Eq. (13), for example. Various electric dipole transitions can then occur between the split $^2P_{1/2}$ term and the split $^2D_{3/2}$ term according to the transition electric dipole selection rules appropriate for the coupling scheme. Overall, therefore, we expect that the novel flux quantum $\hat{B}_\Pi$ splits the original line corresponding to the transition $^2P_{1/2} \rightarrow {}^2D_{3/2}$ in an atom. The details of the splitting pattern depend on which of the various schemes of Section II are chosen. This procedure is similar to the standard theory of the conventional Zeeman effect,[22-24] but in the optical Zeeman effect, $\hat{B}_\Pi$ is a quantum-mechanical operator. In the conventional Zeeman effect, the applied magnetic field $\mathbf{B}_0$ is always a classical, magnetostatic field vector, whose origin is not electromagnetic

A. Quantum Field Theory, Coupled Representation, Eq. (13)

The atomic $^2P_{1/2}$ term is split from Eq. (13) into two levels by the novel operator $\hat{B}_{\Pi}$ of the photon:

$$g_1(F = \tfrac{1}{2}, J_\Pi = 1, J = \tfrac{1}{2}, L = 1, S = \tfrac{1}{2})$$
$$g_2(F = \tfrac{3}{2}, J_\Pi = 1, J = \tfrac{1}{2}, L = 1, S = \tfrac{1}{2})$$

and there is a different g factor for each level. The atomic $^2D_{3/2}$ term is split into the three levels:

$$g_3(F = \tfrac{1}{2}, J_\Pi = 1, J = \tfrac{3}{2}, L = 2, S = \tfrac{1}{2})$$
$$g_4(F = \tfrac{3}{2}, J_\Pi = 1, J = \tfrac{3}{2}, L = 2, S = \tfrac{1}{2})$$
$$g_5(F = \tfrac{5}{2}, J_\Pi = 1, J = \tfrac{3}{2}, L = 2, S = \tfrac{1}{2})$$

each with a different g factor. In general the five g factors (two in the lower term and three in the upper) are all different. Electric dipole transitions within the atom can now occur between the two lower levels and three upper levels with selection rules determined as follows.

The transition electric dipole moment operator is developed using the Wigner-Eckart theorem between coupled states to give

$$\begin{aligned}&\langle SLJJ_\Pi FM_F|\hat{\mu}_0^1|S'L'J'J'_\Pi F'M'_F\rangle \\ &\quad = (-1)^{F-M_F}\begin{pmatrix} F & 1 & F' \\ -M_F & 0 & M'_F \end{pmatrix}\langle SLJJ_\Pi F\|\hat{\mu}_0^1\|S'L'J'J'_\Pi F'\rangle\end{aligned} \tag{34}$$

This procedure yields immediately the selection rules on the Z component of $\hat{\mu}$:

$$\Delta F = 0, \pm 1 \qquad \Delta M_F = 0 \tag{35}$$

Similarly, for X and Y components of $\hat{\mu}$,

$$\Delta F = 0, \pm 1 \qquad \Delta M_F = \pm 1 \tag{36}$$

All selection rules now refer to the net quantum number F.

There are six possible spectral lines generated by electric dipole transitions between the two $^2P_{1/2}$ levels and the three $^2D_{3/2}$ levels, but one of these, from the $F = \frac{1}{2}$ level of the split $^2P_{1/2}$ term to the $F = \frac{5}{2}$ level of the split $^2D_{3/2}$ term, is forbidden by the selection rule (35, 36) just derived, i.e., by the fact that the maximum change in F must be $+1$.

Discussion of the physical meaning of this result is given later. The quantum field theory of the anomalous optical Zeeman effect in the coupled representation splits the original visible frequency spectral line into five, each displaced from the original frequency.

B. Quantum Field Theory, Semicoupled Representation, Eq. (17)

In this case the selection rules on the electric dipole transitions are obtained by the development (for the Z component):

$$\begin{aligned}&\langle SLJM_J; J_\Pi M_{J_\Pi}|\hat{\mu}_0^1\hat{1}_0^0|S'L'J'M'_J; J'_\Pi M'_{J_\Pi}\rangle \\ &\quad = \langle SLJM_J|\hat{\mu}_0^1|S'L'J'M'_J\rangle\langle J_\Pi M_{J_\Pi}|\hat{1}_0^0|J'_\Pi M'_{J_\Pi}\rangle \\ &\quad = (-1)^{J-M_J}\begin{pmatrix} J & 1 & J' \\ -M_J & 0 & M'_J \end{pmatrix}\langle SLJ\|\hat{\mu}_0^1\|S'L'J'\rangle\end{aligned} \tag{37}$$

so that the 3-j symbol is nonzero if and only if

$$\Delta J = 0, \pm 1 \qquad \Delta M_J = 0 \tag{38}$$

Similarly, for X and Y components of $\hat{\mu}$,

$$\Delta J = 0, \pm 1 \qquad \Delta M_J = \pm 1 \tag{39}$$

The Landé g_L factor of Eq. (17) is the same for each level of the $^2P_{1/2}$ term. For each level of the split $^2D_{3/2}$ term the Landé factor is again the same, g_{L1}. Transitions between the levels are controlled by the selection rule $\Delta M_J = 0, \pm 1$. The resulting spectral pattern is three groups of doublets, i.e., six lines. This is recognizable as the same pattern observed in the conventional semiclassical theory of the anomalous Zeeman effect, as illustrated, for example, in Fig. 9.27 of Ref. 22.

C. Quantum Field Theory, Uncoupled Representation, Eq. (27)

In this case the electric dipole (Z component) transition selection rules are determined from the development

$$\begin{aligned}&\langle SM_S; LM_L; J_\Pi M_{J_\Pi}|\hat{1}_0^0\hat{\mu}_0^1\hat{1}_0^0|S'M'_S; L'M'_L; J'_\Pi M'_{J_\Pi}\rangle \\ &\quad = (-1)^{L-M_L}\begin{pmatrix} L & 1 & L' \\ -M_L & 0 & M'_L \end{pmatrix}\langle L\|\hat{\mu}_0^1\|L'\rangle\end{aligned} \tag{40}$$

Assuming, as usual,[22–24] that the spin selection rule

$$\Delta S = 0 \tag{41}$$

s obeyed, we obtain

$$\Delta L = 0, \pm 1 \qquad \Delta M_L = 0 \tag{42}$$

Similarly, for the X and Y components of $\hat{\mu}$,

$$\Delta L = 0, \pm 1 \qquad \Delta M_L = \pm 1 \tag{43}$$

In this case there are no g factors in either term, and both ${}^2P_{1/2}$ and ${}^2D_{3/2}$ terms are split to the same extent.[22] The result is a spectral pattern of three lines, which can be thought of as three coincidental doublets. This is recognizable as the same pattern obtained in the conventional theory of the Paschen-Back effect.[22] In the uncoupled representation of quantum field theory, therefore, the novel $\hat{B}_{\Pi}$ operator is expected to produce the optical Paschen-Back effect.

D. Semiclassical Theory, Eq. (30)

It is straightforward to see that in this case the transition electric dipole moment selection rules are those given by Eqs. (38) and (39), and that the splitting pattern is the same as that in the conventional theory of the anomalous Zeeman effect, consisting of three doublets.

IV. DISCUSSION

In four different schemes we have deduced that the novel property $\hat{B}_{\Pi}$ of the photon[1–5] splits electric dipole transitions occurring at visible frequencies in atoms. It is appropriate to ask which scheme is likely to be the most realistic. It is well known[22] that in the quantum theory of angular momentum coupling, the uncoupled representation leaves the magnitude of the total angular momentum undefined and says nothing about the relative orientation of the contributing individual angular momenta, but defines individual components. The coupled representation defines the total angular momentum but leaves individual components undefined. Either scheme is equally valid and acceptable mathematically. In Section II we found that there is also a third scheme, which we have called the semicoupled representation. All three are valid in the quantum-field theoretical description of the effect of $\hat{B}_{\Pi}$ on atomic transitions.

However, it is independently known that the photon propagates at the speed of light, which implies that the component of $\hat{B}_{\Pi}$ in the azimuthal axis be well defined, because there cannot be any perpendicular components from the theory of relativity. Therefore, it appears that in our coupled representation of Section II, there is a conflict of reasoning, in that the total angular momentum is defined as well as the azimuthal component of the novel field operator of the photon $\hat{B}_{\Pi}$. Therefore, the commutator $[F^2, J_{\Pi_Z}]$ is not zero. However, it is well known[22] that this type of "paradox" can be resolved by remembering that the commutator is an operator, which acts on a wave function, ψ, and if the result

$$[F^2, J_{\pi_Z}]\psi = 0 \tag{44}$$

is true, then F^2 and $M_{J_{\Pi}}$ can be simultaneously well defined in the quantum theory of angular momentum coupling.[22]

This is the mathematical basis for our coupled representation of the problem in Section II. In physical terms, the coupled representation leads to five lines, instead of six as in the semicoupled representation, and this can be tested experimentally to reveal which is the truer representation.

The semicoupled representation treats $\hat{J}$ and $\hat{J}_{\Pi}$ in an uncoupled scheme, so that azimuthal components of both are well defined, but their resultant $\hat{F}$ is not. Therefore, F does not appear in Eq. (17) and there is clear definition of directionality, in that the azimuthal quantum number M_J does appear in Eq. (17) and controls the selection rules as described in Section III. The directionality comes from the presence of the circularly polarized laser, generating the quantity $\hat{B}_{\Pi}$, a laser that propagates in the azimuthal axis Z. In the coupled representation that gives Eq. (13) no azimuthal quantum number appears, but F is well defined and selection rules on F now govern the effect of $\hat{B}_{\Pi}$ on atomic transitions.

In the uncoupled representation of Section II, the only difference from the semicoupled representation is that the operator $2.002\hat{S}$ has been decoupled from $\hat{L}$, leading to the optical Paschen-Back effect. This is therefore a type of strong field limit, in which $2.002\hat{S}$, $\hat{L}$, and $\hat{J}_{\Pi}$ precess independently[22] about the propagation or azimuthal axis Z.

In the semiclassical representation, $\mathbf{B}_{\Pi}$ is a classical field vector, and the treatment of both the anomalous optical Zeeman effect and of the optical Paschen-Back effect becomes the same as conventional theory, leading to the same physical considerations.[22] This is because in the semiclassical representation, $\mathbf{B}_{\Pi}$ is akin to a magnetostatic field, albeit generated by a laser.[1]

Finally, it is straightforward to derive order of magnitude estimates of the splitting from any of the equations (13), (17), (27), and (30), given the relation[1-5]

$$|\mathbf{B}_\Pi| = B_0 = \left(\frac{2I_0}{\varepsilon_0 c^3}\right)^{1/2} \sim 10^{-7} I_0^{1/2} \tag{45}$$

between $|\mathbf{B}_\Pi|$ and the intensity of the laser in watts per square meter. Here, ε_0 is the permittivity in vacuo in S.I. units:

$$\varepsilon_0 = 8.854 \times 10^{-12} J^{-1} C^2 m^{-1} \tag{46}$$

For an intensity of 100 W cm^{-2} (10^6 W m^{-2}) we expect that the novel property $\hat{B}_\Pi$ will shift the original $^2P_{1/2} \rightarrow {}^2D_{3/2}$ transition typically by of the order 1.5×10^6 Hz. This is 5×10^{-5} cm^{-1} (inverse centimeters), and the splitting is expected to be proportional to the square root of the laser's intensity. There should be no splitting if the laser has no degree of circular polarity. These features should help in identifying the effect of the new fundamental photon property $\hat{B}_\Pi$ in which we are interested, and which has recently been proposed theoretically.[1-5]

Acknowledgments

Support is acknowledged from the Leverhulme Trust, Cornell Theory Center, and the Materials Research Laboratory of Penn State University.

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THE OPTICAL FARADAY EFFECT AND OPTICAL MCD

I. INTRODUCTION

This paper continues a series of articles in which the consequences are developed of the recent deduction[1-6] that the photon generates a magnetic flux quantum, an operator

$$\hat{B}_\Pi = B_0 \frac{\hat{J}}{\hbar} \tag{1}$$

Here B_0 is the scalar magnetic flux density amplitude of a beam of N photons (a circularly polarized generator laser), $\hat{J}$ is the photon's angular momentum operator[7] whose eigenvalues are $\pm M_J\hbar$, where M_J is plus or minus one, and where $\hbar$ is the reduced Planck constant. The operator $\hat{B}_\Pi$ changes sign with the circular polarity of the generator laser, is unlocalized in space, and has eigenvalues $\pm M_J B_0$, where B_0 is the laser's scalar flux density amplitude. Its classical equivalent is the axial vector $\mathbf{B}_\Pi$, a novel magnetostatic flux density generated by circularly polarized electromagnetic plane waves.[1-6] The theoretical existence of $\hat{B}_\Pi$ is supported by the experimental evidence for light-induced magnetic effects. The first to be described (in the 1960s) was the inverse Faraday effect,[8-14] and recently it has been shown[15, 16] that NMR resonances are shifted in new and useful ways by the magnetizing effect of circularly polarized argon ion radiation at frequencies far from optical resonance (i.e., where the sample is transparent to the argon ion radiation and does not absorb it). Both these effects can be described in terms of the operator $\hat{B}_{\Pi}$, or its classical equivalent $\mathbf{B}_{\Pi}$ (Refs. 1–6, 17–23). It has also been proposed[4–6] that there exist an optical Zeeman effect, in which $\hat{B}_{\Pi}$ splits singlet electric dipole transition frequencies in atoms; an anomalous optical Zeeman effect for atomic triplet states; and an optical Paschen-Back effect. It has also been proposed theoretically[2, 3] that $\hat{B}_{\Pi}$ can be detected by examining spectrally a circularly polarized laser at visible frequencies reflected from an electron beam.

The existence of $\hat{B}_{\Pi}$ also implies that of an optical Faraday effect, in which it rotates the plane of a linearly polarized probe, an effect that is frequency dependent and gives rise, therefore, to optical magnetic circular dichroism (optical MCD spectroscopy). These effects are developed theoretically in this paper for atoms.

Mason[24] has given an interesting discussion of cause and effect in chirality that includes some pertinent historical analysis of interest here. In 1846, Faraday showed experimentally[25] that a magnetostatic field induces optical activity in flint glass and other isotropic transparent media. In 1884, Pasteur[26] proposed on the basis of this result that the magnetic field represented a source of chirality. It is now known[27] that the magnetically induced optical activity in the Faraday effect has a fundamentally different symmetry[28–31] from that of natural optical activity. Nonetheless, it is interesting for our purpose, following Mason,[24] that le Bel[32] in 1874 had independently proposed that circularly polarized radiation also provides a "chiral force" of the type envisioned by Pasteur emanating from the magnetostatic field used by Faraday. Enantio-differentiating photoreactions were indeed reported by Kuhn and Braun[33] in 1929, and it has been proposed repeatedly (e.g., Bonner[34] that circularly polarized solar irradiation may be responsible for the preponderance in nature of one enantiomer over another. However, Mason[24] favors the universal and parity-violating mechanism of the electroweak force as the origin of this dissymmetry because the electroweak force does not depend on time and location on the earth's surface. (Natural solar radiation is only 0.1% circularly polarized, and equally and oppositely so at dawn and dusk.[24]

It is interesting for our purpose to note that both circular polarity in light and the magnetostatic field have been proposed independently (by le Bel and Pasteur, respectively) as sources of chirality. It had thus been sensed more than one hundred years ago that these two concepts have something in common in their effect on material. Pershan et al.,[11] in their first paper on the experimental demonstration of the magnetizing effect of circularly polarized giant ruby laser radiation, edged toward the concept of

$\hat{B}_{\Pi}$ by describing the effect of the laser as being due to an effective magnetostatic field. In view of these indications, those by Kielich and coworkers,[12, 13] and those by Atkins and Miller,[14] the present author appears to have demonstrated conclusively[1–6] the fact that circularly polarized radiation generates the classical magnetostatic field $\mathbf{B}_{\Pi}$ whose equivalent in quantum field theory is the photon's flux quantum $\hat{B}_{\Pi}$, a novel fundamental property of quantum field theory. As noted elsewhere, the concept of $\hat{B}_{\Pi}$ must be clearly and carefully distinguished from the (standard IUPAC) oscillating **B** field of the electromagnetic plane wave, because the two are quite different.[1–6] We have therefore resolved[1–6] the conjectures of le Bel (1874) and Pasteur (1884) insofar as to show that circularly polarized light can indeed act in the same way as a magnetostatic field, a finding that implies the existence of several novel types of spectroscopy in circumstances where a conventionally applied magnet would be replaced by a circularly polarized laser.

In Section II, the contemporary quantum theory of Faraday's effect of 1846 is developed succinctly for atoms, whereby it becomes relatively straightforward to show that the flux quantum $\hat{B}_{\Pi}$ must also generate Faraday's observation of optical activity in all material, inherently (structurally) chiral or otherwise. In our case, the magnet used by Faraday is replaced by a circularly polarized laser, which generates $\hat{B}_{\Pi}$ and is therefore referred to as the generator laser. Section III develops the frequency dependence of the optical Faraday effect in atoms through the properties of the magnetically (i.e., $\hat{B}_{\Pi}$) perturbed antisymmetric polarizability of conventional contemporary Faraday effect theory,[35] and therefore arrives at expressions in atoms for optical MCD. Finally, a discussion is given of possible experimental configurations and order of magnitudes of the expected optical Faraday effect in terms of the intensity in watts per unit area of the generator laser.

II. OPTICALLY INDUCED FARADAY ROTATION IN ATOMS

The contemporary quantum theory of the Faraday effect is based on the work by Serber,[36] which was the precursor for the A, B, and C terms.[37] This section develops the Serber theory for use with a flux quantum $\hat{B}_{\Pi}$ from the generator laser and shows that the analogy between the optical and conventional Faraday effects is easily forged by using B_{Π} in place of the conventional magnetic field $\mathbf{B}_0$ from a magnet.

The starting point for the theory of the conventional Faraday effect in quantum mechanics is an equation for the rotation of the plane of linearly polarized radiation. This is derived[37] by a consideration of the effect of a magnetostatic field on the antisymmetric part of an atomic or molecular property tensor called the antisymmetric polarizability α''_{ij}:

$$\alpha''_{ij}(B_{0k}) = \alpha''_{ij}(\mathbf{O}) + \alpha''_{ijk} B_{0k} \tag{2}$$

where α''_{ijk} is a perturbation tensor of order three. Before embarking on detailed theoretical development it is instructive to consider the fundamental symmetries of these atomic property tensors (we restrict consideration in this paper to atoms) and to recall that the complex electronic electric polarizability, of which α''_{ij} is the antisymmetric, imaginary component, is derived from time-dependent perturbation theory within whose framework[37] α''_{ij} is a product of two transition electric dipole moment operators. The perturbation tensor α''_{ijk} in this context involves two of these and one transition magnetic dipole moment matrix element. This structure introduces frequency dependence into the conventional Faraday effect, leading to MCD. Similarly, frequency dependence occurs in the optical Faraday effect, leading to optical MCD.

The fundamental symmetries considered are parity inversion (represented by the operator $\hat{P}$) and motion reversal (by the operator $\hat{T}$). In this context α''_{ij} is negative to $\hat{T}$ and positive to $\hat{P}$, while α''_{ijk} is positive to both $\hat{P}$ and $\hat{T}$. Therefore α''_{ij} is finite only in the presence of a $\hat{T}$-negative influence, such as $\mathbf{B}_0$ or $\hat{B}_{\Pi}$, and this influence is mediated by α''_{ijk}, which is finite for all atoms and molecules and is described by the ubiquitous B term.[37] The contemporary theory of Faraday's effect depends on a perturbation of a quantity α''_{ij}, which is itself the result of semiclassical, time-dependent, second-order perturbation theory. The basic reason for this is that the observable in the Faraday effect is an angle of rotation ($\Delta\theta$) (or alternatively a change in ellipticity $\Delta\eta$), which must be calculated from Maxwell's equations or Rayleigh refringent scattering theory.[37] Although the A term is closely related to the Zeeman effect,[38] the observables of the two effects are quite different, being traditionally an angle of rotation (Faraday's effect) and a frequency shift in the visible frequency region (Zeeman's effect). The latter can be described by an energy perturbation, while Faraday's effect needs perturbation of the antisymmetric polarizability, because energy does not appear directly in Maxwell's equations, which are needed to calculate refractive indices and therefrom $\Delta\theta$ and $\Delta\eta$.

With these considerations, the starting point for our development of the optical Faraday effect and optical MCD is the equation[37] for angle of rotation in the conventional quantum theory of the Faraday effect:

$$\Delta\theta \doteqdot \frac{1}{4}\omega\mu_0 c l \left(\frac{N}{d_n}B\right)_{0z} \sum_n \left[\alpha''_{XYZ} - \alpha''_{YXZ} + \frac{m_{nZ}}{kT}(\alpha''_{XY} - \alpha''_{YX})\right] \tag{3}$$

Here, $\Delta\theta$ is the angle of rotation of plane polarized probe radiation of angular frequency ω parallel to the conventionally generated magnetostatic flux density B_z. The quantity N is the total number of molecules per unit volume in a set of degenerate quantum states of the atom, individually designated[37] ψ_n, where d_n is the degeneracy and the sum is over all components of the degenerate set, with $N_d = Nd_n$. In Eq. (3), μ_0 is the vacuum permeability, c is the speed of light, l is the sample length, m_{nZ} is the Z component of the atomic magnetic dipole moment in state n, and kT is the thermal energy per atom. Recall that the atomic property tensors α''_{ij} and α''_{ijk} are derived from semiclassical, time-dependent, perturbation theory and are frequency dependent in general, so that $\Delta\theta$ mapped over a frequency range has the appearance of a spectrum—the conventional MCD spectrum.

Our task here is to incorporate the novel flux quantum $\hat{B}_{\Pi}$ (Refs. 1–6) into Eq. (3), and thus generate the optical Faraday effect and optical MCD. The terms α''_{ij} and α''_{ijk} as used in Eq. (3) are expectation values of the respective quantum-mechanical operators $\hat{\alpha}''_{ij}$ and $\hat{\alpha}''_{ijk}$; in the same way that m_{ni} is an expectation value of the magnetic electronic dipole moment operator $\hat{m}_n$. The quantity B_{0Z} is a classical magnetostatic vector component, and $\Delta\theta$ is an expectation value of the operator $\Delta\hat{\theta}$. It is convenient to transform the appropriate Cartesian components of the operators $\hat{\alpha}''_{ij}$ and $\hat{\alpha}''_{ijk}$ into spherical form,[37,39] using the Condon/Shortley phase convention:

$$\begin{aligned}
\hat{\alpha}''_{XY} &= -\frac{i}{2}\left[\sqrt{2}\,\hat{\alpha}_0^{1\prime\prime} + \left(\hat{\alpha}_2^{2\prime\prime} - \hat{\alpha}_{-2}^{2\prime\prime}\right)\right]\\
\hat{\alpha}''_{YX} &= \frac{i}{2}\left[\sqrt{2}\,\hat{\alpha}_0^{1\prime\prime} - \left(\hat{\alpha}_2^{2\prime\prime} - \hat{\alpha}_{-2}^{2\prime\prime}\right)\right]\\
\hat{\alpha}''_{XYZ} &= -\frac{i}{2}\left[\hat{\alpha}_0^{2\prime\prime} + \frac{1}{\sqrt{6}}\left(\hat{\alpha}_2^{2\prime\prime} + \hat{\alpha}_{-2}^{2\prime\prime}\right) + \frac{1}{\sqrt{3}}\left(\hat{\alpha}_2^{3\prime\prime} - \hat{\alpha}_{-2}^{3\prime\prime}\right)\right]\\
\hat{\alpha}''_{YXZ} &= \frac{i}{2}\left[\hat{\alpha}_0^{2\prime\prime} - \frac{1}{\sqrt{6}}\left(\hat{\alpha}_2^{2\prime\prime} + \hat{\alpha}_{-2}^{2\prime\prime}\right) - \frac{1}{\sqrt{3}}\left(\hat{\alpha}_2^{3\prime\prime} - \hat{\alpha}_{-2}^{3\prime\prime}\right)\right]
\end{aligned} \tag{4}$$

from which

$$\begin{aligned}
\hat{\alpha}''_{XY} - \hat{\alpha}''_{YX} &= -\sqrt{2}\,i\hat{\alpha}_0^{1\prime\prime}\\
\hat{\alpha}''_{XYZ} - \hat{\alpha}''_{YXZ} &= -i\hat{\alpha}_0^{2\prime\prime}.
\end{aligned} \tag{5}$$

Both $\hat{\alpha}''_{ij}$ and $\hat{\alpha}''_{ijk}$ are purely imaginary in the appropriate spherical representation, indicating that the operators $\mathrm{i}\hat{\alpha}_0^1$ and $\mathrm{i}\hat{\alpha}_0^2$ are anti-Hermitian, with purely imaginary eigenvalues.[37] (However, the $\hat{T}$ symmetry of $\mathrm{i}\hat{\alpha}_0^1$ is negative, while that of $\mathrm{i}\hat{\alpha}_0^2$ is positive. Both have positive $\hat{P}$ symmetry.)

The next step in our development for atoms is to replace the vector component B_{0Z} of the conventional quantum theory by the quantum-mechanical operator defined by Eq. (1), which has associated with it the angular momentum quantum number J. An immediate consequence of this replacement is the necessity to consider the magnetic field flux quantum using irreducible tensorial methods of angular momentum coupling theory in quantum mechanics.[39-41] In other words we are now considering a quantized photon angular momentum interacting with an atom, which contains, as usual, quantized orbital and spin electronic angular momentum. Without loss of generality we restrict consideration to atomic singlet states, in which there is a net orbital electronic angular momentum $\hat{L}$, but no net spin angular momentum $\hat{S}$.

With these considerations, Eq. (3) becomes, for the optical Faraday effect

$$\Delta\theta = \langle JM_J; LM_L|\Delta\theta|J'M'_J; L'M'_L\rangle = -\tfrac{1}{4}\omega\mu_0 clN_n \mathrm{i}\langle JM_J|\hat{B}_\Pi|J'M'_J\rangle \times\left(\frac{\langle LM_L|\hat{m}^1_{n0}|L'M'_L\rangle\langle LM_L|\hat{\alpha}_0^1|L'M'_L\rangle}{kT} + \langle LM_L|\hat{\alpha}_0^2|L'M'_L\rangle\right) \tag{6}$$

where we have used an uncoupled representation[39-41] to describe the net angular momentum generated during the interaction of photon and atom. This is justified because the azimuthal components of the operators $\hat{m}_n$ and $\hat{B}_\Pi$ are both well defined in the uncoupled representation,[39-41] whereas in the coupled representation of the same problem the total angular momentum is defined but the individual azimuthal components are not. With these considerations, the expectation value of the angle of rotation in the optical Faraday effect is

$$\Delta\theta = \mp\frac{1}{4}\omega\mu_0 clN_n\,\mathrm{i}M_JB_0\left\{\frac{M_L^2\langle L\|\hat{m}_0^1\|L\rangle\langle L\|\hat{\alpha}_0^1\|L\rangle}{L(L+1)(2L+1)kT} + \frac{[3M_L^2 - L(L+1)]\langle L\|\hat{\alpha}_0^2\|L\rangle}{[L(L+1)(2L+1)(2L+3)(2L-1)]^{1/2}}\right\} \tag{7}$$

ectation value of the photon's $\hat{B}_\Pi$

operator is $\pm M_JB_0$, positive for left circularly polarized radiation and negative for right circularly polarized radiation from the generator laser.

This is an expression for the angle of rotation induced in plane polarized probe radiation in a sample of atoms by a circularly polarized laser generating the flux quantum $\hat{B}_\Pi$. The observation of such a rotation would provide a test for the existence of $\hat{B}_\Pi$. Equation (7) is written out in terms of reduced matrix elements of dipole moment and atomic polarizability operators, matrix elements that are products of electric and magnetic dipole transition dipole moment matrix elements from time-dependent perturbation theory.[37] These introduce frequency dependence into the angle of rotation of the optical Faraday effect. The selection rules governing the various atomic property tensors are as follows:

$$\begin{aligned}&\hat{\alpha}_0^1\colon \Delta L = 0, \quad \Delta M_L = 0\\ &\hat{\alpha}_0^2\colon \Delta L = 0, \quad \pm 2; \quad \Delta M_L = 0\end{aligned} \tag{8}$$

where the $\Delta L = \pm 1$ part is parity forbidden, as in magnetic dipole transitions.

III. OPTICAL MCD: FREQUENCY DEPENDENCE OF Δθ

The origin of frequency dependence in the optical Faraday effect can be traced to semiclassical time-dependent perturbation theory, which produces expressions for the polarizability components as given in the conventional theory of magnetic electronic optical activity.[37] These can be further developed as usual in terms of reduced matrix elements of electric and magnetic transition dipole moment operators. For a given generator laser intensity and frequency, therefore, the optical MCD spectrum is a plot of the $\hat{B}_\Pi$ induced angle of rotation $\Delta\theta$ against the frequency of the linearly polarized probe. Experimentally, this is built up by replacing the conventional magnet of MCD apparatus by the circularly polarized generator laser.

IV. DISCUSSION

Using the concept of $\hat{B}_\Pi$ the theory of the optical Faraday effect can also be developed and understood simply by replacing the magnetic flux density vector component B_{0Z} wherever it occurs in the conventional theory of MCD by the quantity $\pm B_0M_J$, where B_0 is the magnetic flux density amplitude of the generator laser, and $\pm M_J$ are the two possible azimuthal quantum numbers of the photon. Thus, the optical Faraday effect can be developed along the lines of the conventional counterpart in Serber's A, B, and C terms, a convenient description of which is given by Barron[37] in his Eqs. (6.2.2) and (6.2.3). It follows that the optical MCD spectrum, which would test for the existence of $\hat{B}_{\Pi}$, would have the same characteristics as the conventional spectrum. If confirmed experimentally, this would be strong evidence for the photon's fundamental flux quantum $\hat{B}_{\Pi}$ introduced in Refs. 1–6. If the optical MCD spectrum were found to differ from the conventional MCD spectrum, it would indicate the presence of other mechanisms of magnetization by the generator laser, such as the induction of a magnetic dipole moment through[42–45]

$$m_i = {}^{m}\beta^{ee}_{ijk}(E_j E_k^* - E_k E_j^*) \equiv {}^{m}\beta^{ee}_{ijk}\Pi^{(A)}_{jk} \tag{9}$$

where ${}^{m}\beta^{ee}_{ijk}$ is a hyperpolarizability, and $\Pi^{(A)}_{jk}$ is the antisymmetric conjugate product of the generator laser.[42–45]

Finally, for an order of magnitude estimation of the expected angle of rotation in a linearly polarized probe due to a generator laser of intensity $I_0 = 10^6$ W m^{-2}, we use antisymmetric polarizabilities computed ab initio by Manakov et al.[13] in atomic $S = \frac{1}{2}$ ground states as a guide to orders of magnitude. For example, in Cs at 9440 cm^{-1} we have $\hat{\alpha}_0^1 = 3.4 \times 10^{-39}$ C^{2}m^{2}J^{-1}. Focusing attention on the term in $\hat{\alpha}_0^1$ in Eq. (7), and using $N = 6 \times 10^{26}$ molecules m^{-3} and the Bohr magneton for $\hat{m}_0^1$, we obtain for a generator laser delivering at 300 K

$$B_0 \sim 10^{-7} I_0^{1/2} = 10^{-4}\ \mathrm{T} \tag{10}$$

an angle of rotation of 0.8 rad m^{-1}, easily measurable with a spectropolarimeter. This result should be proportional to the square root of the intensity I_0 of the generator laser and inversely proportional to temperature. There is also a contribution from the rank three perturbating tensor of Eq. (7). These features would add to the evidence for the existence of $\hat{B}_{\Pi}$ already available from the inverse Faraday effect[8–14] and light-induced NMR shifts.[15, 16]

Acknowledgments

The Leverhulme Trust, Cornell Theory Center, and the MRL at Penn State are thanked for research support.

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THE PHOTON'S MAGNETOSTATIC FLUX DENSITY $\hat{B}_{\Pi}$: THE INVERSE FARADAY EFFECT REVISITED*

I. INTRODUCTION

Intense, circularly polarized laser pulses produce a net magnetization M_Z (A m^{-1}) in atomic, molecular, and other condensed material such as dilute magnetic semiconductors.[1,2] This magneto-optic property was first proposed by Piekara and Kielich,[3-6] and was demonstrated experimentally in the early 1960s by Pershan et al.[7,9,10] and Shen.[8] Since then, no further experimental work appears to have been reported on the effect. The theory of the inverse Faraday effect rests on the foundations built by Piekara and Kielich,[3-6] and was developed by Pershan et al.[7,9,10] in terms of the antisymmetric part of the tensor $E_i E_j^*$, where E_i is the electric field strength of a circularly polarized laser pulse in V m^{-1} and E_i^* is its own complex conjugate. This antisymmetric intensity is conveniently expressed in vector notation as the cross product $\mathbf{E} \times \mathbf{E}^*$, which is negative[11,12] to motion reversal ($\hat{T}$) and positive to parity inversion ($\hat{P}$). It therefore has the necessary $\hat{P}$ and $\hat{T}$ symmetries of magnetic flux density, which is the qualitative explanation for the ability of a circularly polarized laser to magnetize.

Further development of the theory is due to Kielich,[13-15] Atkins and Miller,[16] Wagnière,[17] Woźniak et al.,[18,19] and Evans et al.[20-22] with computer simulation of the magnetization. These theories all rely on the property $\mathbf{E} \times \mathbf{E}^*$ of the laser pulse. However, it has been shown recently[23-27] that this property, $\mathbf{E} \times \mathbf{E}^*$, is directly proportional to a novel, fundamental, magnetic flux density vector, $\mathbf{B}_{\Pi}$, of the classical electromagnetic field. In quantum-field theory[24] this becomes the novel,

fundamental, and ubiquitous magnetic flux density operator, $\hat{B}_{\Pi}$, of the photon itself. In this paper it is argued that the inverse Faraday effect must be described semiclassically as a combination of terms in all positive integral powers of the classical $\mathbf{B}_{\Pi}$. The original theory, which relies on $\mathbf{E} \times \mathbf{E}^*$, is shown to be equivalent to considering only the term in $|\mathbf{B}_{\Pi}|^2$.

II. DESCRIPTION OF $\mathbf{B}_{\Pi}$

It is straightforward to show that[11,12,17,23-27]

$$\mathbf{E} \times \mathbf{E}^* = 2E_0^2 \, \mathrm{i}\mathbf{k} \tag{1}$$

where $\mathbf{k}$ is a unit axial vector, negative to $\hat{T}$ and positive to $\hat{P}$. This is purely imaginary and proportional to the square of E_0, the scalar electric field strength amplitude of a circularly polarized laser. In free space $E_0 = cB_0$ and

$$\mathbf{E} \times \mathbf{E}^* = 2E_0 c \, \mathrm{i} B_0 \mathbf{k} \equiv 2E_0 c \, \mathrm{i}\mathbf{B}_{\Pi} \tag{2}$$

where B_0 is the scalar magnetic flux density amplitude (tesla) and c is the speed of light. The vector $\mathbf{B}_{\Pi}$ is the product $B_0\mathbf{k}$, which is in units of tesla. From these simple considerations,

$$\mathbf{B}_{\Pi} = \frac{\mathbf{E} \times \mathbf{E}^*}{2E_0 c \, \mathrm{i}} = B_0\mathbf{k} = \frac{E_0}{c}\mathbf{k} = \left(\frac{I_0}{\varepsilon_0 c^3}\right)^{1/2}\mathbf{k} \sim 10^{-7} I_0^{1/2}\mathbf{k}$$
$$= \left(\frac{|\mathbf{N}|}{2\varepsilon_0 c^3}\right)^{1/2}\mathbf{k} \tag{3}$$

Here I_0 is the scalar intensity in W m^{-2}, which in free space is

$$I_0 = \varepsilon_0 c E_0^2 \tag{4}$$

where ε_0 is the free space permittivity. In Eq. (3), $|\mathbf{N}|$ is the scalar magnitude of Poynting's vector

$$\mathbf{N} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B}^* \tag{5}$$

where μ_0 is the free space permeability. The vector $\mathbf{N}$ (Ref. 28) is a flux of energy density, and the novel vector $\mathbf{B}_\Pi$ is a flux of magnetic density. Although $\mathbf{N}$ and $\mathbf{B}_\Pi$ have the same negative $\hat{T}$ symmetry, the former is negative to $\hat{P}$ and the latter, as we have seen, is positive to $\hat{P}$. We note that $\hat{N}$ is nonzero in linear polarization, but $\mathbf{B}_\Pi$ vanishes, because the latter reverses sign with circular polarity, whereas the former does not. Furthermore, $\mathbf{N}$ can be expressed as

$$\mathbf{N} = 2I_0\mathbf{n} \tag{6}$$

where $\mathbf{n}$ is the vector whose scalar magnitude is the real refractive index in the direction of propagation, and which is well known[28, 29] to be proportional to the $\hat{P}$- and $\hat{T}$-negative polar wave vector κ. This must be carefully distinguished from the $\hat{P}$-positive, $\hat{T}$-negative unit axial vector $\mathbf{k}$, which is multiplied by B_0 to form the novel $\mathbf{B}_\Pi$. Note that both $\mathbf{N}$ and $\mathbf{B}_\Pi$ are independent of the phase of the laser, and therefore of its angular frequency ω. In other words, the time averages over many cycles of both $\mathbf{N}$ and $\mathbf{B}_\Pi$ are nonzero, and it follows that $\mathbf{B}_\Pi$ is quite different from the usual oscillating $\mathbf{B}$ field of the electromagnetic plane wave, which vanishes when averaged over time. $\mathbf{B}$ is a complex quantity, with components mutually orthogonal (i.e., in X and Y) to the propagation direction (Z) of the wave, but none in that direction itself. In contrast, $\mathbf{B}_\Pi$ is a purely real quantity,[23-27] and is directed exclusively in Z, with no components in X and Y. Remarkably, its existence appears to have gone unrecognized in the long and illustrious history of the theory of electromagnetic fields.

III. THE ROLE OF $\mathbf{B}_\Pi$ IN THE INVERSE FARADAY EFFECT: SEMICLASSICAL TREATMENT

Using the vector $\mathbf{B}_\Pi$ it becomes straightforward to develop any magnetic effect of the circularly polarized electromagnetic plane wave, because we can now say that such a wave can magnetize material with which it interacts. There exists in nature an optical magnet, which delivers a magnetic flux density in tesla of

$$|\mathbf{B}_\Pi| \sim 10^{-7} I_0^{1/2} \tag{7}$$

Thus, for a circularly polarized laser of intensity $I_0 = 10\,000$ W m^{-2} (1.0 W cm^{-2}) the $\mathbf{B}_\Pi$ field is 10^{-5} T, or 0.1 G, about a tenth of the earth's mean magnetic field.

When $\mathbf{B}_\Pi$ is used, the theory of magnetization by circularly polarized light becomes standard and straightforward, because we have only to adapt the existing semiclassical theory[28-30] of magnetization by an ordinary magnetostatic field, $\mathbf{B}_S$, and replace $\mathbf{B}_S$ everywhere by $\mathbf{B}_\Pi$. Thus, the magnetization is given by

$$M_Z = \frac{1}{\mu}\left(\frac{\kappa}{1+\kappa}\right)B_{\Pi Z} = N\langle m_Z\rangle_U \tag{8}$$

where κ is the volume susceptibility, μ_0 is the vacuum permeability, N is the number density, and $\langle m_Z\rangle$ is the mean magnetic dipole moment. It is assumed here that the total magnetic dipole moment is a sum of permanent and induced components:

$$\begin{aligned} m_Z &= m_Z^{(0)} + m_Z^{(\text{ind})} \\ m_Z^{(\text{ind})} &= \tfrac{1}{2}\xi_{ZZ}B_{\Pi Z}/\mu_0 \end{aligned} \tag{9}$$

where ξ_{ZZ} is the molecular magnetizability.[28-30] The ensemble average appearing in Eq. (8) is assumed to be the usual thermodynamic average[28-30]

$$\langle m_Z\rangle_U = \frac{\int\left(m_Z^{(0)} + m_Z^{(\text{ind})}\right)e^{-U/kT}\,d\Omega}{\int e^{-U/kT}\,d\Omega} \tag{10}$$

where the energy of interaction U is defined by

$$U = -m_Z^{(0)}B_{\Pi Z} - \frac{1}{2}\frac{\xi_{ZZ}}{\mu_0}B_{\Pi Z}^2 \tag{11}$$

The approximation[28-30]

$$\langle m_Z\rangle_U = \langle m_Z\rangle_0 - \frac{1}{kT}\left(\langle m_Z U\rangle_0 - \langle m_Z\rangle_0\langle U\rangle_0\right) + \cdots \tag{12}$$

is used for the thermodynamic average. Here $\langle\ \rangle_0$ denotes ensemble averaging in the absence of U, and $\langle\ \rangle_U$ denotes ensemble in averaging the presence of U. Since $\langle m_Z\rangle_0$ is zero in an initially isotropic material such as a gas or liquid, we have

$$\langle m_Z\rangle_U = -\frac{1}{kT}\langle m_Z U\rangle_0 + \cdots \tag{13}$$

This entirely

the temperature-dependent part of the effect we seek to describe:

$$\langle m_Z\rangle_U = \frac{1}{kT}\left(\langle m_Z^{(0)2}\rangle_0 B_{\Pi Z} + \frac{1}{\mu_0}\langle \xi_{ZZ} m_Z^{(0)}\rangle_0 B_{\Pi Z}^2 + \frac{1}{4\mu_0^2}\langle \xi_{ZZ}^2\rangle_0 B_{\Pi Z}^3\right) + \cdots \tag{14}$$

which shows that the magnetization (A/m) is described by terms in the first three powers of $B_{\Pi Z}$ within the first approximation (12) of the thermodynamic average (10). The term in $B_{\Pi Z}^2$ in Eq. (14) vanishes, however, because the theory of tensor invariants[28-30] shows that the ensemble average $\langle \xi_{ZZ} m_Z^{(0)}\rangle_0$ must vanish in isotropic media (but not in certain crystals). So we are left with terms in $B_{\Pi Z}$ and $B_{\Pi Z}^3$ for which the premultiplying ensemble averages do not vanish in liquids or gases.

In addition to these temperature-dependent terms, there exists the temperature-independent term considered in the usual theory of the inverse Faraday effect,[3-22] which has recently been put in the following simple form by Woźniak et al.[18]:

$$\langle m_Z\rangle_U = \frac{E_0^2}{3}\left({}^m\gamma_{123}^{ee} + {}^m\gamma_{231}^{ee} + {}^m\gamma_{312}^{ee}\right) \tag{15}$$

Here ${}^m\gamma_{ijk}^{ee}$ are molecule fixed-frame components of the appropriate[18, 19] molecular hyperpolarizability tensor. Using

$$E_0 = c|\mathbf{B}_\Pi| \tag{16}$$

it is immediately clear that this term is proportional to $|\mathbf{B}_\Pi|^2$. The complete expression for the inverse Faraday effect within the approximations we have made here is therefore

$$M_Z \doteqdot \frac{Nc^2}{3}\left({}^m\gamma_{123}^{ee} + {}^m\gamma_{231}^{ee} + {}^m\gamma_{312}^{ee}\right)B_{\Pi Z}^2 + \frac{N}{kT}\left(\langle m_Z^{(0)2}\rangle_0 B_{\Pi Z} + \frac{\langle \xi_{ZZ}^2\rangle_0}{4\mu_0^2} B_{\Pi Z}^3\right) \tag{17}$$

IV. ORDER OF MAGNITUDE ESTIMATES

To estimate the various orders of magnitude of the contributing terms in Eq. (17), the magnetic dipole moment is estimated roughly as a tenth of the electronic Bohr magneton, i.e., as 10^{-24} J T^{-1}. A rough order of magnitude approximation to the volume magnetic susceptibility κ is obtained from a model calculation given by Atkins,[28] which gives κ of about 10^{-5}. From this, the magnetizability can be obtained using $\kappa = N\xi_{ZZ}$ where N is the number density.[28] The order of magnitude of the hyperpolarizability ${}^m\gamma_{ijk}^{ee}$ is obtained from the Faraday effect theory of Woźniak et al.[18, 31] as about 10^{-45} A m^4V^{-2} for a typical diamagnetic. For a paramagnetic with a permanent magnetic dipole moment it is assumed that the hyperpolarizability is roughly 100 times bigger, i.e., 10^{-43} A m^4V^{-2}.

In Eq. (17) N is set at 10^{28} molecules for the molar volume in meters cubed (Ref. 18) and kT at 4×10^{-21} J molecule^{-1}, equivalent to 300 K. The order of magnitude of $B_{\Pi Z}$ is set at 1.0 T, corresponding to a pulse of intensity about 3×10^{15} W m^{-2}, available from a contemporary mode-locked laser, which must be accurately circularly polarized.

These rough estimates give an order of magnitude of magnetization (A m^{-1}) of about 2.5 A m^{-1} for the term in $B_{\Pi Z}$, about 2.0 A m^{-1} for the term in $B_{\Pi Z}^3$, and about 30 A m^{-1} for the temperature-independent term proportional to $B_{\Pi Z}^2$. Clearly these figures depend on the estimates we have used for $m_Z^{(0)}$, ξ_{ZZ}, and ${}^m\gamma_{ijk}^{ee}$ but all three terms contribute to the total magnetization. In our estimate, the term in $B_{\Pi Z}^2$ happens to be dominant, but at very low T and with less intense laser pulses, the term in $B_{\Pi Z}$ dominates, provided there is a permanent magnetic dipole moment. (If the latter is zero, there are terms in $B_{\Pi Z}^2$ and $B_{\Pi Z}^3$, but not in $B_{\Pi Z}$.)

Note that the magnetization changes sign with the circular polarity of the laser. The term in $B_{\Pi Z}$ changes sign because the vector $\mathbf{B}_\Pi$ is switched from positive (left) to negative (right). The conjugate product $\mathbf{E} \times \mathbf{E}^*$ changes sign with circular polarization,[18] and the product $\mathbf{E} \times \mathbf{E}^*$ is proportional to $|\mathbf{B}_\Pi|^2\mathbf{k}$, where $\mathbf{k}$ is a unit axial vector.

V. CONCLUSION

The inverse Faraday effect is characterized by a laser-induced magnetization that is proportional to all positive integral powers of $B_{\Pi Z}$. The conventional theory[18] is based solely on a consideration of $\mathbf{E} \times \mathbf{E}^*$, and produces a magnetization proportional to $B_{\Pi Z}^2$ only, from $\mathbf{E} \times \mathbf{E}^*$ multiplied by the sample's hyperpolarizability. We have argued that there is also a magnetization produced by a product of $B_{\Pi Z}$ and the sample's magnetizability.

Acknowledgments

The Leverhulme Trust is thanked for a Fellowship, and the Cornell Theory Center and Penn State Materials Research Laboratory are thanked for additional support.

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THE PHOTON'S MAGNETOSTATIC FLUX QUANTUM: THE OPTICAL COTTON-MOUTON EFFECT

I. INTRODUCTION

The ability of circularly polarized electromagnetic radiation to produce anisotropy in magnetic permeability was first proposed by Piekara and Kielich,[1,2] who systematically described light-induced anisotropy in material electric permittivity ($\Delta\varepsilon$), magnetic permeability ($\Delta\mu$), and refractive index (Δn). In Ref. 1 for example, formulated in the pre-laser era, it was proposed that "On observe alors des changements de ε, μ, ou n, dus à l'action du champ polarisant." We are concerned in this paper with the formulation of an optical Cotton-Mouton effect, a relative of the optical Kerr effect first proposed by Buckingham[3] and classified by Piekara and Kielich in their references. We define the novel optical Cotton-Mouton effect as a change in refractive index (linear dichroism) due to the novel, recently proposed, static magnetic field ($\mathbf{B}_\Pi$) of a circularly polarized electromagnetic plane wave.[4–8] Piekara and Kielich[1,2] described "saturation optique dans un champ optique." This effect later became known as the optical Kerr effect, or Buckingham effect.[3]

This paper is developed from the recent deduction[4–8] that the photon carries a magnetostatic flux quantum, $\hat{B}_\Pi$, whose classical equivalent is a phase-independent magnetic field $\mathbf{B}_\Pi$ generated in a circularly polarized light beam, an axial vector with the symmetry characteristics of a static magnetic flux density (tesla). The latter must be an axial vector positive to the parity inversion operator $\hat{P}$, and negative to the motion reversal operator $\hat{T}$ (Ref. 9). The classical field $\mathbf{B}_\Pi$ of the circularly polarized electromagnetic plane wave is a purely real quantity that is proportional to the antisymmetric (purely imaginary) part of the tensor $E_iE_j^*$, where E_i is the electric field strength of the wave in volts per meter. The scalar part of the tensor $E_iE_j^*$ is proportional to the phase-independent intensity of the plane wave in watts per meter squared, and we have shown elsewhere[4–8] that the vector part of $E_iE_j^*$ (i.e., its antisymmetric part) is proportional to the phase-independent magnetic flux density $\mathbf{B}_\Pi$ and vanishes if there is no degree of circular polarity. Furthermore, we have shown[8] that $\mathbf{B}_\Pi$ can be expressed in terms of the ubiquitous third Stokes parameter S_3 (Ref. 10) and therefore that phenomena such as circular dichroism and ellipticity are fundamentally magnetic.

The definition[4–8] of the $\hat{B}_\Pi$ operator per photon allows a wide range of novel optical/photonic phenomena to be forecasted straightforwardly, on the grounds that circularly polarized electromagnetic radiation can magnetize. This conclusion is independent of the phase of the plane wave, and therefore independent of its angular frequency, ω (rad/s). It follows that the time average of the classical vector $\mathbf{B}_\Pi$ is nonzero. It is emphasized that $\mathbf{B}_\Pi$ is fundamentally different from the usual oscillating $\mathbf{B}$ field of the plane wave[10]: $\mathbf{B}$ vanishes when time averaged, because it is phase dependent, and has no component in the propagation axis Z of the wave. The vector $\mathbf{B}_\Pi$ is directed exclusively in Z, and has no components in X and Y. By expressing the antisymmetric part of the tensor $E_i E_j^*$ as a vector product, $\mathbf{E} \times \mathbf{E}^*$ (refs. 4–8), it becomes clear that $\mathbf{B}_\Pi$ is a relative of the Poynting vector,[8–10] $\mathbf{N} = (\mathbf{E} \times \mathbf{B}^*)/\mu_0$, where μ_0 is the free space permeability. However, the polar vector $\mathbf{N}$ is $\hat{T}$- and $\hat{P}$-negative, whereas the axial vector $\mathbf{B}_\Pi$ is $\hat{T}$-negative, $\hat{P}$-positive,[4–8, 11] a critically important symmetry difference. Accordingly, $\mathbf{N}$ is interpreted physically as a flux of energy density, and $\mathbf{B}_\Pi$ as a flux of magnetic density. Remarkably, $\mathbf{N}$ has been well known for many years, and $\mathbf{B}_\Pi$ appears to be entirely novel.

The flux density vector $\mathbf{B}_\Pi$ is clearly generated in vacuo (i.e., in free space), in direct analogy with $\mathbf{N}$. Both vectors $\mathbf{N}$ and $\mathbf{B}_\Pi$ are generated from solutions of Maxwell's classical equations through vector cross products of the usual, oscillating, phase-dependent $\mathbf{E}$ and $\mathbf{B}$ components of the electromagnetic plane wave solutions, cross products that multiply a vector with a complex conjugate vector, thus removing the phase dependence. For example, the complex conjugate ($\mathbf{E}^*$) of $\mathbf{E}$, a plane wave solution of Maxwell's equations, is also an allowed solution of Maxwell's equations, and the vector product of $\mathbf{E}$ and $\mathbf{E}^*$, two allowed solutions, generates the purely imaginary conjugate product $\Pi^{(A)}$, which is proportional[4–8] to $\mathbf{B}_\Pi$. Similarly, the vector product of $\mathbf{E}$ and $\mathbf{B}^*$ is proportional to $\mathbf{N}$. Therefore, although Maxwell's equations allow no direct, phase-independent solutions in free space, vector products of allowed solutions, such as $\mathbf{N}$ and $\mathbf{B}_\Pi$, are physically meaningful phase-independent quantities whose time averages are nonzero.

It follows that $\mathbf{B}_\Pi$ can interact with material to produce observable effects, again in direct analogy with $\mathbf{N}$. The scalar part of $\mathbf{N}$ is the intensity I_0, and the intensity (for example, of a laser) is clearly a free space quantity that affects and interacts with material. (For example, a sample is heated by intense light, light that travels through a vacuum.) Similarly, $\mathbf{B}_\Pi$ is a free space magnetic flux density that can also affect material. For example, $\mathbf{B}_\Pi$ forms a vector dot product with an electronic or nuclear magnetic dipole moment to give an interaction Hamiltonian (whose expectation value is an observable and measurable energy). This leads to the recently observed phenomenon of optical NMR[12] in which a circularly polarized laser shifts NMR resonances in new and unexpected ways, leading to useful new fingerprints for the analytical laboratory.[13] These shifts were found experimentally to vanish in the uncertainty of measurement when the laser's circular polarization was removed, strong evidence that they depend in an as yet incompletely understood manner on $\mathbf{B}_\Pi$. (There are as many, if not more, mechanisms involving $\mathbf{B}_\Pi$ in optical NMR as there are involving the ordinary magnetostatic field in conventional NMR.)

It is easy to see that if circularly polarized light is simply regarded as an "optical magnet," there should be observable in one way or another all the well-known phenomena of conventionally produced magnetism,[14] such as the Faraday, magnetic circular dichroic, Zeeman, Cotton-Mouton, Gerlach-Stern, Aharonov-Bohm, NMR, and ESR phenomena. Thus far, the $\mathbf{B}_\Pi$ concept has been developed for the optical Zeeman effect,[5] anomalous optical Zeeman effect,[6] the optical Faraday effect,[7] optical effects in Compton scattering,[4] and the inverse Faraday effect,[8] which is bulk magnetization by $\mathbf{B}_\Pi$ of a circularly polarized laser. In general, whenever a magnetic can be used in physics, so can a circularly polarized laser, which generates $\mathbf{B}_\Pi$. The magnitude of $\mathbf{B}_\Pi$ is approximately 10^{-7} $I_0^{1/2}$ in tesla,[4–8] so that an accurately circularly polarized laser of intensity 1.0 W cm^{-2} generates 10^{-5} T, about a tenth of the earth's mean magnetic field. Clearly, pulses of laser radiation of say, up to 10^{16} W m^{-2}, available in principle,[15] generate a substantial 10 T over the duration of the laser pulse. (Normally incoherent radiation, such as daylight, produces no $\mathbf{B}_\Pi$, because there is no mean circular polarization; a linearly polarized laser, however, intense, produces no $\mathbf{B}_\Pi$, because such a laser always contains equal and opposite amounts of right and left circularly polarized light—right and left photons.)

In this paper, an example is given of the straightforward way in which the $\mathbf{B}_\Pi$ vector can be used to anticipate the existence of a novel optical phenomenon—the optical Cotton-Mouton effect. Section II defines $\mathbf{B}_\Pi$ in its classical limit in terms of fundamental constants, and brings out the precise analogy between $\mathbf{B}_\Pi$ and an ordinary magnetostatic flux density, $\mathbf{B}_0$. This allows the semiclassical theory[10] of the Cotton-Mouton effect to be developed straightforwardly in terms of $\mathbf{B}_\Pi$ in section III. The order of magnitude of the linear dichroism (or ellipticity) induced by $\mathbf{B}_\Pi$ is estimated in Section IV.

II. DEFINITION OF THE CLASSICAL $\mathbf{B}_\Pi$ OF A CIRCULARLY POLARIZED LASER

The classical vector $\mathbf{B}_\Pi$ of a circularly polarized laser in free space is obtained straightforwardly[4–8] by a consideration of the $\hat{T}$ and $\hat{P}$ symmetries of the conjugate product—the vector part of $E_i E_j^*$:

$$\mathbf{\Pi}^{(A)} = \mathbf{E} \times \mathbf{E}^* = 2E_0^2 i\mathbf{k} = 2E_0 c i \mathbf{B}_\Pi \tag{1}$$

Here the axial vector $\mathbf{B}_\Pi$ is in tesla, is $\hat{T}$-negative and $\hat{P}$-positive, and is directed in the propagation axis of the laser. Thus, $\mathbf{B}_\Pi$ has the necessary and sufficient characteristics to define a magnetic flux density vector. This simple derivation shows that circularly polarized radiation magnetizes material with which it comes into contact from free space. A relation between $\mathbf{B}_\Pi$ and the Poynting vector $\mathbf{N}$ is obtained straightforwardly from a consideration of[10]

$$\mathbf{B} = \frac{1}{c}\mathbf{n} \times \mathbf{E} \qquad \mathbf{E} = -\frac{c}{n^2}\mathbf{n} \times \mathbf{B} \tag{2}$$

Here, $\mathbf{n}$ is a $\hat{T}$- and $\hat{P}$-negative polar vector, whose scalar magnitude is the refractive index, and which is related to the classical wave vector of the laser by

$$\boldsymbol{\kappa} = \frac{\omega}{c}\mathbf{n} \tag{3}$$

In free space, the scalar magnitude of $\mathbf{n}$ is unity, and it follows that

$$\mathbf{N} = 2I_0\mathbf{n} \tag{4}$$

where the magnitude of $\mathbf{N}$ is defined through the scalar intensity of the laser in W m^{-2}:

$$I_0 = \varepsilon_0 c E_0^2 \tag{5}$$

Here ε_0 is the vacuum permittivity in S.I. units and c is the speed of light. It follows that $\mathbf{B}_\Pi$ is related to the square root of the Poynting vector:

$$\mathbf{B}_\Pi = B_0\mathbf{k} = \frac{E_0}{c}\mathbf{k} = \left(\frac{I_0}{\varepsilon_0 c^3}\right)^{1/2}\mathbf{k} = \left(\frac{|\mathbf{N}|}{2\varepsilon_0 c^3}\right)^{1/2}\mathbf{k} \tag{6}$$

From these simple derivations it follows that the scalar part (or trace) of the tensor $E_iE_j^*$ is responsible for the Poynting vector's magnitude, and that the antisymmetric (vector) part of the tensor $E_iE_j^*$ is responsible for the novel phase-independent magnetic flux density $\mathbf{B}_\Pi$. In quantum field theory it has been shown elsewhere[4–9] that $\mathbf{B}_\Pi$ becomes a novel elementary magnetic field of the photon itself—an operator $\hat{B}_\Pi$.

III. APPLICATION TO THE OPTICAL COTTON-MOUTON EFFECT

Since $\mathbf{B}_\Pi$ has all the characteristics of a magnetostatic flux density, it can be used to describe a variety of novel magneto-optic effects, an example of which is an optical Cotton-Mouton effect, developed in this section with a standard semiclassical approach. The optical Cotton-Mouton effect is the development of linear birefringence in a probe light beam propagating in axis Z through a suitable sample and linearly polarized at 45° to the direction of an applied pump laser in the X axis. The latter is circularly polarized and generates $B_{\Pi X}$. Elliptical polarization in the probe is produced by $B_{\Pi X}$ of the pump, which plays the role of the ordinary magnet of the original effect discovered by Cotton and Mouton[16] in 1907. The pump's $B_{\Pi X}$ produces a phase difference in the two coherent resolved components of the probe, linearly polarized parallel and perpendicular, respectively, to the direction X of $B_{\Pi X}$ of the pump. This phase difference is[10]

$$\delta = \frac{\omega}{c}l(n_\parallel - n_\perp) \tag{7}$$

where $n_\parallel$ and $n_\perp$ are the refractive indices for light linearly polarized parallel and perpendicular to X. The resulting ellipticity is $\delta/2$.

At absorbing wavelengths, the two components $n_\parallel$ and $n_\perp$ are accompanied by two different absorption coefficients, signaling the presence of linear dichroism due to $B_{\Pi X}$ of the circularly polarized pump laser. There is a rotation of the major axis of the polarization ellipse of the probe laser because a difference in amplitude develops between two orthogonal resolved components for which no phase difference exists.[10]

Kielich and Piekara[2] have summarized the various theories of the standard Cotton-Mouton effect, under their classification scheme denoted "optical saturation in a magnetic field." In our case this magnetic field is $B_{\Pi X}$ of the circularly polarized pump laser in direction X. The novel $\mathbf{B}_\Pi$ concept allows these theories to be adapted directly for the optical Cotton-Mouton effect suggested here. We have simply replaced an ordinary magnet with an optical magnetic, which is an intense circularly polarized pump laser, operable at any electromagnetic frequency, from infrared to X-ray regions. Notable theories include those of Raman and Krishnan,[17] Piekara,[18,19] Peterlin and Stuart,[20] Snelman,[21] and the semiclassical approach at Buckingham and Pople.[22,23] Kielich has developed the conventional Cotton-Mouton and related effects in several directions, for example, (1) the theory of the inverse Cotton-Mouton effect,[24] which he described as the induction of magnetic anisotropy by an intense laser beam; (2) the theories in colloids of the inverse Cotton-Mouton effect[25] and the important but neglected Majorana effect[26] in colloids, liquid crystals, and polymers; and (3) general theories of magneto-optics.[27]

These theories can now be recast to great advantage, in principle, using the $\mathbf{B}_\Pi$ concept, or its equivalent for magneto-photonics, the operator $\hat{B}_\Pi$ (Ref. 15) of the photon itself. As an example we take the semiclassical theory of Buckingham and Pople[23] given originally for the ordinary Kerr effect, and adapt it straightforwardly for the optical Cotton-Mouton effect by substituting $\mathbf{B}_\Pi$ for the ordinary static electric field $\mathbf{E}_0$ of the Kerr effect, or the ordinary static $\mathbf{B}_0$ field of the standard Cotton-Mouton effect.[10] In so doing it is convenient to follow the summary given by Barron[10] for the Kerr effect and indicate the simple changes needed for the optical Cotton-Mouton effect along the way.

The starting point is the expression for probe ellipticity in Rayleigh refringent scattering theory[10]:

$$\eta = -\tfrac{1}{4}N\omega\mu_0 cl\left[\alpha'_{XX}(f) - \alpha'_{YY}(f)\right] \tag{8}$$

in terms of laboratory frame components of the real parts of the polarization tensor $\alpha'_{\alpha\beta}$ of a molecules of the sample. Here N is the number of molecules in the sample, ω is the angular frequency of the probe laser, μ_0 is the vacuum magnetic permeability, and l is the sample length in meters through which the probe passes. The $\mathbf{B}_\Pi$ vector of the circularly polarized pump laser generates anisotropy in the sample because $\mathbf{B}_\Pi$ interacts with the permanent and induced magnetic dipole moments in each molecule (or atom). The total magnetic dipole moment per molecule is, accordingly,

$$m_\alpha = m_{0\alpha} + \chi'_{\alpha\beta}B_{\Pi\beta} + \cdots \tag{9}$$

where $m_{0\alpha}$ is the permanent molecular electronic magnetic dipole moment (if nonzero), and $\chi'_{\alpha\beta}$ is the real static susceptibility, a symmetric second-rank property tensor.[10] The dynamic polarizability is perturbed by $\mathbf{B}_\Pi$ of the pump laser as follows:

$$\alpha'_{\alpha\beta}(\mathbf{B}_\Pi) = \alpha'_{\alpha\beta}(0) + \alpha'_{\alpha\beta\gamma} + \alpha'_{\alpha\beta\gamma\delta}B_{\Pi\gamma}B_{\Pi\delta} + \cdots \tag{10}$$

and in the evaluation of η in Eq. (8) an ensemble average is taken of the polarizability tensor components perturbed by $\mathbf{B}_\Pi$ of the pump. In forming this ensemble average, an interaction potential energy is used of the type

$$V(\Omega) = -m_{0X}B_{\Pi X} - \tfrac{1}{2}\chi'_{XX}B^2_{\Pi X} + \cdots \tag{11}$$

From tensor invariant theory[10] the ellipticity is finally obtained, in precise parallel with the theory of the Kerr effect, as

$$\begin{aligned}\eta = -\frac{1}{120}\omega\mu_0 clNB^2_{\Pi X}\Big[&\alpha'_{\alpha\beta\alpha\beta}(f) - \alpha'_{\alpha\alpha\beta\beta}(f)\\ &+ \frac{2}{kT}\left(3\alpha'_{\alpha\beta\alpha}(f)m_{0\beta} - \alpha'_{\alpha\alpha\beta}(f)m_{0\beta}\right)\\ &+ \frac{1}{kT}\left(3\alpha'_{\alpha\beta}(f)\chi'_{\alpha\beta} - \alpha'_{\alpha\alpha}(f)\chi'_{\beta\beta}\right)\\ &+ \frac{1}{k^2T^2}\left(3\alpha'_{\alpha\beta}(f)m_{0\alpha}m_{0\beta} - \alpha'_{\alpha\alpha}(f)m_{0\beta}m_{0\beta}\right)\Big]\end{aligned} \tag{12}$$

which is valid rigorously at transparent frequencies only.

IV. DISCUSSION

For simplicity we consider a sample that has no permanent magnetic dipole moment. For this sample the probably dominant term in Eq. (12) involves a product of the molecular polarizability and molecular susceptibility. The ellipticity developed in the probe is second order in $B_{\Pi X}$, or first order in the intensity I_0 of the pump laser. Accordingly, the sign of η should not be changed by switching the circular polarization of the pump from left to right, thus reversing $\mathbf{B}_\Pi$ (Refs. 4–8). However, if the pump is linearly polarized, $B_{\Pi X}$ and thus η should be zero for all I_0 of the pump.

With these overall considerations and taking a sample molecular electric polarizability[27] of the order $10^{-40}C^2m^2J^{-1}$, a static molecular susceptibility of the order $10^{-24}C^2m^{-4}J^{-1}S^{-1}$, N of the order 10^{26} molecules m^{-3}, l of 1 m, ω about 10^{15} rad s^{-1}, and kT of 4.14×10^{-21} J, corresponding to 300 K, we obtain

$$\eta \doteq 10^{-2}B^2_{\Pi X} \tag{13}$$

Therefore, for a pump laser delivering a $B_{\Pi X}$ pulse of 1.0 T, the ellipticity change is 0.01 rad, or 0.6° m^{-1}. As first discussed by Kielich,[25] this could be enhanced by up to six orders of magnitude in colloidal solution, or in suitable liquid crystals just above the isotropic to mesophase transition, i.e., in a state where the sample is still transparent to pump and probe lasers. The effect of the pump's $\mathbf{B}_\Pi$ pulse can be picked up by a probe using highly developed contemporary timing technology, as in work on the rotation of the elliptical polarization ellipse by a circularly polarized, giant ruby laser pump pulse.[28–30] It therefore appears possible to observe the optical Cotton-Mouton effect as proposed in this work in terms of the novel $\mathbf{B}_\Pi$ vector, whose photon equivalent is the $\hat{B}_\Pi$ operator, the photon's magnetostatic flux density.

Acknowledgments

The Leverhulme Trust, Cornell Theory Center, and the Materials Research laboratory at Penn State are thanked for research support.

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THE PHOTON'S MAGNETOSTATIC FLUX QUANTUM: FORWARD–BACKWARD BIREFRINGENCE INDUCED BY A LASER

I. INTRODUCTION

When a magnetostatic flux density $\mathbf{B}_S$ is applied to an initially isotropic chiral liquid, that liquid develops forward–backward birefringence, otherwise known as magneto-chiral birefringence and magneto-spatial dispersion. The refractive index in the direction $(+Z)$ of forward propagation of a probe laser becomes different from that in the backward direction $(-Z)$. The Kramers-Kronig theorem implies that the same happens to the power absorption coefficient. The presence of this effect in liquids has been proposed several times theoretically, but has never been detected experimentally. The effect appears to have been first proposed by Portigal and Burstein[1] in magnetic crystal symmetries, and was measured by Mankelov et al.[2] The theory was extended by Kielich and Zawodny[3] to crystals with magnetic ordering. Working with liquids, Baranova and Zel'dovich[4] described the refractive index change in circularly polarized probe radiation in terms of the dot product $\mathbf{B}_S \cdot \boldsymbol{\kappa}$, where $\boldsymbol{\kappa}$ is the classical wave vector of the probe, and implied the presence of forward–backward birefringence. The first detailed papers on the subject in chiral liquids are due to Woźniak and Zawodny,[5,6] who defined the molecular point groups able to support the effect, and developed a theory based on electronic distortion and reorientation of the permanent molecular magnetic dipole moment, if nonvanishing. Wagnière and coworkers[7–9] developed the theory of the effect for power absorption as well as refractive index, and Barron and Vrbancich[10] contributed a comprehensive paper on forward–backward birefringence and dichroism in chiral liquids based on time-odd, complex, molecular property tensors. In this work, an unsuccessful attempt to measure the effect experimentally was reported briefly. Woźniak later developed the semiclassical theory of the effect in diamagnetic molecules in the presence of electric[11,12] and optical[13] fields, and later[14] for paramagnetic molecules. Wagnière[15] considered the consequences of using the effect to measure parity violation in atoms, and has also proposed a similar, but much weaker, forward–backward effect, which he named "inverse magneto-chiral birefringence."[16] The latter has recently been developed in the context of frequency dependence by Woźniak et al.[17,18] Barron[19] has reviewed the effect in the context of motion reversal ($\hat{T}$) and parity inversion symmetry ($\hat{P}$).

Recently[20–25] the present author has proposed an optically induced forward–backward birefringence, in which the magnetostatic field is replaced by a circularly polarized pump laser, wherein the antisymmetric component ($\mathbf{\Pi}^{(A)}$) of the intensity tensor of the pump laser is the optical property responsible for the development of forward–backward birefringence in a probe laser directed parallel to the circularly polarized pump. The component $\mathbf{\Pi}^{(A)}$ can be expressed as a $\hat{T}$-negative, $\hat{P}$-positive axial vector directed in the propagation axis Z of the circularly polarized pump. This effect, known as spin chiral dichroism,[20–25] is proportional to the scalar magnitude I_0 of the pump laser intensity, and vanishes if there is no degree of pump circular polarization.

In this paper, we use the recently proposed[26–31] magnetostatic flux density operator $\hat{\mathbf{B}}_{\Pi}$ of the photon to demonstrate straightforwardly that there exists a first-order optically induced forward–backward birefringence in which the traditional $\mathbf{B}_S$ field produced by a strong magnet is replaced by the $\hat{\mathbf{B}}_{\Pi}$ operator of a circularly polarized pump laser pulse. In the classical theory of fields, $\hat{B}_{\Pi}$ becomes the axial flux density vector $\mathbf{B}_{\Pi}$, directed along the propagation axis Z of the pump laser. This new effect is proportional to the square root of the pump laser scalar intensity, i.e., $I_0^{1/2}$. We refer hereafter to this effect as $\hat{B}_{\Pi}$-induced forward–backward anisotropy (BFBA), an effect that is one of a large number of new magneto-photonic phenomena based on the existence of $\hat{B}_{\Pi}$, or its classical equivalent $\mathbf{B}_{\Pi}$. Among these are optical NMR, recently detected experimentally,[32] in which $\hat{B}_{\Pi}$ causes unexpected and characteristic shifts in conventional NMR resonance patterns in N dimensions, optical ESR and the optical Zeeman effects,[27] the optical Faraday effect,[28] the optical Cotton-Mouton effect,[29] and other $\hat{B}_{\Pi}$-induced effects. It has also been shown[30,31] that $\hat{B}_{\Pi}$ is ubiquitous in physical optics, being interpretable in terms of the third Stokes parameter S_3, and is therefore the origin of such well-known phenomena as circular dichroism and ellipticity. The operator $\hat{B}_{\Pi}$ also provides a new fundamental explanation for the de Broglie wave particle duality in photons,[33] in that it can be defined simultaneously in terms of the angular and linear momenta of the photon. The same conclusion applies for other massless leptons that propagate at the speed of light in any reference frame, for example, the neutrino.

In Section II we summarize briefly the pertinent properties of the $\mathbf{B}_{\Pi}$ vector of a circularly polarized pump laser, and relate it to such well-known quantities as the Poynting vector $\mathbf{N}$ (Ref. 34) and the usual electric field strength amplitude (E_0) and magnetic flux density amplitude (B_0) of the classical electromagnetic plane wave. A simple equation relates the scalar magnitude (tesla) B_0 of $\mathbf{B}_{\Pi}$ to the square root of I_0 of the pump laser (watts per square meter). Section III is a straightforward application of the $\mathbf{B}_{\Pi}$ vector to the various theories proposed for forward–backward birefringence due to an ordinary magnetostatic field, $\mathbf{B}_S$. An order of magnitude of the BFBA is given in terms of the intensity I_0 of the pump laser, thus anticipating the feasibility of an experimental investigation of the effect.

II. SUMMARY OF $\mathbf{B}_{\Pi}$ PROPERTIES

The $\mathbf{B}_{\Pi}$ vector is the classical equivalent of the quantum-field operator $\hat{B}_{\Pi}$ (Ref. 26), and can be expressed in terms of the vector cross product $\mathbf{E} \times \mathbf{E}^*$ of the free space electromagnetic plane wave. Here $\mathbf{E}$ is the usual, oscillating, electric field strength vector of the wave, and $\mathbf{E}^*$ is its complex conjugate. We have

$$\mathbf{B}_{\Pi} = \frac{\mathbf{E} \times \mathbf{E}^*}{2E_0 c i} = B_0 \mathbf{k} = \frac{E_0}{c}\mathbf{k} = \left(\frac{I_0}{\varepsilon_0 c^3}\right)^{1/2} \mathbf{k} = \left(\frac{|\mathbf{N}|}{2\varepsilon_0 c^3}\right)^{1/2} \mathbf{k} \quad (1)$$

Here $\mathbf{k}$ is an axial unit vector in the propagation axis of the wave, ε_0 is the vacuum permittivity in S.I. units, and c is the speed of light. From these definitions[26–33] it is clear that the novel $\mathbf{B}_{\Pi}$ is a member of the same class of optical properties as the Poynting vector $\mathbf{N}$:

$$\mathbf{N} = \frac{\mathbf{E} \times \mathbf{B}^*}{\mu_0} \quad (2)$$

where μ_0 is the vacuum permeability in S.I. units and $\mathbf{B}$ is the usual oscillating electromagnetic flux density vector of the plane wave in free space. (Note that the novel $\mathbf{B}_{\Pi}$ is independent of the phase of the plane wave, is directed exclusively in its propagation axis Z and is purely real, reversing its sign with a left to right-switch in circular polarization and vanishing, therefore, in linearly polarized or incoherent radiation such as normal daylight. On the other hand, $\mathbf{B}$ of the plane wave is known[34] to be a complex quantity, phase dependent, with components in X and Y, mutually orthogonal to the propagation axis.) The novel $\mathbf{B}_{\Pi}$ and the well-known $\mathbf{N}$ are both derived by forming vector cross products from oscillating plane wave solutions of the Maxwell equations, but are themselves independent of the phase of the plane wave, and thus of its angular frequency ω. The vector $\mathbf{N}$ is a flux of energy density, and $\mathbf{B}_{\Pi}$ is a flux of magnetic density in tesla (i.e., magnetic flux density). However, the $\hat{P}$ symmetries of $\mathbf{B}_{\Pi}$ and $\mathbf{N}$ are opposite: The former is $\hat{P}$-positive and the latter is $\hat{P}$-negative, implying that the $\mathbf{B}_{\Pi}$ is an axial vector and that $\mathbf{N}$ is a polar vector. The $\hat{T}$ symmetries of both vectors are identical, both are $\hat{T}$-negative,[35-38] a conclusion arrived at after a careful consideration of the $\hat{T}$ symmetries of $\mathbf{E} \times \mathbf{E}^*$ and $\mathbf{E} \times \mathbf{B}^*$. The scalar magnitude of $\mathbf{N}$ is the quantity $2I_0$, which is the trace (or scalar part) of the tensor product $E_i E_j^*$ in tensor subscript notation.[34] The magnitude of $\mathbf{B}_{\Pi}$ is derived from the antisymmetric part of $E_i E_j^*$, which is its vector part.[39]

The following approximate relation is useful for assessing the magnitude of $\mathbf{B}_{\Pi}$ in terms of I_0:

$$|\mathbf{B}_{\Pi}| = B_0 \doteqdot 10^{-7} I_0^{1/2} \tag{3}$$

so that a circularly polarized laser delivering one watt per square centimeter (10 000 W m^{-2}) generates a $\mathbf{B}_{\Pi}$ field of 10^{-5} T, which is 0.1 G or about one-tenth of the earth's mean magnetic field. A laser pulse of 10^{16} generates 10.0 T, equivalent to a contemporary superconducting magnet for the duration of the pulse.

III. EXPRESSIONS FOR FORWARD–BACKWARD ANISOTROPY DUE TO $\mathbf{B}_{\Pi}$

Using the novel $\mathbf{B}_{\Pi}$ vector, it is possible to adapt immediately the key results of previous work on forward–backward birefringence. Adapting the results of Woźniak and Zawodny[5,6] shows, for example, that the magneto-spatial change in the refractive index is proportional to the scalar product $\mathbf{B}_{\Pi} \cdot \boldsymbol{\kappa}$, where $\boldsymbol{\kappa}$ is the wave vector of a circularly polarized probe laser parallel to the pump laser generating $\mathbf{B}_{\Pi}$. By substituting $\mathbf{B}_{\Pi}$ for $\mathbf{B}_S$ in the development of these authors, this change in refractive index can be interpreted through the same electronic distortion mechanism and through the same reorientational process mediated by the permanent magnetic dipole moment. Furthermore, the molecular symmetries (magnetic point groups) mediating the magneto-spatial effect of $\mathbf{B}_{\Pi}$ are the same as those derived by Woźniak and Zawodny for the magneto-spatial effect of an ordinary magnetic field, $\mathbf{B}_S$. The key semiclassical expression of magneto-spatial dispersion due to $\mathbf{B}_{\Pi}$ is the precise analogue of the last term of Eq. (52) of Woźniak and Zawodny[6]:

$$n_{\pm} = -(V + V^{\mathrm{T}})(\mathbf{B}_{\Pi} \cdot \mathbf{S}) + \cdots \tag{4}$$

where $n_{\pm}$ are the real refractive indices measured by the probe laser in the $\pm Z$ directions, the quantities V and V^{T} are defined in terms of molecular property tensors by Eqs. (58) and (59) of Ref. 6, and $\mathbf{s}$ is a unit vector in Z. In an addendum to their original paper,[6] these authors discuss other contributions to their original Eq. (52). They have provided extensive tabulation of the symmetry of the molecular property tensors making up V and V^{T}; but no order of magnitude estimate was made.

Similarly, it is possible to adapt straightforwardly the development of magneto-chiral birefringence due to Barron and Vrbancich.[10] These authors make extensive use of tensor algebra and invariant ensemble averages to describe the phenomenon semiclassically. They provide an approximate estimate of the anticipated order of magnitude of the effect, and also report an attempted experimental observation with a modified Rayleigh interferometer. Our purpose here is to adopt their main results for use with $\mathbf{B}_{\Pi}$ of a circularly polarized pump laser, thus providing immediately a theory of BFBA in chiral liquids. This is achieved by replacing the $\mathbf{B}$ field of the permanent magnet, wherever it occurs, by the $\mathbf{B}_{\Pi}$ field of a circularly polarized pump laser. The symbol $\mathbf{B}$ in the development by Barron and Vrbancich[10] is replaced wherever it occurs in their paper by $\mathbf{B}_{\Pi}$. It is therefore unnecessary to repeat the complicated tensor algebra here and we focus on a result such as their Eq. (3.17a), which gives the magneto-chiral birefringence in terms of appropriate molecular property tensor elements, averaged with the principles of tensor invariants.[10,34] This result is later approximated[10] to base an estimate upon

$$n^{\uparrow\uparrow} - n^{\uparrow\downarrow} \doteqdot \frac{\frac{1}{6}\mu_0 c N B_{\Pi Z} \varepsilon_{\alpha\beta\gamma} m_{0\gamma} G_{\alpha\beta}(f)}{kT} + \cdots$$

where $n^{\uparrow\uparrow} - n^{\uparrow\downarrow}$ is the forward–backward refractive index difference due to a pump laser generating $+B_{\Pi Z}$ and $-B_{\Pi Z}$ (left and right circular polarizations, respectively) to a probe laser in the Z axis. Here $\varepsilon_{\alpha\beta\gamma}$ is the Levi-Civita symbol and $G_{\alpha\beta}(f)$ the appropriate[10] molecular property tensor elements in semiclassical approximation. The tensor $G_{\alpha\beta}(f)$ is supported only by chiral ensembles. The property $m_{0\gamma}$ is an appropriate component of the permanent magnetic dipole moment, and kT is the thermal energy per molecule. Under the conditions discussed on page 728 of Ref. 10:

$$n^{1\uparrow} - n^{1\downarrow} \sim 10^{-7} \tag{6}$$

for a $\mathbf{B}_\Pi$ field of 1.0 T delivered by a pump laser pulse. To generate optically a $\mathbf{B}_\Pi$ of 1.0 T requires a pulse of intensity about 10^{14} W m^{-2} (10^{10} W cm^{-2}). Such an intensity is a practical possibility with a picosecond pulse from an instrument such as a mode-locked dye laser,[35] which is circularly polarized and if necessary, focused. To detect the change $n^{1\uparrow} - n^{1\downarrow}$ experimentally requires the most sensitive type of Rayleigh refractometer,[10] in which a right circularly polarized pump pulse is delivered in one arm of the refractometer and a left circularly polarized pump pulse is delivered simultaneously in the other arm. The sample is chosen optimally as described by Barron and Vbrancich[10] in the context of a conventional magnetostatic **B** from a superconducting magnet.

Woźniak[11-14] and Wagnière[15] have made some interesting developments of the theory of magneto-spatial dispersion. Woźniak has considered the effect of additional electric and optical fields, and Wagnière the possibility of detecting $\hat{p}$ violation with forward–backward effects in atoms. It is straightforward to adapt these theories for use with an optical magnet by replacing the conventional **B** by $\mathbf{B}_\Pi$, in precisely the same manner as considered already. Additionally, it is possible to use a combination of **B** and $\mathbf{B}_\Pi$. These considerations point toward a range of new optically induced forward–backward anisotropy, whose variations with the probe frequency are novel spectroscopic signatures.

The disucssion has been restricted thus far to BFBA to first order in $\mathbf{B}_\Pi$, and in this context, the various phenomena of spin chiral dichroism[20-25] suggested by the present author are effects that are mediated by the antisymmetric conjugate product $\mathbf{E} \times \mathbf{E}^*$, which from Eq. (1) is seen to be proportional to the magnitude of $\mathbf{B}_\Pi$ squared, and to the axial unit vector **k**. These terms therefore also change sign with circular polarity of the laser and mediate forward–backward birefringence. In general, spin chiral anisotropy is present in addition to BFBA, the former being proportional to I_0 and the latter to $I_0^{1/2}$. The molecular property tensors involved in spin chiral anisotropy and BFBA are clearly different properties. This can best be seen from the fact that $\mathbf{B}_\Pi$ forms an interaction energy with a magnetic dipole moment, but $\mathbf{E} \times \mathbf{E}^*$ must form an interaction energy with the antisymmetric part of the electric polarizability.[20-25]

Note that there is no forward–backward birefringence proportional to an ordinary magnetostatic flux density squared because such a quantity is positive to the $\hat{T}$ operator. Forward–backward birefringence needs an influence that is $\hat{T}$-negative, such as $\mathbf{B}_\Pi$, $\mathbf{E} \times \mathbf{E}^*$, or the ordinary **B**. This is

a direct result of Wigner's theorem on motion reversal in the complet experiment, as discussed by Barron[10,34] and the present author.[35]

Acknowledgments

The Cornell Theory Center and Penn State University are thanked for research suppor and the Leverhulme Trust is thanked for a Fellowship.

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THE PHOTON'S MAGNETIC FIELD $\mathbf{B}_\Pi$: THE MAGNETIC NATURE OF ANTISYMMETRIC LIGHT SCATTERING

I. INTRODUCTION

The classical intensity of electromagnetic radiation is a tensor (I_{ij}) proportional to the tensor product $E_iE_j^*$ (Refs. 1–3). Here E_i is a component of the electric field strength and E_j^* denotes its complex conjugate.[4] In free space, the scalar part of the intensity is

$$I_0 = \varepsilon_0 c E_0^2 \tag{1}$$

where ε_0 is the free space permittivity, c the speed of light and E_0 the scalar amplitude of the electric field strength of the electromagnetic plane wave.[5,6] The vector part of $E_iE_j^*$ is conveniently expressed as the conjugate vector cross product:

$$\mathbf{\Pi}^{(A)} = \mathbf{E} \times \mathbf{E}^* = 2E_0^2 i\mathbf{k} \tag{2}$$

which is purely imaginary as a consequence of the fact that $E_iE_j^*$ is a Hermitian tensor of rank two.[7,8] It has recently been shown[9–12] that the conjugate product $\mathbf{\Pi}^{(A)}$ is directly proportional to a novel magnetostatic flux density vector $\mathbf{B}_\Pi$ of the classical electromagnetic plane wave:

$$\mathbf{B}_\Pi = \frac{\mathbf{\Pi}^{(A)}}{2E_0ci} = B_0\mathbf{k} = \frac{E_0}{c}\mathbf{k} = \left(\frac{I_0}{\varepsilon_0 c^3}\right)^{1/2}\mathbf{k} = \left(\frac{|\mathbf{N}|}{2\varepsilon_0 c^3}\right)^{1/2}\mathbf{k} \tag{3}$$

Here $\mathbf{k}$ is a unit axial vector in the propagation axis of the plane wave, B_0 is the plane wave's scalar magnetic flux density amplitude, and $\mathbf{N}$ is the Poynting vector:

$$\mathbf{N} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B}^* \tag{4}$$

Remarkably, the vector $\mathbf{B}_\pi$, a flux of magnetic density (in tesla) independent of the phase of the plane wave, has been overlooked in the long and illustrious history of the classical theory of fields,[8] whereas its close relative $\mathbf{N}$, a flux of energy density, has been well known for many years. Furthermore, the interpretation of $\mathbf{B}_\Pi$ in the quantum theory of fields leads straightforwardly[10] to the conclusion that the photon generates on the most fundamental level a magnetic flux density operator

$$\hat{B}_\Pi = B_0\frac{\hat{J}}{\hbar} \tag{5}$$

directly proportional to its angular momentum $\hat{J}$, a well-known boson operator.[13] Here $\hbar$ is the unit of angular momentum in quantum mechanics, the reduced Planck constant.

Quite generally, therefore, the ubiquitous antisymmetric part of the electromagnetic intensity (denoted by I_{ij}^-) can be rewritten in terms of the novel magnetic vector $\mathbf{B}_\Pi$, leading immediately to novel insights about all processes in physical optics that depend on I_{ij}^-.

In this paper we illustrate this conclusion with reference to the antisymmetric part of Rayleigh scattering from molecular liquids, a process first considered by Placzek[14] in 1934. Section II defines the antisymmetric part of the scattered intensity in terms of the scattered $\mathbf{B}_\Pi$ vector, adapting the arguments of Knast and Kielich.[7] Section III continues the development in terms of Rayleigh refringent scattering theory, used to relate the incoming and scattered magnetic vectors $\mathbf{B}_\Pi$. Section IV is a discussion of these results, leading to the conclusions that antisymmetric light scattering in general can be reinterpreted fundamentally as a purely magnetic process, whereby the incoming $\mathbf{B}_\Pi$ (or $\hat{B}_\Pi$ of quantum field theory) interacts with the molecular ensemble forming the scattering volume, and is scattered as the vector $\mathbf{B}_{\Pi S}$. The two magnetic vectors $\mathbf{B}_\Pi$ and $\mathbf{B}_{\Pi S}$ are related through the molecular property tensors of the scattering volume.

II. DEFINITION OF THE SCATTERED $\mathbf{B}_{\Pi S}$ VECTOR

It is convenient to define the scattered $\mathbf{B}_{\Pi S}$ vector in terms of the development of Knast and Kielich,[7] who have also tabulated extensively the magnetic point group symmetries of the antisymmetric part of the molecular and atomic polarizability. In so doing we define the intensity tensor of the incoming light as

$$I_{ij} = \varepsilon_0 c E_i E_j^* \tag{6}$$

from which it follows that the vector part of the intensity, the antisymmetric component I_{ij}^- of Knast and Kielich[7] can be expressed simply as the purely imaginary axial vector:

$$\begin{aligned} \mathbf{I}^- &\equiv \tfrac{1}{2}\varepsilon_{ijk}(I_{ij} - I_{ji}) = \mathrm{i} I_0 k \\ &= \varepsilon_0 c^2 E_0 \mathrm{i}\mathbf{B}_\Pi \end{aligned} \tag{7}$$

where $\mathbf{k}$ is a unit axial vector in the propagation direction. From these definitions we deduce immediately that the incoming $\mathbf{B}_\Pi$ vector is proportional to the square root of the incoming I_0:

$$\mathbf{B}_\Pi = \left(\frac{I_0}{\varepsilon_0 c^2 E_0}\right)\mathbf{k} = \left(\frac{I_0}{\varepsilon_0 c^3}\right)^{1/2}\mathbf{k} \tag{8}$$

The antisymmetric part of the scattered light intensity tensor is defined by Knast and Kielich to be:

$$I_{ijs}^-(t) = \left(\frac{\omega_0}{c}\right)^4 \langle M_i(\mathbf{r}, t_0) M_j^*(\mathbf{r}, t)\rangle_{t_0}^- \tag{9}$$

where

$$M_i(\mathbf{r}, t_0) = \mu_i^{(\mathrm{p})}(t_0)\exp[\mathrm{i}(\Delta\boldsymbol{\kappa}\cdot\mathbf{r}(t_0))] \tag{10}$$

where $\mu_i^{(\mathrm{p})}$ is the ith component of the electric dipole moment induced in a molecule p by the light's electric field strength vector, as usual in the theory of scattering.[15,16] The quantity $\Delta\boldsymbol{\kappa}$ is a difference in wave vectors of incident and scattered light, as usual, and the summation extends over all the N molecules of the volume. The angular brackets in Eq. (9) denote a time-correlation function[17] of the fluctuating quantity M_i. It is well known from the theory of nonequilibrium statistical mechanics[17] that the normalized time-correlation function starts at unity and evolves to zero with time t. Its Fourier transform is a spectral function of angular frequency ω. In Rayleigh scattering[15-17] the spectrum is the experimental Rayleigh bandshape, which is Fourier transformed to give the time-correlation function. The scattered antisymmetric intensity $I_{ijk}^-(k)$ is therefore frequency dependent, and forms the antisymmetric part of the Rayleigh scattering spectrum. However, $I_{ijs}^-(t)$ is directly proportional to the scattered $\mathbf{B}_{\Pi S}$ vector, (or in quantum-field theory[10] the scattered flux density operator $\hat{B}_{\Pi S}$) and we reach the significant conclusion that the spectrum of antisymmetric Rayleigh scattering is a graph of the scattered $\beta_{\Pi S}(\omega)$ plotted against the change in frequency $(\omega - \omega_0)$, where ω_0 is the incoming laser frequency. Antisymmetric Rayleigh scattering is therefore a process that can be described entirely in terms of the vector $\mathbf{B}_\Pi$.

This conclusion can be underlined by expressing the fundamental equation (9) in terms of a mean magnetic dipole moment $\langle \mathbf{m}_S(t)\rangle$, formed at time t from the antisymmetric part of the time correlation function $\langle M_i(\mathbf{r}, t_0) M_j^*(\mathbf{r}, t)\rangle_{t_0}$. This development is based on the relation

$$\langle \mathbf{m}_S(t)\rangle = \xi\langle \mathbf{M}(\mathbf{r}, t_0) \times \mathbf{M}^*(\mathbf{r}, t)\rangle_{t_0} \tag{11}$$

where ξ is a proportionality coefficient; i.e., the vector cross product of two nonidentical electric dipole moment vectors is proportional to a magnetic dipole moment. This conclusion can be illustrated by the following simple model.

Express the electric dipole moments $\boldsymbol{\mu}$ and $\boldsymbol{\mu}^*$ as products of electronic charge e and position vectors $\mathbf{r}$ and $\mathbf{r}^*$. Then the cross product can be expressed as an area

$$\boldsymbol{\mu} \times \boldsymbol{\mu}^* = e^2\mathbf{r} \times \mathbf{r}^* = e^2 A\mathbf{k} \tag{12}$$

Considering the simple model of the motion of charge with instantaneous linear velocity v around a circle of radius r, the magnetic dipole moment is known[18] to be proportional to the product IA, where A is the area of the circle, and I is the quantity $ev/(2\pi r)$. It follows that, in general, a magnetic dipole moment is proportional to area, and therefore to the cross product of $\boldsymbol{\mu}$ and $\boldsymbol{\mu}^*$. The same conclusion is derived for the cross product of transition electric dipole moments by Atkins and Miller.[19] It is also a consequence of the fact that antisymmetric electric polarizability is proportional to the cross product of transition electric dipole moments, and therefore has the same symmetry[20] as a magnetic dipole moment.

From these considerations, we reach the equation

$$\mathbf{m}_S(t) = \frac{1}{\xi}\left(\frac{c}{\omega_0}\right)^4 \varepsilon_0 c^2 E_{0s} \mathbf{B}_{\Pi S}(t) \qquad (13)$$

showing that the magnetic dipole moment $\mathbf{m}_S(t)$ is proportional to the scattered $\mathbf{B}_{\Pi S}(t)$ vector at time t. Fourier transformation leads immediately to the conclusion that the magnetic dipole moment at frequency ω is proportional to the scattered $\mathbf{B}_{\Pi S}(\omega)$ vector at the frequency ω of the antisymmetric Rayleigh scattering spectrum.

III. REFRINGENT SCATTERING APPROACH

In this section we adapt straightforwardly Rayleigh refringent scattering theory to provide an expression linking the incoming $\mathbf{B}_\Pi$ and the scattering $\mathbf{B}_{\Pi S}$ in terms of a parameter Ξ_Z, which is defined in terms of the molecular property tensors of the scattering volume. The scattered light intensity tensor is defined in semiclassical Rayleigh refringent scattering theory by[20]

$$\begin{aligned} I_{\alpha\beta}^{(s)} &= \varepsilon_0 c E_\alpha^{(S)} E_\beta^{*(S)} \\ &= \varepsilon_0 c \left(\frac{\omega^2 \mu_0}{4\pi R}\right)^2 a_{\alpha\gamma} a_{\beta\delta}^* E_\gamma^{(0)} E_\delta^{*(0)} \end{aligned} \qquad (14)$$

where

$$E_\alpha^{(s)} = \frac{\omega^2 \mu_0}{4\pi R} \exp\left[i\omega\left(\frac{R}{c} - t\right)\right] a_{\alpha\beta} E_\beta^{(0)} \qquad (15)$$

s the scattered electric field detected in the wave zone at a point d at a listance R from the molecular origin. The origin of scattered light is considered to be the characteristic radiation field generated by the oscillating electric and magnetic multipole moments induced in a molecule by the electromagnetic fields of the incident light wave. Here $a_{\alpha\beta}$ is the scattering ensor, a molecular property of the scattering volume for particular ncident and scattered directions given by unit vectors $\mathbf{n}^{(0)}$ and $\mathbf{n}^{(S)}$. In Eq. 14) ω is the angular frequency of the incoming wave, whose electric field trength vector is denoted $\mathbf{E}^{(0)}$, so that the scattered intensity tensor $I_{\alpha\beta}^{(S)}$ is expressed in terms of the incident intensity tensor $I_{\gamma\delta}^{(0)}$ by

$$I_{\alpha\beta}^{(S)} = \left(\frac{\omega^2 \mu_0}{4\pi R}\right)^2 a_{\alpha\gamma} a_{\beta\delta}^* I_{\gamma\delta}^{(0)} \qquad (16)$$

It is immediately clear, therefore, that the incident $\mathbf{B}_\Pi$ in antisymmetric scattering can be expressed in terms of the scattered $\mathbf{B}_{\Pi S}$ in a similar way. Thus, we arrive at the conclusion that antisymmetric light scattering, in general, is a process whereby the incoming $\mathbf{B}_\Pi$ is transformed into a scattered $\mathbf{B}_{\Pi S}$; i.e., antisymmetric Rayleigh scattering is a purely magneto-optic process. This argument is developed by consideration of the antisymmetric (vector) part of the scattered intensity tensor

$$\begin{aligned} I_\varepsilon^{(AS)} &= \varepsilon_0 c \Pi_\varepsilon^{(AS)} \\ \Pi_\varepsilon^{(AS)} &= \frac{1}{2}\left(\frac{\omega^2 \mu_0}{4\pi R}\right)^2 \varepsilon_{\alpha\beta\varepsilon}\left(E_\alpha^{(S)} E_\beta^{*(S)} - E_\beta^{(S)} E_\alpha^{*(S)}\right) \end{aligned} \qquad (17)$$

For the Z component

$$\Pi_Z^{(AS)} = 2E_0^{(0)2} i \Xi_Z \qquad (18)$$

where

$$\begin{aligned} \Xi_Z = \frac{\omega^4 \mu_0^2}{32\pi^2 R^2 i} &[(a_{XX} a_{YX}^* - a_{YX} a_{XX}^*) + (a_{XY} a_{YY}^* - a_{YY} a_{XY}^*) \\ &+ i(a_{XX} a_{YY}^* - a_{YX} a_{XY}^*) - i(a_{XY} a_{YX}^* - a_{YY} a_{XX}^*)] \end{aligned} \qquad (19)$$

is in general a complex quantity. Using the result

$$\Pi_Z^{(A)} = 2E_0^{(0)2} i \qquad (20)$$

it follows that

$$\Pi_Z^{(AS)} = \Pi_Z^{(A)} \Xi_Z \qquad (21)$$

and that

$$B_{\Pi S Z} = \Xi_Z B_{\Pi Z} \qquad (22)$$

For forward scattering, there is no component of $\mathbf{B}_{\Pi S}$ other than $B_{\Pi S Z}$; but there are components of $\mathbf{B}_{\Pi S}$ in X, Y, and Z, depending on the scattering angle. These are all generated from the incoming $B_{\Pi Z}$ by tensor multiplication with $\Xi_{\alpha\beta}$ in its second-rank tensor form.

Without loss of generality we concentrate on forward scattering in the rest of this section, so that[20]

$$\mathrm{Re}\left(a^f_{\alpha\beta}\right) = \alpha'_{\alpha\beta}(f) + \zeta'_{\alpha\beta\gamma}(f)n'_\gamma + \alpha''_{\alpha\beta}(g) + \zeta''_{\alpha\beta\gamma}(g)n \tag{23}$$

$$\mathrm{Im}\left(a^f_{\alpha\beta}\right) = -\alpha''_{\alpha\beta}(f) - \zeta''_{\alpha\beta\gamma}(f)n'_\gamma + \alpha'_{\alpha\beta}(g) + \zeta'_{\alpha\beta\gamma}(g)n_\gamma + \cdots \tag{24}$$

where $\alpha'_{\alpha\beta}$ and $\alpha''_{\alpha\beta}$ are respectively the real and imaginary parts of the molecular polarizability tensor[20] and $\zeta'_{\alpha\beta\gamma}$ and $\zeta''_{\alpha\beta\gamma}$ are those of the zeta tensor defined by Barron.[20] In the forward direction, the process becomes one of antisymmetric spectral absorption, in which the incoming and outgoing $B_{\Pi Z}$ and $B_{\Pi SZ}$ define the absorption coefficient:

$$A^{(A)} \propto \frac{I_Z^{(s)}}{I_Z^{(0)}} = \frac{B_{\Pi SZ}}{B_{\Pi Z}} = \Xi_Z \tag{25}$$

After ensemble averaging[20]

$$\langle \Xi_Z \rangle = \frac{\omega^4 \mu_0^2}{192\pi^2 R^2}\left(a_{\alpha\alpha}a^*_{\beta\beta} - \alpha_{\alpha\beta}a^*_{\alpha\beta}\right) \tag{26}$$

and

$$B_{\Pi SZ} = \langle \Xi_Z \rangle B_{\Pi Z} \tag{27}$$

describes the process in terms of tensor invariants[20] of the molecular ensemble constituting the scattering volume.

IV. DISCUSSION

The historical development and experimental evidence for antisymmetric Rayleigh scattering has been reviewed in detail by Barron.[20] In this work and that of Knast and Kielich,[7] the magnetic nature of the phenomenon is implied indirectly, for example, through the fact that the process is described with the antisymmetric part of molecular property tensors such as the electric polarizability. Using the relation

$$\alpha''_k = \tfrac{1}{2}\varepsilon_{ijk}\left(\alpha''_{ij} - \alpha''_{ji}\right) \tag{28}$$

this tensor can be described as an axial vector, α''_k (Refs. 21–25), which has the same symmetry as a magnetic dipole moment, and whose irreducible representations in various molecular point groups are the same as those of the magnetic dipole moment or angular momentum. Knast and Kielich,[7] in their Eq. (36), also point out that the vector α''_k is negative to the motion reversal operator $\hat{T}$, and it follows[21–25] that it can form an interaction energy only with another $\hat{T}$-negative property, the antisymmetric conjugate product $\mathbf{E} \times \mathbf{E}^*$ of Eq. (2) of this paper. This argument shows that $\mathbf{E} \times \mathbf{E}^*$ must have the symmetry of a magnetic field, since α''_k has the symmetry of a magnetic dipole moment. In fact, as discussed in the introduction, $\mathbf{E} \times \mathbf{E}^*$ is proportional to the novel magnetic field $\mathbf{B}_\Pi$, the classical equivalent of the operator $\hat{B}_\pi$ for each individual photon. Equation (13) of this paper now shows that antisymmetric scattering can be thought of as the induction by the scattered magnetic field $\mathbf{B}_{\Pi S}$ of the magnetic dipole moment $\langle \mathbf{m}_S(t) \rangle$ (which is the value at t of a time-correlation function), and this result can also be generalized in quantum field theory, or the theory of magneto-photonics. The Fourier transform of $\langle m_S(t) \rangle$ is a point on the spectrum of scattered light at the frequency ω. The magnetic dipole moment $\langle \mathbf{m}_S \rangle$ has the same irreducible representations in the appropriate point groups[7] as the antisymmetric polarizability considered by Knast and Kielich,[7] and $\mathbf{B}_{\Pi S}(t)$ has the same symmetry as the antisymmetric part of the scattered intensity, denoted in tensor notation by I^-_{ij} by these authors.[7] It follows that the same conclusions arrived at by Knast and Kielich[7] for the properties of the antisymmetric polarizability hold for the novel magnetic dipole moment $\langle \mathbf{m}_S(t) \rangle$. For example, $\langle \mathbf{m}_S(t) \rangle$ is nonzero only in the presence of a $\hat{T}$-negative influence, which in Eq. (13) is the magnetic field $\mathbf{B}_{\Pi S}(t)$.

Another conclusion that becomes immediately obvious in our magnetic interpretation of forward antisymmetric Rayleigh light scattering is that it involves circular polarization. The magnetic fields $\mathbf{B}_\Pi$ and $\mathbf{B}_{\Pi S}$ vanish if there is no degree of circular polarization, respectively in the incoming and scattered radiation. These findings are reinforced by the arguments, summarized in Section 3.5.3 of Ref. 20, for Rayleigh scattering in the near forward direction from refringent scattering theory. In this case, the degree of circular polarization is directly proportional to the pseudoscalar magnitude[12] of the scattered $\mathbf{B}_{\Pi S}$, which is the third Stokes parameter of the scattered radiation. Thus, in purely antisymmetric, near forward Rayleigh scattering, if the incident beam is completely circularly polarized, so is the scattered beam. This is summarized in our terms by Eq. (27), in which $B_{\Pi Z}$ and $B_{\Pi SZ}$ are both well defined, and in which the coefficient $\langle \Xi_Z \rangle$ is a finite ensemble average. Clearly, if $B_{\Pi Z}$ is zero (no degree of circular polarization in the incoming beam), then $B_{\Pi SZ}$ is also zero, because the molecular ensemble average $\langle \Xi_Z \rangle$ is nonzero in general. Therefore, there is no near forward scattering.

V. CONCLUSION

The phenomenon of antisymmetric light scattering has been interpreted in terms of the novel incident and scattered magnetostatic flux density vectors $\mathbf{B}_{\Pi}$ and $\mathbf{B}_{\Pi S}$, respectively. This shows that antisymmetric scattering is a purely magneto-optic phenomenon, giving information on the nature of the scattered $\mathbf{B}_{\Pi S}$ vector. In magneto-photonics, the vector $\mathbf{B}_{\Pi}$ is replaced by the operator $\hat{B}_{\Pi}$, and the appropriate quantum theory must be employed.

Acknowledgments

The Leverhulme Trust is thanked for a Fellowship and additional research support is acknowledged from the Cornell Theory Center and the Materials Research Laboratory of Penn State University.

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MANIFESTLY COVARIANT THEORY OF THE ELECTROMAGNETIC FIELD IN FREE SPACETIME, PART 1: ELECTRIC AND MAGNETIC FIELDS AND MAXWELL'S EQUATIONS

I. INTRODUCTION

It has recently been shown[1–5] that there exist longitudinal solutions of Maxwell's equations in free spacetime which are independent of the phase ϕ of the traveling plane wave. These longitudinal electric and magnetic fields, denoted $\mathbf{E}^{(3)}$ and $\mathbf{B}^{(3)}$, respectively, are consistent with the conclusion of quantum electrodynamics that there exist four photon polarizations in free spacetime, one timelike ((0)), two transverse spacelike ((1) and (2)), and one longitudinal spacelike ((3)).[6,7] However, the existence of four photon polarizations has to date been regarded[7] as being in conflict with the deduction that the photon can have only two helicities, $+1$ and -1. This in turn has led to the arbitrary assertion that only the two transverse spacelike polarizations (1) and (2) can be "physically meaningful" in free spacetime. The timelike ((0)) and longitudinal spacelike ((3)) are conventionally discarded as physically meaningless. This implies that the theory of the electromagnetic field in free spacetime loses manifest covariance.[7] This fundamental difficulty is well described by Ryder,[7] from whose Chapter 4 we quote the following: "the electromagnetic field, like any massless field, possesses only two independent components, but is covariantly described by a (potential) four vector A_μ. In choosing two of these components as the physical ones, and thence quantizing them, we lose manifest covariance. Alternatively, if we wish to keep covariance, we have two redundant components."

Clearly, if the theory of the electromagnetic field in free spacetime is to be made manifestly covariant and therefore rigorously consistent with special relativity, then all four photon polarizations must be physically meaningful. This implies that electric and magnetic fields in vacuo must be four vectors, E_μ and B_μ, respectively, in spacetime. The difficulty with this notion to date appears to have been the preconception that any longitudinal solution of Maxwell's equations in free spacetime must necessarily be phase dependent, so that the longitudinal spacelike solution cannot be solenoidal. This means that Gauss's theorem in differential form is violated by such a solution.[8-15] However, with the recent discovery[1-5] that the longitudinal solutions to Maxwell's equations in vacuo are not phase dependent, the conflict with Gauss's theorem disappears, and one of the most intractable difficulties of electromagnetic field theory is removed. In so doing, the very basis of electrodynamics is changed profoundly, because at present the subject is based on the existence in vacuo of a potential four vector A_μ, whose four-curl gives the antisymmetric electromagnetic field tensor $F_{\mu\nu} = -F_{\nu\mu}$ in spacetime. The components of $F_{\mu\nu}$ contain no explicit reference to the timelike component of the four vectors E_μ and B_μ, and the longitudinal components that appear in $F_{\mu\nu}$ are evidently discarded as unphysical. To maintain manifest covariance the timelike and longitudinal components must be retained, and must have physical meaning. In other words, the electric and magnetic parts of the electromagnetic plane wave in free space are treated conventionally as three vectors in Euclidean space, and not as manifestly covariant four vectors in pseudo-Euclidean spacetime. This reveals an internal inconsistency in electrodynamics in vacuo, in that the d'Alembert equation

$$\Box A_\mu = 0 \tag{1}$$

allows four photon polarizations, but the Maxwell equations

$$\frac{\partial F_{\mu\nu}}{\partial x_\nu} = 0 \qquad \frac{\partial \tilde{F}_{\mu\nu}}{\partial x_\nu} = 0 \tag{2}$$

$$(x \equiv (X, Y, Z, ict))$$

link only the spacelike components of E_μ and B_μ. They make no explicit reference to their timelike components $E^{(0)}$ and $B^{(0)}$. (In Eq. (2), the Maxwell equations are stated in terms of the four divergence of $F_{\mu\nu}$ and of its dual, $\tilde{F}_{\mu\nu}$ (Ref. 7).) A consistent, manifestly covariant, and rigorous theory of electrodynamics in vacuo must link E_μ and B_μ to the tensor $F_{\mu\nu}$, which is the four-curl of A_μ.

In Section II of this paper, a brief review is given of the phase-independent longitudinal components $E^{(3)}$ and $B^{(3)}$ of the electromagnetic plane wave in vacuo. These are identified with the conclusions of quantum field theory[7] that physical photon states in a manifestly covariant gauge such as the Lorentz gauge are described as admixtures of operators of the field, namely the creation and annihilation operators.

Section III links the four vectors E_μ and B_μ to the four tensor $F_{\mu\nu}$, and shows that E_μ and B_μ take the form of a Pauli Lubanski vector and pseudovector, respectively, in spacetime. These are well defined[7] within the inhomogeneous Lorentz group (or Poincaré group). This leads in turn to the conclusion that the two photon helicities, +1 and −1, can be reconciled rigorously with four physically meaningful photon polarizations, because the helicities can be described equally well in terms either of (1) and (2) polarizations or of (0) and (3) polarizations. This is consistent with our earlier[1,2] conclusion that one photon generates the longitudinal magnetic field component:

$$\mathbf{B}^{(3)} = \langle \psi | \hat{B}^{(3)} | \psi \rangle = \frac{B^{(0)}}{\hbar} \langle \psi | \hat{J} | \psi \rangle \tag{3}$$

where $|\psi\rangle$ is an eigenstate of the photon and where the eigenvalues of the operator $\hat{J}$ are $M_J \hbar$; $M_J = +1$ and -1, the photon helicities. The result (3) is generalized in Section III through the definition of E_μ and B_μ as Pauli-Lubanski types in spacetime.

Section IV deals with some consequences in vacuum electrodynamics of the existence of manifestly covariant E_μ and B_μ, with four physically meaningful components. The Maxwell equations, in particular the differential form of Gauss's theorem, are developed covariantly in terms of E_μ and B_μ. Specifically, Gauss's theorem in differential form becomes

$$\frac{\partial E_\mu}{\partial x_\mu} = 0 \quad \text{or} \quad \nabla \cdot \mathbf{E} + \frac{1}{c}\frac{\partial E^{(0)}}{\partial t} = 0 \tag{4a}$$

and

$$\frac{\partial B_\mu}{\partial x_\mu} = 0 \quad \text{or} \quad \nabla \cdot \mathbf{B} + \frac{1}{c}\frac{\partial B^{(0)}}{\partial t} = 0 \tag{4b}$$

The electromagnetic energy and energy flux densities in vacuo are expressed in terms of products of E_μ and B_μ, showing that the (3) and (0) polarizations do not make explicit contributions to either on a time-averaged basis. The four Stokes parameters, however, are profoundly affected by the manifest covariance of E_μ and B_μ, in that it is no longer sufficient to describe S_0, S_1, S_2, and S_3 in terms of Pauli matrices.[17] It is shown that the covariant description of the Stokes parameters in vacuo can be obtained through the use of Dirac matrices.[18] This description maintains the fundamental relation

$$S_0^2 = S_1^2 + S_2^2 + S_3^2 \tag{5}$$

on the Poincaré sphere, while at the same time showing that S_1 and S_2 become different in a description based on E_μ and B_μ rather than on the usual transverse spacelike **E** and **B**. A new term appears in both S_1 and S_2 due to the existence of physically meaningful (0) and (3) states of the electromagnetic field. The parameters S_0 and S_3, on the other hand, are unaffected.

Section V is a discussion of the available experimental evidence for $\mathbf{B}^{(3)}$, and suggests several experimental tests of its physical existence when the electromagnetic field interacts with matter.

II. THE LONGITUDINAL SOLUTIONS OF MAXWELL'S EQUATIONS IN FREE SPACETIME: (0) AND (3) POLARIZATIONS

Longitudinal solutions of Maxwell's equations in vacuo appear not to have been considered as physically meaningful in the great majority of standard texts. Jackson[8] simply states that the differential form of Gauss's theorem demands that phase-dependent solutions are transverse. The possibility of phase-independent solutions appears not to be considered. It is frequently considered[8–15] that a plane, monochromatic, electromagnetic wave traveling in Z (the propagation axis) in vacuo is simply the sum of two coherent waves linearly polarized in the orthogonal axes X and Y. Atkins[11] and Landau and Lifshitz[12] similarly consider only transverse fields, and thus transverse polarizations, in a Cartesian or circular basis. Whitner,[9] however, mentions briefly and without further development that "Plane waves are an important example but they do constitute a special case; we must not conclude that all electromagnetic waves are transverse." Similarly, other authors[8–15] in classical and quantum electrodynamics in vacuo make little or no mention of longitudinal solutions.

Recently, however, Evans[1–4] and Farahi and Evans[5] have systematically considered the theory of phase-independent longitudinal electric and magnetic fields, which are solutions to the free spacetime Maxwell equations and thus obey the differential form of Gauss's theorem in free

spacetime. This work has developed rapidly from the observation[1] th spacelike components of the plane wave in vacuo are interrelated by

$$\mathbf{B}^{(3)} = \frac{\mathbf{E}^{(1)} \times \mathbf{E}^{(2)}}{\mathrm{i}cE_0} \qquad (6$$

where

$$\mathbf{E}^{(1)} \equiv E_0 \hat{\mathbf{e}}^{(1)} \mathrm{e}^{\mathrm{i}\phi} \qquad (7a$$

$$\mathbf{E}^{(2)} \equiv E_0 \hat{\mathbf{e}}^{(2)} \mathrm{e}^{-\mathrm{i}\phi} \qquad (7b$$

are the transverse electric field components. Here

$$\hat{\mathbf{e}}^{(1)} = \frac{\mathbf{i} - \mathrm{i}\mathbf{j}}{\sqrt{2}}$$

$$\hat{\mathbf{e}}^{(2)} = \frac{\mathbf{i} + \mathrm{i}\mathbf{j}}{\sqrt{2}}$$

are unit vectors in the circular basis,[19] where **i** and **j** are unit cartesia vectors in X and Y, orthogonal to the propagation axis Z. Here ϕ is tl phase of the traveling monochromatic plane wave, defined by

$$\phi = \omega t - \boldsymbol{\kappa} \cdot \mathbf{r}$$

where ω is its angular frequency at instant t and $\boldsymbol{\kappa}$ its wave vector position **r** in Euclidean space. We have from Eqs. (6) and (7)

$$\mathbf{B}^{(3)} = B_0 \hat{\mathbf{e}}^{(3)} \qquad (8$$

with

$$\hat{\mathbf{e}}^{(1)} \times \hat{\mathbf{e}}^{(2)} = \mathrm{i}\hat{\mathbf{e}}^{(3)} \qquad (9$$

and the well-known free spacetime relation[8–15]

$$E_0 = cB_0 \qquad (10$$

With $B_0 \equiv B^{(0)}$ we write Eq. (8) as

$$B_0 - |\mathbf{B}^{(3)}| = 0 \qquad (11$$

where $|\mathbf{B}^{(3)}|$ denotes the scalar magnitude of the vector $\mathbf{B}^{(3)}$.

From Eqs. (6), (7), and (11) it is clear that field polarizations (0) and (3) are not independent of polarizations (1) and (2). Furthermore, by considering the results of quantum field theory,[7] polarization (0) can be identified as being timelike in a manifestly covariant description, and (3) as longitudinal spacelike. The field $\mathbf{B}^{(3)}$ is consistent with Maxwell's equations in vacuo and therefore with Gauss's theorem. The key to this result is that the phase ϕ has been removed in the conjugate product (6). The electric counterpart of Eq. (11) is, in general:

$$E^{(0)} - |\mathbf{E}^{(3)}| = 0;\ E^{(0)} \propto E_0 \tag{12a}$$

$$\mathbf{E}^{(3)} = \mathbf{E}^{(0)}\hat{\mathbf{e}}^{(3)} \tag{12b}$$

Equations (12) also represent solutions of Maxwell's equations in vacuo.

Equations (11) and (12) are, furthermore, related[5] through conservation of electromagnetic energy by the Euclidean space equation:

$$\mathbf{E}^{(3)} \times \mathbf{B}^{(2)} = \mathbf{B}^{(3)} \times \mathbf{E}^{(2)} \tag{13}$$

showing that if $\mathbf{B}^{(3)}$ is real, as in Eq. (6), $\mathbf{E}^{(3)}$ is imaginary. Relations such as (6) and (11) to (13) show that there are only two independent states for **E** and **B** because from Eq. (6) either of states (1) and (2) can be expressed in terms of (3); and the latter can be expressed in terms of (0) through Eq. (11) for $\mathbf{B}^{(3)}$ and Eq. (12) for $\mathbf{E}^{(3)}$. Finally, states (0) for **E** and **B** are related by Eq. (10) and states (3) by Eq. (13). This result, that there are only two independent states out of the four possible, (0) to (3), is evidently the classical expression of the fact[7] that the massless gauge field possesses only two independent components, but is at the same time covariantly described by a four vector, A_μ made up of four physically meaningful polarization states (0), (1), (2), and (3).

Thus far, we have used a conventional, classical description in terms of spacelike vectors **E** and **B**, but have introduced the novel $\mathbf{E}^{(3)}$ and $\mathbf{B}^{(3)}$. By using the relativistic quantum description of the electromagnetic field[7] we now introduce the concept of electric and magnetic field four vectors E_μ and B_μ, respectively.

It is well known[7] that the quantization of Eq. (1) in the Lorentz gauge proceeds through a condition derived by Gupta and Bleuler in the early days[6] of quantum field theory:

$$\frac{\partial \hat{A}_\mu^{(+)}}{\partial x_\mu}|\boldsymbol{\psi}\rangle = 0 \tag{14}$$

where $\hat{A}_\mu^{(+)}$ is the operator equivalent of A_μ and acts on a photon eigenstate $|\boldsymbol{\psi}\rangle$. Equation (14) leads to the result[7] that physical photon states are admixtures of (0) and (3) photon polarizations in such a way that

$$(\hat{a}^{(0)} - \hat{a}^{(3)})|\psi\rangle = 0 \tag{15a}$$

$$\langle\psi|\hat{a}^{(0)+}\hat{a}^{(0)}|\psi\rangle = \langle\psi|\hat{a}^{(3)+}\hat{a}^{(3)}|\psi\rangle \tag{15b}$$

where $\hat{a}$ and $\hat{a}^+$ are annihilation and creation operators, respectively. Furthermore, the energy density of the quantized field is proportional[7] to the sum

$$\sum_{\lambda=0}^{3} \left(\hat{a}^{(\lambda)+}\hat{a}^{(\lambda)} - \hat{a}^{(0)+}\hat{a}^{(0)}\right) \tag{16}$$

and from Eq. (15b)[7] the contribution of the longitudinal ((3)) and timelike ((0)) states cancel. It will be shown in this section that the classical but manifestly covariant equivalents of Eqs. (15) and (16) are obtained from the four vectors $\mathbf{E}_\mu$ and $\mathbf{B}_\mu$.

We use the well-known relations[19,20] (in S.I. units),

$$\hat{E}^{(0)} = \left(\frac{2\hbar\omega}{\varepsilon_0 V}\right)^{1/2}\hat{a}^{(0)} \qquad \hat{E}^{(3)} = \left(\frac{2\hbar\omega}{\varepsilon_0 V}\right)^{1/2}\hat{a}^{(3)}$$
$$\hat{B}^{(0)} = \left(\frac{2\mu_0\hbar\omega}{V}\right)^{1/2}\hat{a}^{(0)} \qquad \hat{B}^{(3)} = \left(\frac{2\mu_0\hbar\omega}{V}\right)^{1/2}\hat{a}^{(3)} \tag{17}$$

to link the annihilation operators in states (0) and (3) to the equivalent field operators. Here ε_0 is the permittivity and μ_0 the permeability of the vacuum state, $\hbar$ is the reduced Planck constant, and V the quantization volume. From Eqs. (15a) and (17),

$$(\hat{E}^{(0)} - \hat{E}^{(3)})|\psi\rangle = 0 \qquad (\hat{B}^{(0)} - \hat{B}^{(3)})|\psi\rangle = 0 \tag{18}$$

and from Eq. (15b),

$$\langle\psi|\hat{E}^{(0)+}\hat{E}^{(0)}|\psi\rangle = \langle\psi|\hat{E}^{(3)+}\hat{E}^{(3)}|\psi\rangle$$
$$\langle\psi|\hat{B}^{(0)+}\hat{B}^{(0)}|\psi\rangle = \langle\psi|\hat{B}^{(3)+}\hat{B}^{(3)}|\psi\rangle \tag{19}$$

The classical equivalent of Eq. (18) is

$$E^{(0)} - |\mathbf{E}^{(3)}| = 0 \qquad B^{(0)} - |\mathbf{B}^{(3)}| = 0 \tag{20a}$$

and that of Eqs. (19) is

$$\begin{aligned} \mathbf{E}^{(0)2} &= \mathbf{E}^{(3)} \cdot \mathbf{E}^{(3)} \\ \mathbf{B}^{(0)2} &= \mathbf{B}^{(3)} \cdot \mathbf{B}^{(3)} \end{aligned} \tag{20b}$$

where $\mathbf{E}^{(3)} = \langle \psi | \hat{\mathbf{E}}^{(3)} | \psi \rangle$, etc. Equation (20a) is identical with Eq. (11), and Eq. (20b) is consistent with Eqs. (11) and (12a). However, Eq. (19) was derived from a quantized counterpart, Eq. (15a), which is manifestly covariant in that physical photon states are admixtures of states (0) and (3) of the quantum field.[7] It follows that classical field states in vacuo are also admixtures of the classical (0) and (3) polarizations as defined by Eqs. (19) and (20). From this we arrive at two fundamentally important conclusions:

1. The electric and magnetic components of the electromagnetic field in vacuo are manifestly covariant four vectors in spacetime, E_μ and B_μ, respectively, all of whose four components must be physically meaningful.
2. From Eq. (19) the fields $\mathbf{E}^{(3)} = E^{(0)}\hat{\mathbf{e}}^{(3)}$ and $\mathbf{B}^{(3)} = B^{(0)}\hat{\mathbf{e}}^{(3)}$ are independent of the phase of the traveling plane wave, which is consistent with Eq. (6) and the development thereof.

The four physical states of the classical, manifestly covariant, electromagnetic field are formed from the (0) and (3) admixtures $E^{(0)} - |\mathbf{E}^{(3)}|$ and $B^{(0)} - |\mathbf{B}^{(3)}|$ and from the well-known transverse ((1) and (2)) components.[8–15] Although Maxwell's phenomenological equations of the 1860s are conventionally accepted as being consistent with special relativity, the electric and magnetic fields that they relate in vacuo are purely spacelike. The field potentials in terms of which **E** and **B** are described in the conventional theory[8–15] are, on the other hand, taken to be components of the potential four vector in spacetime:

$$A_\mu \equiv (\mathbf{A}, +\mathrm{i}\phi) \tag{21}$$

where **A** is the spacelike (vector) potential and ϕ is the timelike (scalar) potential. The four-curl of A_μ conventionally produces the electromagnetic field tensor in spacetime (see Appendix A):

$$F_{\mu\nu} = -F_{\nu\mu} = \varepsilon_0 \begin{bmatrix} 0 & cB_Z & -cB_Y & -\mathrm{i}E_x \\ -cB_Z & 0 & cB_X & -\mathrm{i}E_Y \\ cB_Y & -cB_X & 0 & -\mathrm{i}E_Z \\ \mathrm{i}E_X & \mathrm{i}E_Y & \mathrm{i}E_Z & 0 \end{bmatrix} \tag{22}$$

The difficulty with $F_{\mu\nu}$ and with the conventional theory is that $F_{\mu\nu}$ contains no explicit reference to the timelike components of E_μ and B_μ. These are removed by the mathematical nature of the four-curl:

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} \tag{23}$$

A method must be found to relate E_μ and B_μ to the four potential A_μ, thus making the theory rigorously self-consistent and manifestly covariant.

It is reasonable to base this method in contemporary quantum field theory,[7] in which the electromagnetic field is an example of a massless gauge field (as opposed to a spinor field), described in general by the Poincaré group. The latter incorporates three Lorentz rotation generators (J_i), three Lorentz boost generators (K_i), and four generators of spacetime translation (P_μ). Details are well summarized in Ref. 7. The difference between the Poincaré and Lorentz groups is that the former incorporates the generator of spacetime translations, defined by

$$P_\mu \equiv i\frac{\partial}{\partial x_\mu} \tag{24}$$

and which is proportional through a factor $\hbar$ to the momentum energy four vector operator. The Pauli-Lubansky pseudovector in spacetime, W_μ, characterizes the Poincaré group by forming its second (spin) Casimir invariant $W_\mu W_\mu$. (The first (mass) Casimir invariant is formed from $P_\mu P_\mu$.) The Pauli Lubansky pseudovector is defined by

$$W_\mu = -\tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma}J_{\nu\rho}P_\sigma \tag{25}$$

where $\varepsilon_{\mu\nu\rho\sigma}$ is the totally antisymmetric spacetime tensor of rank four, and where the four tensor $J_{\nu\rho}$ is

$$J_{\mu\sigma}(\mu, \sigma = 0, \ldots, 3)\begin{bmatrix} J_{ij} = -J_{ji} = \varepsilon_{ijk} J_k \\ J_{i0} = -J_{0i} = -K_i \end{bmatrix}$$

$$i, j, k = 1, 2, 3 \tag{26}$$

III. THE LINK BETWEEN E_μ, B_μ, $F_{\mu\nu}$, AND A_μ

The photon helicity is defined in the lightlike condition applied to Eq. (25), in which condition W_μ becomes proportional to P_μ, so that the helicity is a number, $+1$ (Ref. 7). The opposite value, -1, is given by considerations[7] of parity inversion. We show in this section that E_μ and B_μ can be defined in terms of $F_{\mu\nu}$, and therefore of the four curl of A_μ, by an equation whose structure is the same as that of Eq. (25). Thus, E_μ and B_μ are identified as a Pauli Lubansky vector and pseudovector, respectively. This procedure succeeds in expressing electric and magnetic four vectors in terms of a single potential four vector, and covariantly describes the conventional[8–15] relations between **E**, **B**, **A**, and ϕ in vacuo.

The primary basis of the derivation is the observation that $J_{\mu\nu}$ has the same structure as $F_{\mu\nu}$, both being antisymmetric four tensors of the type (26). The $ij = -ji$ components of $F_{\mu\nu}$ are therefore identified as being proportional to rotation generators of the Poincaré group.[7] With this observation, it becomes obvious that the spacelike electric components in Eq. (22) are proportional to boost generators of the Poincaré group.[7] Pure boost Lorentz transformations[7] are those connecting two inertial frames moving at a relative speed v. A Lorentz rotation is a four vector rotation in spacetime. Therefore, the conventional assertion that $cB^{(3)}$ and $E^{(3)}$ in $F_{\mu\nu}$ are physically meaningless is tantamount to asserting that one out of three rotation generators and one out of three boost generators are physically meaningless. This is a reductio ad absurdum, and a vivid demonstration of the fact that the conventional assertion that $E^{(3)}$ and $B^{(3)}$ are unphysical is flawed fundamentally, i.e.,is geometrically unsound.

Secondly, Eq. (3), which we have derived elsewhere[2] using independent considerations in the quantum field, shows that $\hat{B}^{(3)}$ is directly proportional to the photon's quantized angular momentum boson operator $\hat{\mathbf{J}}$. Classically, the rotation generators $\mathbf{J}_X$, $\mathbf{J}_Y$, and $\mathbf{J}_Z$ of the Poincaré group are matrices of numbers which obey the commutation relations

$$[\mathbf{J}_X, \mathbf{J}_Y] = \mathrm{i}\mathbf{J}_Z \tag{27}$$

and cyclic permutations thereof. These are immediately recognizable to be the commutators of quantized angular momentum within the factor $\hbar$. This suggests that $F_{\mu\nu}$ is proportional to $J_{\mu\nu}$ through a four scalar invariant of spacetime. For example, components 12 and 21 of $F_{\mu\nu}$ are respectively $cB^{(3)}$ and $-cB^{(3)}$, and all other off-diagonal components of $F_{\mu\nu}$ have the same dimensions as the 12 and 21 components. The 12 and 21 components of $J_{\mu\nu}$ are the angular momenta $J^{(3)}$ and $-J^{(3)}$ within a factor $\hbar$. The 21 and 12 component proportionality is therefore embodied in Eq. (3).

Thirdly, the contemporary quantum field description of photon helicity in terms of W_μ and P_μ clearly involves the concept of spacetime translation within the Poincaré group, introduced by Wigner[21] in 1939, and whose generator, as we have seen, is P_μ. This is missing from the Lorentz group.[7] The concept of spacetime translation is also missing from the Maxwell equations, which do not deal explicitly in the timelike field polarization (0). Spacetime translation is implied in d'Alembert's equation (1),[7] but if and only if all four field polarizations are taken to be physically meaningful. To see this, recall (1) that the four-curl (23) removes the (0) polarization, and (2) that the Maxwell equations (2) are equations[7] in $F_{\mu\nu}$ and its dual $\tilde{F}_{\mu\nu}$. From the proportionality of $F_{\mu\nu}$ to $J_{\mu\nu}$ it becomes clear, however, that the components of $F_{\mu\nu}$ must be either rotation or boost generators of the Poincaré group, and there is no reference within $F_{\mu\nu}$ to spacetime translation. Photon helicity, on the other hand, is described in terms of the proportionality and orthogonality in spacetime of W_μ to P_μ (Ref. 7) in the lightlike condition. Therefore, the description of electric and magnetic components in vacuo in terms of $F_{\mu\nu}$ is inconsistent with the contemporary description of helicity. This inconsistency can be remedied if and only if electric and magnetic components of the electromagnetic field in vacuo are manifestly covariant four vectors E_μ and B_μ.

Fourthly, defining a photon state $|k\rangle$ in the lightlike condition, the photon helicity (λ) in contemporary thought[7] is given by the condition

$$(W_\mu - \lambda P_\mu)|k\rangle = 0 \tag{28}$$

so that for the massless photon, λ is a number $(+1)$, which is the ratio of W_μ to P_μ and which has the dimensions of angular momentum,[7] provided that P_μ has the dimensions of linear momentum/energy by multiplication by $\hbar$. For lightlike particles with no mass, such as the photon[7],

$$k_\mu \equiv (0, 0, k, -\mathrm{i}k) \tag{29}$$

which can be regarded as a unit four vector

$$\delta_\mu \equiv (0, 0, 1, -\mathrm{i}) \tag{30}$$

describing a massless particle moving at the speed of light in the Z spacelike axis (the propagation axis of the electromagnetic wave). Equation (30) can be incorporated into Eq. (25) by dividing the left and right sides of Eq. (25) by $\hbar$, so that W_μ, $J_{\nu\rho}$, and P_σ become numbers. This is consistent with the definition of rotation and boost generators as matrices of numbers (Eqs. (2.65)–(2.67) of Ref. 7).

With these considerations, we are led to the following fundamental definitions of B_μ and E_μ in terms of $F_{\nu\rho}$ and δ_σ:

$$cB_\mu = -\frac{\mathrm{i}}{2\varepsilon_0}\varepsilon_{\mu\nu\rho\sigma}F_{\nu\rho}\delta_\sigma \tag{31a}$$

$$E_\mu = \frac{1}{2\varepsilon_0}\varepsilon_{\mu\nu\rho\sigma}\tilde{F}_{\nu\rho}\delta_\sigma \tag{31b}$$

In these equations, we recall that if $\varepsilon_{0123} = 1$, then its other nonzero elements are $+1$ and -1, according as to whether ε_{0123} can be generated by an even or odd number of subscript pair permutations. Thus, for example,

$$\begin{array}{lll} \varepsilon_{3120} = -1 & \varepsilon_{3210} = 1 & \varepsilon_{2310} = -1 \\ \varepsilon_{3120} = -1 & \varepsilon_{1320} = 1 & \varepsilon_{1230} = -1 \end{array} \tag{32}$$

and so on. All elements of $\varepsilon_{\mu\nu\rho\sigma}$ are zero in which two or more subscripts are equal. The elements of $F_{\nu\rho}$ are labeled explicitly as

$$F_{\nu\rho}(\nu, \rho = 0, 1, 2, 3) = \begin{bmatrix} 11 & 12 & 13 & 10 \\ 21 & 22 & 23 & 20 \\ 31 & 32 & 33 & 30 \\ 01 & 02 & 03 & 00 \end{bmatrix} \tag{33}$$

With these definitions it is verified by tensor algebra (Appendix B) that the real elements (labeled (1), (2), (3), and (0)) of the magnetic and electric field four vectors

$$\begin{aligned} E_\mu &\equiv (E^{(1)}, E^{(2)}, E^{(3)}, -\mathrm{i}E^{(0)}) \\ B_\mu &\equiv (B^{(1)}, B^{(2)}, B^{(3)}, -\mathrm{i}B^{(0)}) \end{aligned} \tag{34}$$

are given by Eqs. (31). (The dual $\tilde{F}_{\mu\nu}$ of $F_{\rho\sigma}$ is obtained by the well-known[7] dual transformation $\tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}$.)

Equation (31a) covariantly defines B_μ as a Pauli-Lubansky pseudovector, and Eq. (31b) covariantly defines E_μ as a Pauli-Lubansky vector. These definitions imply several properties of both B_μ and E_μ:

1. Since $F_{\mu\nu}$ is the four-curl of A_ν (Eq. (23)), Eqs (31) covariantly relate B_μ and E_μ to A_μ in spacetime.
2. Since E_μ and B_μ are four vectors in Minkowski spacetime, it follows in pseudo-Euclidean geometry that $E_\mu E_\mu$ and $B_\mu B_\mu$ are constants in spacetime, and that

$$\frac{\partial E_\mu}{\partial x_\mu} = \text{constant} \qquad \frac{\partial B_\mu}{\partial x_\mu} = \text{constant} \tag{35}$$

3. Equation (31a) is dual with Eq. (31b), because under the dual transformation of fields

$$F_{\rho\sigma} \to \tilde{F}_{\mu\nu} \qquad E_\mu \to -\mathrm{i}cB_\mu \tag{36}$$

4. The parity inversion $\hat{P}$ and motion reversal $\hat{T}$ symmetries of B_μ are consistent with those of $F_{\nu\rho}$ and δ_σ, bearing in mind that the latter is the unit generator of spacetime translation. Since Eq. (31b) is dual with Eq. (31a), its symmetries are consistent with those of Eq. (31a). B_μ is a pseudovector because its spacelike component **B** is positive to $\hat{P}$ and negative to $\hat{T}$. E_μ is a vector because **E** is negative to $\hat{P}$ and positive to $\hat{T}$.
5. From the properties of the Pauli-Lubansky pseudovectors and vectors,[7] both E_μ and B_μ are orthogonal to δ_μ in spacetime:

$$B_\mu\delta_\mu = 0 \qquad E_\mu\delta_\mu = 0 \tag{37}$$

6. Both E_μ and B_μ are defined in Eqs. (31) in terms of the unit generator of spacetime translations δ_σ, allowing E_μ and B_μ to be covariantly and consistently interpreted in terms of helicity in the lightlike condition.[7]
7. Since E_μ and B_μ are defined covariantly, the timelike components $E^{(0)}$ and $B^{(0)}$, respectively, are both explicitly and implicitly stated to be physically meaningful in spacetime.
8. The products $B_\mu B_\mu$ and $E_\mu E_\mu$ are both Casimir invariants[7] of the Poincaré group, specifically Casimir invariants of the second kind, or

"spin" invariants. The product $\delta_\mu\delta_\mu$ is a Casimir invariant of the first kind ("mass" invariant). This deduction follows from the definition (31) of B_μ and E_μ as Pauli-Lubansky types. In the lightlike condition, i.e., for the massless electromagnetic gauge field,

$$\delta_\mu\delta_\mu = 0 \qquad E_\mu E_\mu = 0 \qquad B_\mu B_\mu = 0 \tag{38}$$

Equation (38) is a classical statement of the fact that the photon in the quantum field is massless and possesses spin.

9. From Eqs. (37) and (38), E_μ and B_μ are both orthogonal and proportional to δ_μ in spacetime. The proportionality constant (a scalar in spacetime) expresses the helicity of the electromagnetic gauge field.

It can be verified explicitly that the fundamental conditions (37) and (38) are satisfied by the circularly polarized transverse components of Eq. (7) in combination with the longitudinal components of Eqs. (8) and (12b). For example,

$$\begin{aligned} E_\mu E_\mu &\equiv E^{(1)2} + E^{(2)2} + E^{(3)2} - E^{(0)2} \\ &= E^{(0)2}\left(\hat{\mathbf{e}}^{(1)} \cdot \hat{\mathbf{e}}^{(1)} e^{2i\phi} + \hat{\mathbf{e}}^{(2)} \cdot \hat{\mathbf{e}}^{(2)} e^{-2i\phi} + \hat{\mathbf{e}}^{(3)} \cdot \hat{\mathbf{e}}^{(3)} - 1\right) \\ &= E^{(0)2}\left(\hat{\mathbf{e}}^{(1)} \cdot \hat{\mathbf{e}}^{(1)} e^{2i\phi} + \hat{\mathbf{e}}^{(2)} \cdot \hat{\mathbf{e}}^{(2)} e^{-2i\phi}\right) \\ &= \frac{E^{(0)2}}{2}\left((\mathbf{i} - \mathbf{ij}) \cdot (\mathbf{i} - \mathbf{ij}) e^{2i\phi} + (\mathbf{i} + \mathbf{ij}) \cdot (\mathbf{i} - + \mathbf{ij}) e^{-2i\phi}\right) \\ &= 0 \end{aligned} \tag{39}$$

IV. CONSEQUENCES FOR VACUUM ELECTRODYNAMICS

Equations (31) covariantly define the four vectors E_μ and B_μ in spacetime. This means that the fundamentals of vacuum electrodynamics are changed. One immediate consequence is that Eqs. (35) restate the Gauss theory in covariant form. Using the polarizations defined in Eqs. (7), (8), and (12b) it is clear that the constant in Eqs. (35) is zero and that the Gauss theorem in differential form is covariantly written as

$$\frac{\partial E_\mu}{\partial x_\mu} = 0 \quad \text{or} \quad \nabla \cdot \mathbf{E} + \frac{1}{c}\frac{\partial E^{(0)}}{\partial t} = 0 \tag{40}$$

and

$$\frac{\partial B_\mu}{\partial x_\mu} = 0 \quad \text{or} \quad \nabla \cdot \mathbf{B} + \frac{1}{c}\frac{\partial B^{(0)}}{\partial t} = 0 \tag{41}$$

Equations (40) and (41) replace the conventional spacelike statements of the Gauss theorem in differential form in vacuo:

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \cdot \mathbf{B} = 0 \tag{42}$$

Therefore, it becomes clear that the covariant definitions (30) lead to the following covariant statements of the Maxwell equations in vacuo:

$$\begin{aligned} \frac{\partial F_{\mu\nu}}{\partial x_\mu} &= 0 \qquad \frac{\partial \tilde{B}_\mu}{\partial x_\mu} = 0 \\ \frac{\partial \tilde{F}_{\mu\nu}}{\partial x_\mu} &= 0 \qquad \frac{\partial B_\mu}{\partial x_\mu} = 0 \end{aligned} \tag{43}$$

in which $\tilde{B}_\mu$ is the dual of $\mathbf{B}_\mu$, and $\tilde{F}_{\mu\nu}$ that of $\mathbf{F}_{\rho\sigma}$. The Maxwell equations in the form (43) are covariantly consistent with the d'Alembert equation (1), which was the starting point of our development.

The vacuum electromagnetic energy density is, from Eqs. (31), covariantly defined in S.I. units as (see Appendix C)

$$U = \frac{1}{2}\left(\varepsilon_0 E_\mu E_\mu + \frac{1}{\mu_0} B_\mu B_\mu\right) \tag{44}$$

where μ_0 is the permeability in vacuo. (Note that it is not consistent to refer to the vacuum as "free space"; it is covariantly described as "free spacetime.") From the example of Eqs. (39), it is clear that field polarizations (0) and (3), although physically meaningful, do not contribute to U, so that Eq. (44) happens to reduce to the conventional[8–15] spacelike definition of U:

$$U(\text{conventional}) \equiv \frac{1}{2}\left(\varepsilon_0 \mathbf{E} \cdot \mathbf{E} + \frac{1}{\mu_0} \mathbf{B} \cdot \mathbf{B}\right) \tag{45}$$

This deduction is consistent with the quantum field theory leading[7] to Eq. (16).

Similarly, the vacuum electromagnetic flux density (the conventional, spacelike, Poynting vector[8-15]) is covariantly defined from Eqs. (30) as the four vector product of E_μ and B_μ, the four tensor, in S.I. units:

$$S_{\mu\nu} = \frac{1}{\mu_0}(E_\mu B_\nu - B_\mu E_\nu) \tag{46}$$

The conventional statement of the law of conservation of electromagnetic energy in vacuo is the Poynting theorem,[8-15] expressed through the continuity equation:

$$\nabla \cdot \mathbf{S} + \frac{1}{c^2}\frac{\partial U}{\partial t} = 0 \tag{47}$$

This is already Lorentz covariant in structure, because it is an equation in the four divergence of the Poynting four vector $S_\mu = (S, -\mathrm{i}S^{(0)})$, i.e.,

$$\frac{\partial S_\mu}{\partial x_\mu} = 0 \tag{48}$$

However, the conventional definition (47) implies that the two spacelike components of the Poynting four vector S_μ orthogonal to the propagation direction of the electromagnetic wave in vacuo must vanish. The definition (47), although Lorentz covariant, is not necessarily manifestly covariant, because it is based on the conventional[8-15] assumption that **E** and **B** are spacelike and transverse.

In a manifestly covariant description it is necessary to relate the Poynting four vector of Eq. (48) to the Poynting four tensor $S_{\mu\nu}$ formed from the vector product in spacetime of the novel four vectors E_μ and B_μ. It is reasonable to propose that this relation is

$$S_\mu = \frac{\mathrm{i}}{2}\varepsilon_{\mu\nu\rho\sigma}S_{\nu\rho}\delta_\sigma \tag{49}$$

where $\varepsilon_{\mu\nu\rho\sigma}$ and δ_σ have the same meaning as in Eq. (31). Explicitly,

$$S^{(1)} \equiv S_1 = \frac{\mathrm{i}}{2}(\varepsilon_{1230}S_{23}\delta_0 + \varepsilon_{1320}S_{32}\delta_0 + \varepsilon_{1203}S_{20}\delta_3 + \varepsilon_{1023}S_{02}\delta_3) \tag{50a}$$

$$S^{(2)} \equiv S_2 = \frac{\mathrm{i}}{2}(\varepsilon_{2310}S_{31}\delta_0 + \varepsilon_{2130}S_{13}\delta_0 + \varepsilon_{2013}S_{01}\delta_3 + \varepsilon_{2103}S_{10}\delta_3) \tag{50b}$$

$$S^{(3)} \equiv S_3 = \frac{\mathrm{i}}{2}(\varepsilon_{3210}S_{21}\delta_0 + \varepsilon_{3120}S_{12}\delta_0)$$

$$-\mathrm{i}S^{(0)} \equiv -\mathrm{i}S_0 = \frac{\mathrm{i}}{2}(\varepsilon_{0123}S_{12}\delta_3 + \varepsilon_{0213}S_{21}\delta_3) \tag{50c}$$

$$\text{with} \quad \delta_3 = 1 \quad \delta_0 = -\mathrm{i} \quad (\delta_1 = \delta_2 = 0)$$

$$\begin{array}{llll} \varepsilon_{0123} = 1 & \varepsilon_{0213} = -1 & \varepsilon_{3120} = -1 & \varepsilon_{3210} = -1 \\ \varepsilon_{2130} = 1 & \varepsilon_{2310} = -1 & \varepsilon_{2013} = 1 & \varepsilon_{2103} = -1 \\ \varepsilon_{1230} = -1 & \varepsilon_{1320} = 1 & \varepsilon_{1203} = 1 & \varepsilon_{1023} = -1 \end{array} \tag{50d}$$

Equations (50a) and (50b) show that in this definition, the Poynting four vector in spacetime develops components in the spacelike axes orthogonal to the propagation axis (3).

The definition (49) of the manifestly covariant Poynting vector introduces the unit generator of spacetime translations, δ_σ, for an electromagnetic wave traveling in vacuo in the spacelike axis (3). In direct analogy with our fundamental definitions, Eqs. (31), of E_μ and B_μ, S_μ is thereby defined within the Poincaré group rather than the Lorentz group, and spacetime translation is included explicitly in the definition. This means that the manifestly covariant Poynting vector is also a Pauli-Lubansky vector within the Poincaré group in spacetime. Note that from Eqs. (50c) and (50d),

$$\begin{aligned} S^{(3)} - |S^{(0)}| &= 0 \\ S^{(3)} \cdot S^{(3)} - |S^{(0)}|^2 &= 0 \end{aligned} \tag{51}$$

i.e., the conventional, spacelike Poynting vector, which has only one spacelike component, (3), and no timelike component, (0), becomes within the structure of Eq. (49) a physical state that is an admixture of (3) and (0) polarizations. The other spacelike components of S_μ, i.e., (1) and (2), also become physically meaningful through Eqs. (50a) and (50b). At an instant in spacetime, these components (1) and (2) are experimental observables. However, observations (Appendix 3) of the electromagnetic energy flux density, known as the Poynting vector,[8-15] are made by the observer with an instrument such as a power meter, which gives only the time-averaged value of the Poynting vector. The components $S_{23}, S_{32}, S_{20}, S_{02}, S_{31}, S_{13}, S_{01}, S_{10}$ disappear upon time averaging (Appendix 3) because they are made up of products of one phase-dependent component and one that is phase independent. The components S_{21} and S_{12}, on the other hand, are products of two phase-dependent components, one of which is the complex conjugate of the other, so that the phase disappears in the product, which is thereby nonzero after time averaging. For these reasons, the conventional Poynting theorem (47), which is not manifestly covariant, happens to be an adequate description of the law of conservation of electromagnetic energy, but only on a time-averaged basis. If it were experimentally possible to observe electromagnetic energy flux density in an instant in spacetime, then the components S_{23} and so on would contribute explicitly to the law of conservation of energy. Clearly, if S_μ is a Pauli-Lubansky vector in the Poincaré group, then the product is a Casimir invariant of type two of the Poincaré group, and S_μ is orthogonal to δ_μ in spacetime.

The description of the electromagnetic field polarization in vacuo through the four Stokes parameters in terms of E_μ and B_μ requires a modification[22] of the conventional description[23] based on Pauli matrices:

$$S_0 = [E_X E_Y]\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}E_X^*\\ E_Y^*\end{bmatrix} \tag{52a}$$

$$S_1 = [E_X E_Y]\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}E_X^*\\ E_Y^*\end{bmatrix} \tag{52b}$$

$$S_2 = [E_X E_Y]\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}E_X^*\\ E_Y^*\end{bmatrix} \tag{52c}$$

$$S_3 = [E_X E_Y]\begin{bmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{bmatrix}\begin{bmatrix}E_X^*\\ E_Y^*\end{bmatrix} \tag{52d}$$

so that

$$S_0^2 = S_1^2 + S_2^2 + S_3^2 \tag{53a}$$

$$[S_1, S_2] = \mathrm{i}S_3 \tag{53b}$$

Within a factor $\hbar$, the Stokes parameters obey the commutation rules of quantized angular momentum, and form a four vector $S_\mu \equiv (\mathbf{S}, \mathrm{i}S_0)$. (The Stokes vector S_μ should not be confused with the Poynting vector, unfortunately also denoted $\mathbf{S}$ in the conventional literature.) It follows from Eq. (53b) that the Pauli matrices also obey the angular momentum commutation rules. Equations (52) omit the longitudinal and timelike polarizations of the four vector E_μ, and for a manifestly covariant description, these must be included in the basic definition of the four Stokes parameters (real numbers in the conventional theory[8-15]). The generalization of S_0 to S_3 must conform with Eqs. (53), and S_0, which is proportional to the electromagnetic energy density in vacuo,[8-15] must conform to our earlier results (16) and (39). It is natural to propose the replacement in Eqs. (52) of the field vectors $[E_X, E_Y]$ and $\begin{bmatrix}E_X^*\\ E_Y^*\end{bmatrix}$ by their manifestly covariant equivalents (four vectors), and to replace the Pauli matrices by Dirac matrices (angular momentum operators[24]). The latter obey the same commutation rules and their structure is that of a "doubled" (4×4) Pauli matrix. The following generalization is manifestly covariant and conforms with Eq. (53):

$$S_0 = \left[E_X E_Y \frac{E_Z}{2} \frac{-\mathrm{i}E^{(0)}}{2}\right]\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}E_X^*\\ E_Y^*\\ E_Z/2\\ -\mathrm{i}E^{(0)}/2\end{bmatrix} \tag{54a}$$

$$S_1 = \left[E_X E_Y \frac{E_Z}{2} \frac{-\mathrm{i}E^{(0)}}{2}\right]\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{bmatrix}\begin{bmatrix}E_X^*\\ E_Y^*\\ E_Z/2\\ -\mathrm{i}E^{(0)}/2\end{bmatrix} \tag{54b}$$

$$S_2 = \left[E_X E_Y \frac{E_Z}{2} \frac{-\mathrm{i}E^{(0)}}{2}\right]\begin{bmatrix}0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}E_X^*\\ E_Y^*\\ E_Z/2\\ -\mathrm{i}E^{(0)}/2\end{bmatrix} \tag{54c}$$

$$S_3 = \left[E_X E_Y \frac{E_Z}{2} \frac{-\mathrm{i}E^{(0)}}{2}\right]\begin{bmatrix}0 & -\mathrm{i} & 0 & 0\\ \mathrm{i} & 0 & 0 & 0\\ 0 & 0 & 0 & -\mathrm{i}\\ 0 & 0 & \mathrm{i} & 0\end{bmatrix}\begin{bmatrix}E_X^*\\ E_Y^*\\ E_Z/2\\ -\mathrm{i}E^{(0)}/2\end{bmatrix} \tag{54d}$$

(The factor $\frac{1}{2}$ follows from the definitions, Eqs. (31) and Appendix B, Eq. (B.7).) Explicitly written out, the covariant Stokes parameters for one sense of circular polarization become

$$S_0 = E_X E_X^* + E_Y E_Y^* = E^{(0)2} \quad (55a)$$

$$S_1 = E_X E_X^* - E_Y E_Y^* + \tfrac{1}{4}\left(E_Z^2 + E^{(0)2}\right) = \tfrac{1}{2}E^{(0)2} \quad (55b)$$

$$S_2 = E_X E_Y^* + E_Y E_X^* - \frac{i}{4}\left(E_Z E^{(0)} + E^{(0)} E_Z\right) = -\frac{1}{2} i E^{(0)2} \quad (55c)$$

$$S_3 = -i\left(E_X E_Y^* - E_Y E_X^*\right) = E^{(0)2} \quad (55d)$$

We find that the conventional result $S_1 = S_2 = 0$ in circular polarization[8-15] is replaced by

$$S_1 = iS_2 = \tfrac{1}{2}E^{(0)2} = \tfrac{1}{2}\mathbf{E}^{(3)} \cdot \mathbf{E}^{(3)} \quad (56)$$

Our covariant theory leaves the value of S_0 unchanged, as required, and finally, S_3 is also unchanged. Significantly, the results (55) can be expressed entirely in terms of $\mathbf{E}^{(3)}$ (or $\mathbf{B}^{(3)}$):

$$S_0 = |S_3| = \mathbf{E}^{(3)} \cdot \mathbf{E}^{(3)} \quad (57)$$

together with Eq. (56). Since $E^{(0)} = cB^{(0)}$ in free spacetime, Eqs. (56) and (57) can be represented in terms of $\mathbf{B}^{(3)}$. In particular,

$$|S_3| = c^2 \mathbf{B}^{(3)} \cdot \mathbf{B}^{(3)} = c^2 B^{(0)} |\mathbf{B}^{(3)}| \quad (58)$$

a result derived previously[1-5] through the relation

$$\mathbf{B}^{(3)} = \frac{\mathbf{E}^{(1)} \times \mathbf{E}^{(2)}}{icE^{(0)}} \quad (59)$$

It is interesting to note that the following eigenvalue (operator type but classical) equation consistently reconciles the existence of only one photon helicity for one sense of circular polarization:

$$\begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix} \begin{bmatrix} E_X \\ E_Y \\ E_Z/2 \\ -iE^{(0)}/2 \end{bmatrix} = \lambda^{(i)} \begin{bmatrix} E_X \\ E_Y \\ E_Z/2 \\ -iE^{(0)}/2 \end{bmatrix} \quad (60)$$

The eigenvalues are

$$\lambda^{(1)} = \frac{-iE_Y}{E_X} = -1 \qquad \lambda^{(2)} = \frac{iE_X}{E_Y} = -1$$
$$\lambda^{(3)} = \frac{-E^{(0)}}{E_Z} = -1 \qquad \lambda^{(4)} = \frac{-E_Z}{E^{(0)}} = -1 \quad (61)$$

For the opposite sense of circular polarization, E_Y and E_Z change sign and the four eigenvalues $\lambda^{(i)}$ becomes $+1$. Equation (60) therefore provides the result

$$\lambda^{(1)} = \lambda^{(2)} = \lambda^{(3)} = \lambda^{(4)} = \mp 1 \quad (62)$$

for different senses of circular polarization, and reconciles the existence of four different field polarizations with only two different field helicities. (The four polarizations are right and left circular spacelike, longitudinal spacelike, and timelike. The two helicities are $+1$ and -1.) In the conventional theory,[8-15] Eq. (60) becomes

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} E_X \\ E_Y \end{bmatrix} = \lambda^{(i)} \begin{bmatrix} E_X \\ E_Y \end{bmatrix} \quad (63)$$

i.e., helicities $\lambda^{(3)}$ and $\lambda^{(4)}$ are missing, and the remaining two "transverse" helicities are generated by a Pauli matrix rather than a Dirac matrix.

It is therefore concluded that the structures of the Stokes parameters are changed in the manifestly covariant description of electrodynamics in vacuo, and therefore so is the fundamental specification of the polarization characteristics of light: the Hermitian polarization density, or coherency matrix of Born and Wolf[13] and the polarization tensor of Landau and Lifshitz.[12] Specifically, the Stokes parameters S_1 and S_2 no longer vanish in circular polarization, and this is a direct consequence of the covariant nature of E_μ, in that its longitudinal and timelike components now contribute to a purely real, nonzero, S_1, and a purely imaginary S_2 with the opposite sign. Conventionally,[8-15] S_1 and S_2 are nonzero only in elliptical polarization. They can be described in terms of excess of linear polarization, and conventionally it is considered that there is no excess of linear polarization when the beam is fully right or left circularly polarized. However, in the covariant description, there is an additioinal longitudinal component in the propagation axis of the beam, even in a completely circularly polarized beam. The longitudinal components $\mathbf{E}^{(3)}$ and $\mathbf{B}^{(3)}$

vanish, however, if the beam has an equal amount of transverse right and transverse left circularly polarized components. In this state of transverse linear polarization, the imaginary contribution to S_2 vanishes, but the real contribution to S_1 doubles. This can be interpreted to mean that although $\mathbf{E}^{(3)}$ changes sign between right and left transverse circular polarization, its square $E^{(3)2}$ evidently does not. It is $E^{(3)2}$ that contributes to S_1. This emphasizes the fact that the Stokes parameters are quadratic in the electric part of the electromagnetic field.

S_3 is unchanged in the covariant description, because S_3 is defined in this description by

$$\mathbf{B}^{(3)} = \frac{\mathbf{E}^{(1)} \times \mathbf{E}^{(2)}}{\mathrm{i}cE^{(0)}} = B^{(0)}\mathbf{k} \tag{64a}$$

$$B^{(0)} = \left(\frac{\varepsilon_0}{I_0 c}\right)^{1/2} |S_3| \tag{64b}$$

V. DISCUSSION

The covariant description of the electromagnetic field in vacuo shows that there are physically meaningful fields $\mathbf{B}^{(3)}$ and $\mathbf{E}^{(3)}$ that satisfy Maxwell's equations. These fields do not appear explicitly in the conventional theory,[8–15] and are assumed to be physically meaningless. It is therefore necessary to identify experiments that can distinguish between the conventional theory and the manifestly covariant theory of the electromagnetic field. One immediately obvious consequence of $\mathbf{B}^{(3)}$ is that circularly polarized electromagnetic radiation can magnetize matter. Before embarking on a development of these properties, however, we show that effects such as natural optical activity, the electrical Kerr effect, and the development of ellipticity in an initially circularly polarized light beam can be explained in terms of changes in $\mathbf{B}^{(3)}$ as they traverse a sample. The essential reason for this is that whenever the Stokes parameter S_3 appears in physical optics, it signals (vide supra) the existence of $\mathbf{B}^{(3)}$, to whose magnitude it is directly proportional:

$$|\mathbf{B}^{(3)}| = \left(\frac{\varepsilon_0}{I_0 c}\right)^{1/2} |S_3| = \frac{|S_3|}{c^2 B^{(0)}} \tag{65}$$

Therefore, S_3 can be replaced whenever it occurs by the scalar quantity

$$\pm c^2 B_0 |\mathbf{B}^{(3)}| \equiv \pm c^2 \mathbf{B}^{(3)} \cdot \mathbf{B}^{(3)} \tag{66}$$

In material media, as opposed to free space, Kielich[25] has shown that linear and nonlinear optical activity depends on S_3, and in the Rayleigh theory[26] of natural optical activity in chiral media, it is well known that whatever the nature of the several molecular property tensors participating in the polarization and magnetization of the material, the observable of circular dichroism has pseudoscalar symmetry and is proportional to the third Stokes parameter. For different enantiomers for a given sense of transverse circular polarization, or for one enantiomer for different sense of transverse circular polarization,

$$\frac{I_R - I_L}{I_R + I_L} = \pm \frac{S_3}{S_0} \tag{67}$$

where I_R and I_L are the intensities of right and left components transmitted by structurally chiral material, with

$$I_0 = I_R + I_L \tag{68}$$

for the transmitted total beam intensity. From Eqs. (67) and (68) we derive the result (with $S_0 = c^2 B^{(0)2}$),

$$\pm \frac{S_3}{S_0} = \pm \frac{|\mathbf{B}^{(3)2}|}{B^{(0)2}} = \frac{I_R - I_L}{I_R + I_L} \tag{69}$$

which reveals the fundamental origin of the phenomenon of circular dichroism at all electromagnetic frequencies, because it shows that the observable $(I_R - I_L)$ is proportional to $|\mathbf{B}^{(3)2}|$.

The origin of circular dichroism, therefore, resides in the photon's longitudinal magnetostatic flux quantum $\hat{B}^{(3)}$, whose expectation value is $\mathbf{B}^{(3)}$.

The observable $I_R - I_L$ is therefore a spectral consequence of the interaction of $\mathbf{B}^{(3)}$ with structurally chiral material. From Eq. (69), $I_R - I_L$ is proportional to the real pseudoscalar $\pm|\mathbf{B}^{(3)}|$ after they emerge from the chiral material through which the beam has passed, i.e., after interaction has occurred between the flux quantum $\hat{B}^{(3)}$ and the appropriate molecular property tensors.[26] For one photon, the observable $I_R - I_L$ provides an experimental measure of the transmitted elementary $\mathbf{B}^{(3)}$ at each frequency. Although $\mathbf{B}^{(3)}$ itself is independent of frequency, the interacting molecular property tensor is not. Semiclassical perturbation theory[26] gives, for linear optical activity,

$$\frac{S_3}{S_0} = \frac{|\mathbf{B}^{(3)2}|}{B^{(0)2}} \tanh[\omega\mu_0 clN\zeta''_{XYZ}(g)] \tag{70}$$

where μ_0 is the permeability in vacuo, ω the angular frequency of the beam, l the sample path length, and ζ''_{XYZ} a combination[26] of molecular property tensors, which may be electric and/or magnetic in nature. For nonlinear optical activity, Eq. (70), as shown by Kielich,[25] contains additional terms.

Therefore, every time natural optical activity is observed with $I_R - I_L$, as in circular dichroism, the quantity $\mathbf{B}^{(3)}$, has been measured. In this context, a covariant description of the electromagnetic field is one that identifies the phenomenon of circular dichroism with the longitudinal field $\mathbf{B}^{(3)}$, showing that the latter is physically meaningful and is, indeed, well measured in the literature although not explicitly recognized as a magnetic field. In the conventional description on the other hand, natural optical activity is measured by S_3/S_0, which is given by

$$\frac{S_3}{S_0} = \frac{-i(E_X E_Y^* - E_Y E_X^*)}{E_X E_X^* + E_Y E_Y^*} \tag{71}$$

and $\mathbf{E}^{(3)}$ and $\mathbf{B}^{(3)}$ are conventionally supposed to be physically meaningless. However, S_3/S_0 is, of course, also expressible in the covariant description by Eq. (71), showing that the covariant description is both simpler and more complete than the conventional one. The conventional assertion that $\mathbf{B}^{(3)}$ be physically meaningless conflicts with Eq. (6), and becomes unsustainable, because $\mathbf{E}^{(1)} \times \mathbf{E}^{(2)}$ is a physically meaningful quantity directly proportional to S_3. It is more complete, more revealing, and more "natural" to describe optical activity as changes in $\mathbf{B}^{(3)}$ as a medium is traversed by a light beam. In other words, the phenomenon of natural optical activity is definitive experimental evidence for the existence of $\mathbf{B}^{(3)}$ in physical optics.

More generally, it can be shown that any phenomenon in optics that involves S_3 must involve $\mathbf{B}^{(3)}$ in its quantized or classical forms, whichever is the more appropriate to a given situation. Throughout the contemporary literature[27] that there are many of these optical phenomena, one commonplace example being the development of ellipticity in an initially circularly polarized light beam. For example, in the electric Kerr effect,[28] beam

ellipticity (η) is expressed in terms of S_3, and is induced with an electric field in a probe laser. The electric Kerr effect is therefore

$$\frac{|\mathbf{B}^{(3)2}|}{B^{(0)2}} = \sin(2\eta) \tag{72}$$

where η is the ellipticity developed in the transmitted probe as a result of the application of an electric field to a sample. This is experimental evidence for the existence of the longitudinal field $\mathbf{B}^{(3)}$. Note that for an initially linearly polarized beam, $\mathbf{B}^{(3)}$ is zero, and so Eq. (72) shows that the development of ellipticity in the Kerr effect is a direct consequence of the interaction of $\mathbf{B}^{(3)}$ with the medium through which the probe laser has passed.

Rayleigh refringent scattering theory[26] shows that the third Stokes parameter S_3 is associated with a change $d\eta/dZ$ in ellipticity in a beam passing through a sample of thickness Z. Therefore, $d\eta/dZ$ measures changes in $\mathbf{B}^{(3)}$ as it traverses the sample thickness Z. We arrive at the generally valid conclusion that ellipticity in the electromagnetic plane wave is directly related to $\mathbf{B}^{(3)}$, and that the development of ellipticity can be expressed in terms of these fields. The scalar magnitude of $\mathbf{B}^{(3)}$ is $|\mathbf{B}^{(3)}|$, respectively, associated with the timelike polarization (0) of the electromagnetic field. The timelike polarization always appears as an admixture with the longitudinal polarization, and both are physically meaningful because they are observed in fundamental optical phenomena. Equation (70), for example, shows that circular dichroism is related to the molecular property tensor sum represented by ζ'', which is made up of the Rosenfeld tensor and the electric quadrupole tensor. Note carefully, however, that ζ'' is a material property, while $\mathbf{B}^{(3)}$ is a property of free spacetime, which interacts with matter. The definition of S_3 in free spacetime in terms of $\mathbf{B}^{(3)}$ is obviously unaffected by any material property because $\mathbf{B}^{(3)}$ is associated with fundamental photon polarizations. In Eq. (70) we have used the result that $|\mathbf{B}^{(3)2}|$ is directly proportional to the Stokes parameter S_3 in free spacetime, and have replaced the Stokes parameter by a term proportional to $|\mathbf{B}^{(3)2}|$.

In summary, there is copious experimental evidence for the existence of $\mathbf{B}^{(3)}$, which is a physically meaningful magnetic field in free spacetime. Through Eq. (6), the conventional description is supplanted in physical optics by the more complete and more rigorous covariant description, i.e., by a description that is fully compatible with the theory of special relativity. Although the conventional description is self-consistent up to a point, the key equation (6) of this paper shows that it lacks the polarizations (3) and (0), which are present in the quantum field[7] but usually wrongly asserted to be physically meaningless. Note that Eq. (6) is invariant to the fundamental symmetries of physics[29]: charge conjugation $\hat{C}$, parity inversion $\hat{P}$, and motion reversal $\hat{T}$ and is therefore a rigorously self-consistent equation of electrodynamics in free spacetime.

It is also interesting to note[29] that in the field of high-energy particle physics, experimental evidence exists for timelike photons that can be produced in electron positron annihilation processes at extremely high energy. This presumably means that in such processes the concommitant magnetic and electric field amplitudes $B^{(0)}$ and $E^{(0)}$ exist independently, and are therefore physically meaningful.

Having established with available data the existence of $\mathbf{B}^{(3)}$, it is now possible to reinterpret known optical phenomena and to predict with some degree of confidence the existence of hitherto unmeasured optical phenomena based on $\mathbf{B}^{(3)}$. The everyday phenomenon of optical absorption is described by the Beer-Lambert law:

$$\alpha(\bar{\nu}) = \frac{1}{d} \log_e \frac{I_0}{I} \tag{73}$$

Here I_0 is the incident beam intensity, I the transmitted beam intensity, and d the sample length. α is the power absorption coefficient[30] in neper m^{-1}, and this quantity can be reinterpreted in terms of $\mathbf{B}^{(3)}$, because the zeroth Stokes parameter S_0 is proportional to beam intensity. Therefore, simple optical absorption is a process that can be interpreted in terms of the longitudinal electric and/or magnetic fields of the electromagnetic plane wave, an interpretation that is just as valid as the usual one[30] in terms of the transverse components $\mathbf{E}^{(1)}$ or $\mathbf{E}^{(2)}$. In general, since all four Stokes parameters in covariant electrodynamics can be expressed in terms of $\mathbf{B}^{(3)}$, all optical phenomena involving beam polarization or optical coherence processes in linear physical optics can also be described in terms of these longitudinal fields.

In nonlinear optics,[31] the light beam is used to induce phenomena in material media (e.g., molecular matter such as liquids), phenomena that depend nonlinearly on the electric and magnetic components of the intense laser beam. A large number of such phenomena have been observed in the past thirty years,[31] and the theory of such processes has been systematically developed by Kielich and coworkers,[32] following early inroads by Piekara and Kielich,[33] who were among the first to consider systematically statistical molecular theories of optically induced phenomena in isotropic dielectric and diamagnetic media. These earlier theories are, of course, formulated in terms of the transverse spacelike components of our covariant description, and should be modified to take into account the existence of $\mathbf{B}^{(3)}$ in free spacetime. These fields are expected to produce observable magnetization and polarization when they interact with matter. For laser beams that are intense enough, various optical saturation phenomena due to $\mathbf{B}^{(3)}$ should occur. A classic work such as the early paper by Kielich[34] on frequency and spatially variable electric and magnetic polarization induced in nonlinear media by electromagnetic fields should be covariantly developed, so that the Born-Infeld electrodynamics[35] to which it refers can be extended to include $\mathbf{B}^{(3)}$ within a manifestly covariant structure. Terms such as $\mathbf{E} \times \mathbf{E}^*$ in the work by Kielich[34] can be replaced by $\mathbf{B}^{(3)}$, for example, thus predicting birefringence effects proportional to the square root of intensity, in addition to the traditional effects proportional to intensity, such as the inverse Faraday effect.[36]

In another classic paper by Kielich,[37] on nonlinear processes resulting from multipole interaction between molecules and electromagnetic fields, it would be interesting to explore the role played by $\mathbf{B}^{(3)}$ in the various nonlinear optical phenomenon proposed in this work, for example, (1) a covariant reformulation of the Dirac theory to describe the absorption of a flux quantum $\hat{B}^{(3)}$; (2) a covariant scattering theory for $\mathbf{B}^{(3)}$; (3) the role of $\mathbf{B}^{(3)}$ in the nonlinear optical processes where linear superposition is lost; (4) investigations of the probability of an n photon process with magnetic transitions involving an incoming $\hat{\mathbf{B}}^{(3)}$ flux quantum; (5) scattering theory involving the classical $\mathbf{B}^{(3)}$. Again, in the theory of nonlinear light scattering from colloidal media,[38] $\mathbf{B}^{(3)}$ is expected to play a basic part in defining the depolarization ratio, since, as we have seen, $\mathbf{B}^{(3)}$ is proportional to $I_R - I_L$. In general, in Rayleigh refringent scattering theory, the Stokes parameters in our covariant description enter in terms of $\mathbf{B}^{(3)}$, so that the longitudinal field is fundamental to any description. The role of $\mathbf{B}^{(3)}$ in the Majorana effect,[39] and intensity dependent optical circular birefringence[40] is also fundamental. The interesting phenomenon of ellipse self-rotation by a circularly polarized laser[41] is also fundamentally dependent on the longitudinal field $\mathbf{B}^{(3)}$.

More recently, the phenomena associated with light squeezing in quantum electrodynamics have become prevalent in the literature[42] and in this context Tanaś and Kielich[19] have systematically investigated the effect of squeezing on a large number of optical phenomena, including the effect on the four Stokes operators, the quantum equivalent of the four Stokes parameters.[43] It was deduced that the parameters S_1 and S_2 are in general affected by squeezing, and it would be interesting to develop this result in a manifestly covariant description, where classically, as we have seen, the four Stokes parameters are affected in basic structure, and new terms are added to S_1 and S_2. The field $\mathbf{B}^{(3)}$ also plays a role in light self-squeezing in Kerr media, discovered by Kielich et al.[44] and in general in all nonlinear quantum electrodynamics, fundamentally changing the structure of the theory.

For example, Frey et al.[45] have recently observed azimuth rotation due to an intense laser beam (the optical Faraday effect), and this has been shown by Farahi and Evans[5] to be a linear function of the square root of laser intensity, i.e., to be linearly dependent on the magnitude of $\mathbf{B}^{(3)}$. This is the first experimental evidence for the ability of $\mathbf{B}^{(3)}$ to magnetize a material, in this case a magnetic semiconductor.[45] Magnetization by a circularly polarized light beam has been observed as the inverse Faraday effect,[36] and recently, as laser-induced shifts in NMR spectra.[46] Light shifts in atomic spectra have also been observed experimentally[47] and can be reinterpreted in terms of $\mathbf{B}^{(3)}$. In general, a large number of phenomena can be reinterpreted in terms of longitudinal[48] fields in vacuo phenomena that are at present attributed solely to the transverse fields $\mathbf{E}^{(1)}$ and $\mathbf{E}^{(2)}$. In theory, optical effects due to $\mathbf{B}^{(3)}$ can be identified and separated from the concommitant effects due to $\mathbf{E}^{(1)}$ and $\mathbf{E}^{(2)}$, or $\mathbf{B}^{(1)}$ and $\mathbf{B}^{(2)}$, because the former are expected to be proportional to the square root of laser intensity (and integral powers thereof), and the latter to even powers only of laser intensity.

APPENDIX A: CARTESIAN AND CIRCULAR REPRESENTATIONS

The subscripts in the matrix in Eq. (22) are conventionally[8–15] given in the Cartesian basis, (X, Y, Z), while circular polarization is described in the circular basis $((1), (2), (3))$. Any physical phenomenon should be independent of the basis (i.e., laboratory frame of reference) used in its description, and in this paper the link between the two representations is given in terms of the following unit vector equations. Superscripts (1), (2), and (3) refer respectively to the first and second sense of transverse circular polarization, and the longitudinal polarization:

$$\hat{e}^{(1)} \equiv \frac{1}{\sqrt{2}}(\mathbf{i} - \mathrm{i}\mathbf{j}) \tag{A.1}$$

$$\hat{e}^{(2)} = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathrm{i}\mathbf{j}) \tag{A.2}$$

$$\hat{e}^{(3)} \equiv \mathbf{k} \tag{A.3}$$

where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are Cartesian unit vectors in X, Y, and Z, respectively. Thus,

$$\hat{\mathbf{e}}^{(1)} \times \hat{\mathbf{e}}^{(2)} = \mathrm{i}\hat{\mathbf{e}}^{(3)} = \mathrm{i}\mathbf{k} \tag{A.4}$$

The circular basis is used in Eq. (34) to define E_μ and B_μ in terms of polarizations (0), (1), (2), and (3), which are respectively timelike, transverse circular spacelike (1) and (2), and longitudinal spacelike. In Eq. (33), $F_{\nu\rho}$ is accordingly defined in the circular basis. In Appendix B, however, the explicit demonstration of Eqs. (31) is carried out in the Cartesian basis. Equations (31) are, of course, valid in any frame of reference fixed in the laboratory. The longitudinal spacelike and timelike components are the same in the Cartesian and circular basis, while the transverse components can be interrelated with Eqs. (A.1) and (A.2). Equation (22) has been obtained from a Cartesian representation of A_μ, the four potential, using a four curl, Eq. (23), in the Cartesian frame for the spacelike components.

APPENDIX B: EXPLICIT DEMONSTRATION OF EQUATIONS (31)

In this Appendix we provide an explicit demonstration of the self-consistency of Eqs. (31), both for the B_μ and E_μ vectors, because these equations form the basis of our manifestly covariant theory of vacuum electrodynamics. From Eqs. (31a) and the definition of δ_σ in Eq. (30),

$$cB_1 = -\frac{\mathrm{i}}{2\varepsilon_0}(\varepsilon_{1230}F_{23}\delta_0 + \varepsilon_{1320}F_{32}\delta_0 + \varepsilon_{1203}F_{20}\delta_3 + \varepsilon_{1023}F_{02}\delta_3) \tag{B.1}$$

$$cB_2 = -\frac{\mathrm{i}}{2\varepsilon_0}(\varepsilon_{2310}F_{31}\delta_0 + \varepsilon_{2130}F_{13}\delta_0 + \varepsilon_{2013}F_{01}\delta_3 + \varepsilon_{2103}F_{10}\delta_3) \tag{B.2}$$

$$cB_3 = -\frac{\mathrm{i}}{2\varepsilon_0}(\varepsilon_{3210}F_{21}\delta_0 + \varepsilon_{3120}F_{12}\delta_0) \tag{B.3}$$

$$-\mathrm{i}cB_0 = -\frac{\mathrm{i}}{2\varepsilon_0}(\varepsilon_{0123}F_{12}\delta_3 + \varepsilon_{0213}F_{21}\delta_3) \tag{B.4}$$

th

$$\delta_1 = \delta_2 = 0 \qquad \delta_3 = 1 \qquad \delta_0 = -\mathrm{i}$$

$$\begin{array}{llll}
\varepsilon_{0123} = 1 & \varepsilon_{0213} = -1 & \varepsilon_{3120} = -1 & \varepsilon_{3210} = 1 \\
\varepsilon_{2130} = 1 & \varepsilon_{2310} = -1 & \varepsilon_{2013} = 1 & \varepsilon_{2103} = -1 \\
\varepsilon_{1230} = -1 & \varepsilon_{1320} = 1 & \varepsilon_{1203} = 1 & \varepsilon_{1023} = -1
\end{array}$$

$$\begin{array}{ll}
F_{23} = c\varepsilon_0 B_X = -F_{32} & F_{20} = -\mathrm{i}\varepsilon_0 E_Y = -F_{02} \\
F_{31} = c\varepsilon_0 B_Y = -F_{13} & F_{01} = \mathrm{i}\varepsilon_0 E_X = -F_{10} \\
F_{21} = -c\varepsilon_0 B_Z = -F_{12} &
\end{array}$$

or one sense of circular polarization, we have

$$E_X = \frac{E_0}{\sqrt{2}} \qquad E_Y = -\frac{\mathrm{i}E_0}{\sqrt{2}} \qquad B_X = \frac{\mathrm{i}B_0}{\sqrt{2}} \qquad B_Y = \frac{B_0}{\sqrt{2}} \tag{B.5}$$

$$cB_Y = E_X \qquad cB_X = -E_Y \tag{B.6}$$

that in Eqs. (B.1) and (B.2),

$$\begin{aligned} cB_1 &= cB_X - E_Y = 2cB_X \\ cB_2 &= cB_Y + E_X = 2cB_Y \end{aligned} \tag{B.7}$$

nd so the left sides become magnetic components in vacuo with the acuum relation $E_0 = cB_0$.

Similarly, the dual of $F_{\mu\nu}$ is the four tensor[8-15],

$$\tilde{F}_{\mu\nu} \equiv -\varepsilon_0 \begin{bmatrix} 0 & \mathrm{i}E_Z & -\mathrm{i}E_Y & -cB_X \\ -\mathrm{i}E_Z & 0 & \mathrm{i}E_X & -cB_Y \\ \mathrm{i}E_Y & -\mathrm{i}E_X & 0 & -cB_Z \\ cB_X & cB_Y & cB_Z & 0 \end{bmatrix} \tag{B.8}$$

nd in Eq. (31b),

$$E_1 = \tfrac{1}{2}\left(\varepsilon_{1230}\tilde{F}_{23}\delta_0 + \varepsilon_{1320}\tilde{F}_{32}\delta_0 + \varepsilon_{1203}\tilde{F}_{20}\delta_3 + \varepsilon_{1023}\tilde{F}_{02}\delta_3\right) \tag{B.9}$$

$$E_2 = \tfrac{1}{2}\left(\varepsilon_{2310}\tilde{F}_{31}\delta_0 + \varepsilon_{2130}\tilde{F}_{13}\delta_0 + \varepsilon_{2013}\tilde{F}_{01}\delta_3 + \varepsilon_{2103}\tilde{F}_{10}\delta_3\right) \tag{B.10}$$

$$E_3 = +\tfrac{1}{2}\left(\varepsilon_{3210}\tilde{F}_{21}\delta_0 + \varepsilon_{3120}\tilde{F}_{12}\delta_0\right) \tag{B.11}$$

$$-\mathrm{i}E_0 = -\tfrac{1}{2}\left(\varepsilon_{0123}\tilde{F}_{12}\delta_3 + \varepsilon_{0213}\tilde{F}_{21}\delta_3\right) \tag{B.12}$$

With the relations (B.5) and (B.6) it can be shown that the components in Eqs. (B.9) and (B.10) are the electric components $2E_X$ and $2E_Y$, with

$$\tilde{F}_{12} = -\tilde{F}_{21} = \mathrm{i}\varepsilon_0 E_Z \qquad \tilde{F}_{13} = -\tilde{F}_{31} = -\mathrm{i}\varepsilon_0 E_Y$$

$$\tilde{F}_{10} = -\tilde{F}_{01} = -c\varepsilon_0 B_X \qquad \tilde{F}_{23} = -\tilde{F}_{32} = \mathrm{i}\varepsilon_0 E_X$$

Similarly, it may be checked explicitly that

$$\begin{aligned} B_\mu \delta_\mu &\equiv |B_1|\,|\delta_1| + |B_2|\,|\delta_2| + |B_3|\,|\delta_3| - |B_0|\,|\delta_0| \\ &= 0 + 0 + B_Z - B_Z \\ &= 0 \end{aligned}$$

APPENDIX C: THE ELECTRODYNAMICAL ENERGY DENSITY AND TIME-AVERAGED ENERGY DENSITY, OR INTENSITY, I_0

Equation (38) produces the free spacetime result

$$E_\mu E_\mu = 0 \tag{C.1}$$

This is interpreted to mean that the scalar product of the two four vectors E_μ and E_μ is zero in the lightlike condition. In the conventional theory[8-15] the equivalent of Eq. (C.1) is

$$\mathbf{E} \cdot \mathbf{E} = 0 \tag{C.2}$$

Equations (C.1) and (C.2) do not mean, however, that the time-averaged electromagnetic energy density I_0 is zero in vacuo. The quantity I_0 (W m^{-2}) is defined covariantly by

$$\begin{aligned} I_0 &= \varepsilon_0 c E^{(0)2} \\ &\equiv \tfrac{1}{2}\varepsilon_0 c E_\mu E_\mu^* \end{aligned} \tag{C.3}$$

where E_μ^* is the complex conjugate of E_μ in vacuo. Explicitly,

$$\begin{aligned} E_\mu &\equiv (E^{(1)}, E^{(2)}, E^{(3)}, -\mathrm{i}E^{(0)}) \\ E_\mu^* &\equiv (E^{(1)*}, E^{(2)*}, E^{(3)}, -\mathrm{i}E^{(0)}) \end{aligned} \tag{C.4}$$

and

$$E_\mu E_\mu^* = \frac{E^{(0)2}}{2}((\mathbf{i} - \mathrm{i}\mathbf{j}) \cdot (\mathbf{i} + \mathrm{i}\mathbf{j}) + (\mathbf{i} + \mathrm{i}\mathbf{j}) \cdot (\mathbf{i} - \mathrm{i}\mathbf{j})) \qquad \text{(C.5)}$$
$$= 2E^{(0)2}$$

This result shows that $E_\mu E_\mu^*$ is covariantly described because it is a constant in free spacetime. Equation (C.3) is known as the time-averaged energy density[8–15] or beam *intensity*. This is invariant to Lorentz transformation and is a scalar quantity. Note that although E_μ^* is defined in Eq. (C.4) as the complex conjugate of E_μ, the sign of the timelike component $-\mathrm{i}E^{(0)}$ does not change, because the operation $E_\mu \to E_\mu^*$ takes place in a fixed frame of reference $(X, Y, Z, -\mathrm{i}ct)$ in pseudo-Euclidean spacetime. Finally, $E^{(3)}$ is defined as having no imaginary part, and is invariant under $E_\mu \to E_\mu^*$. Thus, $E^{(3)}$ and $-\mathrm{i}E^{(0)}$ do not contribute to I_0.

APPENDIX D: SIMPLE LORENTZ TRANSFORMATION OF E_μ AND B_μ

The simple Lorentz transformation of the four vector E_μ is given covariantly by

$$E'^{(0)} = E^{(0)}\xi; \qquad E'^{(3)} = E^{(3)}\xi \qquad E^{(2)} = E^{(2)\prime} \qquad E^{(1)} = E^{(1)\prime} \qquad \text{(D.1)}$$

and, similarly, the transformation of B_μ is

$$B'^{(0)} = B^{(0)}\xi \qquad B'^{(3)} = B^{(3)}\xi \qquad B^{(2)} = B^{(2)\prime} \qquad B^{(1)} = B^{(1)\prime} \qquad \text{(D.2)}$$

The transform is from the covariantly defined frame $(X, Y, Z, -\mathrm{i}ct)$ to $(X', Y', Z', -\mathrm{i}ct')$; which translates along Z at speed v relative to the former. In Eqs. (D.1) and (D.2),

$$\xi = \frac{1 - v/c}{(1 - v^2/c^2)^{1/2}} \qquad \text{(D.3)}$$

This is referred to as a simple Lorentz transformation because there is no rotation and no translation generator considered. In other words, the origin of frame $(X, Y, Z, -\mathrm{i}ct)$ does not translate, and no rotations are considered in spacetime. For the electromagnetic plane wave in vacuo,

$v = c$, and Eqs. (D.1) and (D.2) give

$$E^{(0)} = E^{(0)\prime} \qquad B^{(0)} = B^{(0)\prime}$$
$$E^{(3)} = E^{(3)\prime} \qquad B^{(3)} = B^{(3)\prime} \qquad \text{(D.4)}$$

which confirm that the equations

$$|\mathbf{E}^{(3)}| = E^{(0)} \qquad |\mathbf{B}^{(3)}| = B^{(0)} \qquad \text{(D.5)}$$

are invariant to the simple Lorentz transformation. The results (D.1) to (D.4) confirm that the E_μ and B_μ fields are the same for all v, because $v = c$ in vacuo, and c is the universal constant of special relativity. Therefore, all four components of both E_μ and B_μ are formally invariant to the simple Lorentz transformation.

It is important to note that this result is fully consistent with, but contains additional information compared with, the standard approach,[12] which applies the simple Lorentz transformation to the four potential vector A_μ and to the second rank tensor $F_{\mu\nu}$. In S.I. units the standard approach gives the well-known result

$$E'_X = \frac{E_X - vB_Y}{(1 - v^2/c^2)^{1/2}}$$
$$E'_Y = \frac{E_Y + vB_X}{(1 - v^2/c^2)^{1/2}} \qquad \text{(D.6)}$$
$$E'_Z = E_Z$$

and using the free space relations

$$cB_Y = E_X \qquad cB_X = -E_Y \qquad \text{(D.7)}$$

we obtain

$$E'_X = \xi E_X \qquad E'_Y = \xi E_Y \qquad E_Z = E'_Z \qquad \text{(D.8)}$$

For $v = 0$, these equations show that the three spacelike components of E_μ (and of B_μ) are separately invariant to Lorentz transformation, but say nothing about the timelike component $\mathbf{E}^{(0)}$ or its relation to $\mathbf{E}^{(3)}$. For this, a more complete theory, as in this paper, is needed. Since the simple Lorentz transformation does not involve the generator of translations, it is an incomplete description of the properties of the electromagnetic field.

The photon is never at rest, but being massless, always moves at the velocity of light c, implying that the origin of frame $(X, Y, Z, -ict)$ also moves at c. The generator of spacetime translations is automatically required, therefore, for a description of the photon, since the latter always translates at c in any frame of reference. Since c is a universal constant, the assumption that there is a frame $(X', Y', Z', -ict')$ which moves at v relative to $(X, Y, Z, -ict)$ conflicts with Einstein's second principle. In other words it is not possible for the photon to define a frame moving at a speed v relative to one that is moving at speed c

Acknowledgments

Many interesting letters were received from S. Kielich of Adam Mickiewicz University, Poznań, Poland, during the course of this work, and are gratefully acknowledged. Many interesting e-mail discussions are acknowledged with K. A. Earle of Cornell University.

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MANIFESTLY COVARIANT THEORY OF THE ELECTROMAGNETIC FIELD IN FREE SPACETIME, PART 2: THE LORENTZ FORCE EQUATION

I. INTRODUCTION

In Part 1 of this series a manifestly covariant theory was developed for the electromagnetic field in free spacetime, in which[1] the electric and magnetic fields are treated as four vectors E_μ and B_μ, all of whose components are physically meaningful. This is a departure from the conventional approach suggested by the recent discovery[2-5] that the longitudinal ((3)) and transverse ((1) and (2)) components of the electromagnetic field are linked by[2]

$$\mathbf{B}^{(3)} = \frac{\mathbf{E}^{(1)} \times \mathbf{E}^{(2)}}{icE_0} \tag{1}$$

Here $\mathbf{B}^{(3)}$ is the longitudinal magnetic field of the electromagnetic plane wave, and $\mathbf{E}^{(1)}$ and $\mathbf{E}^{(2)}$ define the transverse electric fields through

$$\mathbf{E}^{(1)} = E_0\hat{\mathbf{e}}^{(1)}\,e^{i\phi} \tag{2a}$$

$$\mathbf{E}^{(2)} = E_0\hat{\mathbf{e}}^{(2)}\,e^{-i\phi} \tag{2b}$$

where E_0 (proportional to the timelike polarization $E^{(0)}$ is the scalar amplitude of the plane wave, and c is the speed of light in free spacetime, the universal constant of special relativity. The unit vectors $\hat{\mathbf{e}}^{(1)}$ and $\hat{\mathbf{e}}^{(2)}$ are defined in the circular basis[1] by

$$\hat{\mathbf{e}}^{(1)} = \frac{\mathbf{i} - i\mathbf{j}}{\sqrt{2}} \tag{3a}$$

$$\hat{\mathbf{e}}^{(2)} = \frac{\mathbf{i} + i\mathbf{j}}{\sqrt{2}} \tag{3b}$$

$$\hat{\mathbf{e}}^{(1)} \times \hat{\mathbf{e}}^{(2)} = i\hat{\mathbf{e}}^{(3)} \tag{3c}$$

where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in X and Y, mutually orthogonal to the propagation axis Z of the electromagnetic plane wave. The phase ϕ is defined as usual[6-15] by

$$\phi = \omega t - \boldsymbol{\kappa} \cdot \mathbf{r} \tag{4}$$

where $\boldsymbol{\kappa}$ is the wave vector at position $\mathbf{r}$, and ω is the angular frequency at instant t in free spacetime.

Equation (1) is the key to the theory of covariant electrodynamics, essentially because the novel longitudinal field $\mathbf{B}^{(3)}$ is independent of the phase of the plane wave, and thus satisfies the conventionally defined Gauss theorem in differential form. Equation (1) is invariant to the fundamental symmetries, charge conjugation $\hat{C}$, parity inversion $\hat{P}$, and motion reversal $\hat{T}$, i.e., the right and left sides have the same $\hat{C}$, $\hat{P}$, and $\hat{T}$ symmetries.[1-5] Furthermore, the numerator on its right side is proportional[16-18] to the antisymmetric part of the light intensity tensor I_{ij}. The latter is known to be proportional to the third Stokes parameter S_3 and to mediate experimentally observable phenomena, such as antisymmetric light scattering[16-18] and the inverse Faraday effect,[19] and can therefore be considered a nonzero property of a circularly polarized electromagnetic wave in free space. Inter alia, $\mathbf{B}^{(3)}$ from Eq. (1) is similarly nonzero in free space, because the denominator on the right side of Eq. (1) is nonzero for finite $E^{(0)}$. It is more logical to state that the right side of Eq. (1) is nonzero because $\mathbf{B}^{(3)}$ is nonzero rather than the other way around, because $\mathbf{B}^{(3)}$ is a fundamental solution of Maxwell's equations in free spacetime. The quantity $\mathbf{E}^{(1)} \times \mathbf{E}^{(2)}$ is, on the other hand, built up from a cross product of fundamental transverse electric fields. It is clear, however, that if the antisymmetric part of the light intensity is nonzero, then $\mathbf{B}^{(3)}$ is nonzero. In other words, $\mathbf{B}^{(3)}$ is the source of the antisymmetric part of light intensity and all concommitant experimental phenomena. It is worth noting in the context of $\hat{C}$ symmetry[19] that

$$\hat{C}(A_\mu) = -A_\mu \tag{5}$$

where A_μ is the well-known potential four vector in free spacetime. The $\hat{C}$ symmetries of A_μ, E_μ, and B_μ are all negative, so that the concomitant fields of the photon change sign under $\hat{C}$. Although the photon is stated[19] to be its own antiparticle, the antiphoton, generated by $\hat{C}$ from the photon, is associated with electric and magnetic fields of the opposite sign. For this reason, the antiphoton is a distinct entity from the photon. Furthermore, all four components of A_μ, E_μ, and B_μ must change sign under $\hat{C}$; i.e., all four polarizations (0), (1), (2), and (3) change sign. On the other hand, spacelike quantities, such as the propagation vector $\boldsymbol{\kappa}$, by definition are unaffected by $\hat{C}$, so that the $\hat{C}$ operator produces an electromagnetic wave propagating in the same direction, but with all four polarizations reversed. The electromagnetic wave produced in vacuo by $\hat{C}$ defines the antiphoton in the quantum field, a distinct entity from the photon. The fact that the concommitant fields are reversed in sign does not mean that $\mathbf{B}^{(3)}$ of Eq. (1) violates $\hat{C}$ symmetry. In the same way, Eq. (5) does not mean that A_μ violates $\hat{C}$ symmetry in vacuo. We conclude that Eq. (1) satisfies $\hat{C}$, $\hat{P}$, and $\hat{T}$ invariance in vacuo, and is a legitimate equation of electrodynamics.

In Part 1 of this series the vectors E_μ and B_μ were defined in terms of the electromagnetic field four tensor[6–15] $F_{\mu\nu}$, and its dual, $\tilde{F}_{\mu\nu}$. It was shown[1] that both E_μ and B_μ are Pauli-Lubanski types within the Poincaré group (the inhomogeneous Lorentz group), and that the products $E_\mu E_\mu$ and $B_\mu B_\mu$ form Casimir invariants of the Poincaré group. The Maxwell equations, Poynting theorem, and Stokes parameters were derived in manifestly covariant form, and it was shown that phenomena such as natural optical activity, ellipticity, and the electric Kerr effect can be expressed in terms of changes in B_μ (or its electric counterpart E_μ). It was shown that optical absorption can be defined in terms of B_μ and E_μ, and suggestions were made for experiments to detect the magnetizing effect of B_μ and the polarizing effect of E_μ as an electromagnetic wave interacts with matter. In this paper (part 2), the Lorentz force equation is investigated in manifestly covariant form; i.e., a manifestly covariant theory is given of the interaction of an electromagnetic wave with the electron.

In Section II, the Lorentz force equation is derived from the covariant definitions of E_μ and B_μ, and expressed in terms of its magnetic and electric components. Section III examines the individual terms in the equation and shows that in manifestly covariant form, the Lorentz equation contains extra terms that, in principle, produce experimentally observable effects on the electron. There follows a discussion that suggests possible experiments for the detection of the extra manifestly covariant forces on the electron.

II. DERIVATION OF THE MANIFESTLY COVARIANT LORENTZ FORCE EQUATION

Our aim is to derive the equation describing the interaction of E_μ and B_μ with an electron, this being a manifestly covariant description of the interaction of an electromagnetic wave with particulate matter. In Part 1, the four vectors E_μ and B_μ were defined as[1]

$$E_\mu = \tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\tilde{F}_{\nu\rho}\delta_\sigma \tag{6a}$$

$$B_\mu = -\frac{\mathrm{i}}{2c}\varepsilon_{\mu\nu\rho\sigma}F_{\nu\rho}\delta_\sigma \tag{6b}$$

where $F_{\nu\rho}$ is the four curl of A_ν in free spacetime, and $\tilde{F}_{\nu\rho}$ is its dual. The unit tensors $\varepsilon_{\mu\nu\rho\sigma}$ and δ_σ are respectively the totally antisymmetric unit tensor in four dimensions and the unit generator of spacetime translations.[1, 20] These quantities are written out for reference as follows:

$$E_\mu \equiv (E^{(1)}, E^{(2)}, E^{(3)}, -\mathrm{i}E^{(0)}) \tag{7a}$$

$$B_\mu \equiv (B^{(1)}, B^{(2)}, B^{(3)}, -\mathrm{i}B^{(0)}) \tag{7b}$$

$$\delta_\mu \equiv (0, 0, 1, -\mathrm{i}) \tag{7c}$$

$$F_{\mu\nu} \equiv \begin{bmatrix} 0 & cB^{(3)} & -cB^{(2)} & -\mathrm{i}E^{(1)} \\ -cB^{(3)} & 0 & cB^{(1)} & -\mathrm{i}E^{(2)} \\ cB^{(2)} & -cB^{(1)} & 0 & -\mathrm{i}E^{(3)} \\ \mathrm{i}E^{(1)} & \mathrm{i}E^{(2)} & \mathrm{i}E^{(3)} & 0 \end{bmatrix} \tag{7d}$$

$$\tilde{F}_{\mu\nu} \equiv \begin{bmatrix} 0 & -\mathrm{i}E^{(3)} & \mathrm{i}E^{(2)} & cB^{(1)} \\ \mathrm{i}E^{(3)} & 0 & -\mathrm{i}E^{(1)} & cB^{(2)} \\ -\mathrm{i}E^{(2)} & \mathrm{i}E^{(1)} & 0 & cB^{(3)} \\ -cB^{(1)} & -cB^{(2)} & -cB^{(3)} & 0 \end{bmatrix} \tag{7e}$$

The need to define E_μ and B_μ as four vectors in spacetime is a direct consequence of Eq. (1), because the latter implies that there is a relation between the transverse and longitudinal spacelike components of the electromagnetic wave in vacuo. The conventional assertion that longitudinal components be "unphysical"[6–15] is no longer tenable in view of Eq. (1), because if $\mathbf{E}^{(1)}$ and $\mathbf{E}^{(2)}$ be physically meaningful, then so must $\mathbf{B}^{(3)}$. It has been demonstrated[1–5] that the existence of $\mathbf{B}^{(3)}$ implies the existence of $\mathbf{E}^{(3)}$, and quantum field theory[20] leads to

$$\begin{gathered} |\mathbf{B}^{(3)}| - B^{(0)} = 0 \qquad |\mathbf{E}^{(3)}| - E^{(0)} = 0, \\ \mathbf{B}^{(3)} \cdot \mathbf{B}^{(3)} = B^{(0)2} \qquad \mathbf{E}^{(3)} \cdot \mathbf{E}^{(3)} = E^{(0)2} \end{gathered} \tag{8}$$

i.e., that physical states are admixtures of polarizations (3) and (0). Therefore, all four polarizations are physically meaningful. This is consistent with the fact that A_μ has four components.

The Lorentz force equation can be expressed covariantly by

$$f_\mu = F_{\mu\nu}J_\nu \tag{9}$$

where f_μ is a force four vector[6] and J_μ is the charge current four vector. In the conventional theory this is taken to be an adequate description of the interaction of the electric and magnetic components of the electromagnetic field with an electron. The conventional approach, however, assumes that the longitudinal and timelike components of these fields are unphysical, which means essentially that the longitudinal component is set to zero. In view of Eqs. (8) this is an illogical procedure, because if $E^{(3)}$ or $B^{(3)}$ be zero, then so must $E^{(0)}$ and $B^{(0)}$, but the latter are also proportional to the amplitudes of transverse components such as $\mathbf{E}^{(1)}$ and $\mathbf{E}^{(2)}$, and in defining these, $E^{(0)}$ is obviously not zero. The conventional approach is therefore logically inconsistent. In the manifestly covariant theory,[1] on the other hand, this inconsistency is remedied. The inconsistency of the conventional approach is "hidden" by the mathematical nature of the four curl, which defines $F_{\mu\nu}$ as

$$F_{\mu\nu} \equiv \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} \tag{10}$$

since this four curl leaves the timelike components of E_μ and B_μ undefined, i.e., the matrix $F_{\mu\nu}$ contains only the spacelike components on its off diagonals. The conventional antisymmetric tensor $F_{\mu\nu}$ contains no reference to the timelike polarizations $E^{(0)}$ and $B^{(0)}$. It follows, therefore, that the Lorentz force equation in covariant form (9) cannot be manifestly covariant, because $F_{\mu\nu}$ is used to define the Lorentz force vector f_μ. Manifest covariance means that the physically meaningful polarizations (0) and (3) must be taken into consideration when calculating the force on the electron.

This is achieved by solving Eqs. (6a), (6b), and (9) simultaneously as follows. We note firstly the definitions of J_μ and f_μ:

$$J_\mu \equiv \left(\rho\frac{v^{(1)}}{c}, \rho\frac{v^{(2)}}{c}, \rho\frac{v^{(3)}}{c}, \mathrm{i}\rho\right) \tag{11a}$$

$$f_\mu \equiv \left(f^{(1)}, f^{(2)}, f^{(3)}, f^{(0)}\right) \tag{11b}$$

where ρ is the charge density, and $v^{(1)}$, $v^{(2)}$, and $v^{(3)}$ are the spacelike velocity components of the electron. The inverse of J_μ is defined so that

$$J_\mu J_\mu^{-1} = 1 \tag{12}$$

Multiplying both sides of Eq. (9) from the right by J_ν^{-1} we obtain

$$F_{\mu\nu} = f_\mu J_\nu^{-1} \tag{13}$$

so that in Eq. (6b)

$$cB_\mu = -\frac{\mathrm{i}}{2}\varepsilon_{\mu\nu\rho\sigma} f_\nu J_\rho^{-1}\delta_\sigma \tag{14}$$

Multiplying both sides from the right by δ_σ^{-1} yields

$$cB_\mu\delta_\sigma^{-1} = -\frac{\mathrm{i}}{2}\varepsilon_{\mu\nu\rho\sigma} f_\nu J_\rho^{-1} \tag{15}$$

and multiplying both sides of this equation from the right by J_ρ gives

$$cB_\mu J_\rho \delta_\sigma^{-1} = -\frac{\mathrm{i}}{2}\varepsilon_{\mu\nu\rho\sigma} f_\nu \tag{16}$$

Here we have used the fact that

$$J_\rho\delta_\sigma^{-1} = \delta_\sigma^{-1} J_\rho \tag{17}$$

Finally, multiplying both sides of Eq. (16) from the right by δ_σ gives the magnetic part of the Lorentz force equation in manifestly covariant form:

$$cB_\mu J_\rho = -\frac{\mathrm{i}}{2}\varepsilon_{\mu\nu\rho\sigma} f_\nu \delta_\sigma \tag{18}$$

Similarly, the electric part of the Lorentz force equation is

$$E_\mu J_\rho = \tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma} g_\nu \delta_\sigma \tag{19}$$

where g_μ is defined through the dual $\tilde{F}_{\mu\nu}$ of the electromagnetic field tensor $F_{\mu\nu}$:

$$g_\mu \equiv \tilde{F}_{\mu\nu} J_\nu \tag{20}$$

III. COMPONENTS OF THE MANIFESTLY COVARIANT LORENTZ EQUATION

In this section the structure of the tensor Eqs. (18) and (19) is investigated for individual terms, and the result is compared with the conventional Lorentz force equation for an electromagnetic plane wave interacting with

TABLE I
Summary: Generalized Lorentz Equation

Force	Components
f_0	E_1J_1i, E_2J_2i
	New E_3J_3i
f_1	$-2cB_2J_3$
	$2cB_2\rho = 2E_1\rho$
	New $2cB^{(0)}J_2 = 2cB^{(3)}J_2$
f_2	$2cB_1J_3$
	$-2cB_1\rho = 2E_2\rho$
	New $-2cB^{(3)}J_1 = -2cB^{(0)}J_1$
f_3	$c(B_2J_1 - B_1J_2)$
	New $-E_3\rho$

an electron. Note that in the conventional approach, only the two transverse components (1) and (2) exist, the components (0) and (3) are discarded as unphysical. In our manifestly covariant approach, components (0), (1), (2), and (3) are physically meaningful. Explicit calculations are given, and the individual results from Eq. (18) are presented in Table I.

A. Components f_1 and f_2

The Lorentz force on the electron due to B_0 is given as follows:

$$\begin{aligned} cB_0J_2 &= -\frac{i}{2}\varepsilon_{0123}f_1\delta_3 \\ cB_0J_1 &= -\frac{i}{2}\varepsilon_{0213}f_2\delta_3 \end{aligned} \tag{21}$$

and with the definitions $B_0 = -iB^{(0)}$ and $\delta_3 \equiv 1$ we obtain the 1 and 2 components of f_μ

$$\begin{aligned} f_1 &= 2cB^{(0)}J_2 = 2\rho v_2 B^{(0)} \\ f_2 &= -2cB^{(0)}J_1 = -2\rho v_1 B^{(0)} \end{aligned} \tag{22}$$

Similarly, it may be shown that

$$\begin{aligned} f_1 &= 2cB^{(3)}J_2 \\ f_2 &= -2cB^{(3)}J_1 \end{aligned} \tag{23}$$

so that

$$\begin{aligned} f_1 &= 2cB^{(0)}J_2 = 2cB^{(3)}J_2 \\ f_2 &= -2cB^{(0)}J_2 = -2cB^{(3)}J_2 \end{aligned} \tag{24}$$

which is consistent with Eq. (8), i.e., with the quantum theoretical result that physical photon states are admixtures of (0) and (3) polarizations such that Eq. (8) holds.

Clearly, the forces in Eq. (24) are absent from the conventional Lorentz equation. It is seen by inspection of Eqs. (22) and (23) that they have precisely the same form as the equations of motion[8] of a charge in a static magnetic field $\mathbf{B}^{(3)}$, whose magnitude is equal to $B^{(0)}$. This is consistent with the phase-independent definition of $\mathbf{B}^{(3)}$, Eq. (1), although $\mathbf{B}^{(3)}$ is generated by a photon traveling at the speed of light and cannot be regarded as a conventional magnetostatic field. It is a longitudinal magnetic field which travels with the photon at the speed of light. In principle, an experiment can be devised for measuring these extra forces on the electron in the manifestly covariant theory. This possibility is discussed further below.

Additional contributions to f_1 and f_2 arise from the timelike component of the charge current four vector J_μ, but unlike the contributions from Eq. (24), these are also present in the conventional theory. They arise as follows:

$$\begin{aligned} cB_1J_0 &= -\frac{i}{2}\varepsilon_{1203}f_2 \\ f_2 &= -2cB_1\rho \end{aligned} \tag{25}$$

and using the relation in circular polarization,

$$cB_1 = -E_2 \tag{26}$$

we obtain

$$f_2 = 2E_2\rho \tag{27}$$

Similarly,

$$f_1 = 2cB_2\rho \tag{28}$$

and using

$$cB_2 = E_1 \tag{29}$$

we obtain

$$f_1 = 2E_1\rho \tag{30}$$

B. The f_0 and f_3 Forces

The timelike f_0 components from Eq. (18) are obtained by

$$\begin{aligned} f_0 &= -2cB_1J_2i = E_2J_2i \\ f_0 &= 2cB_2J_1i = E_1J_1i \end{aligned} \tag{31}$$

and correspond to the well-known[8] $\mathbf{E} \cdot \mathbf{J}$ force from the conventional Lorentz force equation in covariant from. Therefore, the manifestly covariant Eq. (18) provides no new terms in $\mathbf{E} \cdot \mathbf{J}$.

The f_3 component is obtained from

$$f_0 + \mathrm{i}f_3 = \frac{2}{\mathrm{i}}cB_1J_2 \tag{32}$$

together with

$$f_0 + \mathrm{i}f_3 = -\frac{2}{\mathrm{i}}cB_2J_1 \tag{33}$$

Solving Eqs. (32) and (33) simultaneously gives

$$\begin{aligned} f_0 &= 0 \\ f_3 &= c(B_2J_1 - B_1J_2) \end{aligned} \tag{34}$$

i.e., there is no contribution to f_0 from the B_2 and B_1 components interacting with J_1 and J_2 respectively, and the overall f_3 component is the same as that in the conventional Lorentz force equation. Finally, there are manifestly covariant forces:

$$\begin{aligned} f_0 &= F_{03}J_3 = \mathrm{i}E_3J_3 \\ f_3 &= F_{30}J_0 = -E_3\rho \end{aligned} \tag{35}$$

direct from Eq. (9), the equation from which (18) is derived using (6a) and (6b).

These results are summarized in Table I, which shows that there are additions to the conventional f_0, f_1, f_2, and f_3 in the manifestly covariant description. From Table I, Eq. (18) reduces to the conventional Lorentz equation if $\mathbf{B}^{(3)} = \mathbf{0}$. However, this assumption of the conventional approach is illogical, because it conflicts with Eq. (1).

IV. DISCUSSION

The question arises immediately as to whether the extra terms in Table I marked "new" are observable experimentally. This type of observation might be able to distinguish between the conventional theory and the manifestly covariant approach based on Eq. (1). It is probably difficult to isolate a single electron in a vacuum in order to test the theory directly, but the use of electron beams may be feasible. The resultant force on a single electron due to an electromagnetic plane wave is given in the manifestly covariant approach by a combination of Eqs. (9) (the conventional equation) and (18), taking into account thereby the existence of Eqs. (1) and (8). In the conventional approach, the force is described by the spacelike components of Eq. (9) alone, and Eqs. (1), (8), and (18) are not considered.

The conventional calculation of the trajectory of an electron in a monochromatic, circularly polarized plane wave is a standard problem (e.g., Landau and Lifshitz,[10] p. 118), in which there are no linear forces such as $-E_3\rho$ of the manifestly covariant theory. Presumably, such a force would cause the linear deflection of an electron beam when the latter is acted upon by a circularly polarized electromagnetic beam, such as an X-ray beam. However, this assumption does not consider statistical effects in either the electromagnetic or electron beam. Extra Lorentz precession terms due to f_1 and f_2 are expected in the manifestly covariant theory. If extra forces can be observed unequivocally, this would add to the considerable experimental evidence for the manifestly covariant theory reviewed in Part 1 of this series, evidence from sources such as absorption of circularly polarized light, circular dichroism, ellipticity, and the electric Kerr effect.

Acknowledgments

The author acknowledges many interesting letters from Stanislaw Kielich during the course of this work, and many interesting discussions with F. Farahi and other members of the physics department at UNCC.

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MANIFESTLY COVARIANT THEORY OF THE ELECTROMAGNETIC FIELD IN FREE SPACETIME, PART 3: $\hat{C}$, $\hat{P}$, AND $\hat{T}$ SYMMETRIES

I. INTRODUCTION

It has recently been observed[1–5] that there exists an equation of electrodynamics in vacuo that defines a longitudinal magnetic field, $\mathbf{B}^{(3)}$, which is independent of the phase of the electromagnetic plane wave, thus showing for the first time that there exist physically meaningful longitudinal solutions to Maxwell's equations in vacuo. Parts 1 and 2 of this series[1, 2] developed the theory of manifestly covariant electrodynamics from this basic observation, and recent work by Farahi and Evans[4] has shown that the existence of $\mathbf{B}^{(3)}$ implies the existence of its longitudinal electric counterpart $\mathrm{i}\mathbf{E}^{(3)}$. In Part 1[1] it was shown that $\mathrm{i}\mathbf{E}^{(3)}$ and $\mathbf{B}^{(3)}$ do not contribute to the electromagnetic energy density, and that Poynting's theorem can be expressed in terms of four, rather than two, polarizations. The existence of four photon polarizations, (0), (1), (2), and (3), was reconciled with two photon helicities, +1 and −1, by noting[1] that the helicities can be defined in terms either of (0) and (3) or of (1) and (2). Here (0) denotes the timelike photon polarization, (1) and (2) the transverse spacelike, and (3) the longitudinal spacelike. In Part (2), the Lorentz force equation was expressed in manifestly covariant form.

In this paper (Part 3), the fundamental symmetries of physics are applied to the basic equation

$$\mathbf{B}^{(3)} = \frac{\mathbf{E}^{(1)} \times \mathbf{E}^{(2)}}{\mathrm{i} E_0 c} \tag{1}$$

of manifestly covariant electrodynamics (MCE). Here $\mathbf{B}^{(3)}$ is linked[1–5] to the transverse, oscillating, electric fields $\mathbf{E}^{(1)}$ and $\mathbf{E}^{(2)}$ of the plane wave in vacuo, where c is the speed of light. Here $\mathbf{E}^{(1)}$ is the complex conjugate of $\mathbf{E}^{(2)}$:

$$\mathbf{E}^{(1)} \equiv E_0 \hat{\mathbf{e}}^{(1)} \mathrm{e}^{\mathrm{i}\phi} \tag{2a}$$

$$\mathbf{E}^{(2)} \equiv E_0 \hat{\mathbf{e}}^{(2)} \mathrm{e}^{-\mathrm{i}\phi} \tag{2b}$$

where

$$\phi = \omega t - \boldsymbol{\kappa} \cdot \mathbf{r} \tag{3}$$

is the phase of the plane wave, with, as usual, ω as the angular frequency at instant t, $\boldsymbol{\kappa}$ the wave vector at position $\mathbf{r}$. The circular basis[6, 7] is used to define the unit vectors $\hat{\mathbf{e}}^{(1)}$ and $\hat{\mathbf{e}}^{(2)}$:

$$\hat{\mathbf{e}}^{(1)} = \frac{1}{\sqrt{2}}(\mathbf{i} - \mathrm{i}\mathbf{j}) \tag{4a}$$

$$\hat{\mathbf{e}}^{(2)} = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathrm{i}\mathbf{j}) \tag{4b}$$

where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in axes X and Y of the Cartesian frame (X, Y, Z).

In Section II, it is shown that Eq. (1) is invariant under the following conditions:

1. The charge conjugation operator $\hat{C}$, which changes the sign of charge in classical electrodynamics, and in particle physics produces the antiparticle from the original particle
2. The parity inversion operator $\hat{P}$
3. The motion reversal operator $\hat{T}$

In other words, the left and right sides of Eq. (1) remain balanced after application of $\hat{C}$, $\hat{P}$, and $\hat{T}$ to each variable on both sides. Equation (1) is therefore a legitimate equation of electrodynamics, and $\mathbf{B}^{(3)}$ has the $\hat{C}$, $\hat{P}$, and $\hat{T}$ symmetries, and units, of magnetic flux density. $\mathbf{B}^{(3)}$ is also a solution of Maxwell's equations[1–5] in vacuo, and is therefore a real, physically meaningful, longitudinal magnetic field with polarization (3). It has been shown in Parts 1 and 2 of this series that as a direct consequence, electrodynamics (both classical and quantum) must be made manifestly covariant in nature.

In Section II, the fundamental symmetries $\hat{C}$, $\hat{P}$, and $\hat{T}$ are applied to electromagnetic radiation in vacuo, represented by the helicity λ and the potential four vector A_μ. These are the two fundamental elements of the electromagnetic plane wave in vacuo. The helicity λ is negative to $\hat{P}$, and is a number, $+1$ or -1. In contemporary quantum field theory[8] λ is defined for the massless electromagnetic gauge field as the ratio of the Pauli-Lubansky pseudovector W_μ to the generator of spacetime translations P_μ. It is related in the lightlike condition to the second (spin) Casimir invariant of the inhomogeneous Lorentz group (or Poincaré group). The first (mass) Casimir invariant is zero for the electromagnetic field, and so λ is the only nonzero quantity that is invariant to the most general type of Lorentz transformation in the theory of special relativity. The Lorentz invariant spacetime character of the electromagnetic wave is described therefore in terms of λ. In the quantum field the photon is described by two helicities, $+1$ and -1. On the other hand, the concommitant electrodynamic properties of the electromagnetic field in vacuo are described by d'Alembert's equation:

$$\Box A_\mu = 0 \tag{5}$$

where $\Box$ is the d'Alembertian and A_μ the potential four vector. The electric and magnetic parts of the electromagnetic field can be described in terms of A_μ (Refs. 9–14). It is therefore necessary and sufficient to describe electromagnetism in vacuo in terms of the fundamental spacetime quantity λ, and the fundamental electrodynamic quantity A_μ. Section III therefore considers $\hat{C}$, $\hat{P}$, and $\hat{T}$ symmetry applied to λ and A_μ, and defines the response of the electromagnetic field to $\hat{C}$, $\hat{P}$, and $\hat{T}$ in terms of λ and A_μ. Specifically, it is shown that nonzero longitudinal solutions of Maxwell's equations are consistent with $\hat{C}$, $\hat{P}$, and $\hat{T}$ in vacuo. Finally, a detailed discussion is given of the correct way in which to apply $\hat{C}$, $\hat{P}$, and $\hat{T}$ to manifestly covariant electrodynamics, addressing some misconceptions in the recent literature.[15]

II. THE $\hat{C}$, $\hat{P}$, AND $\hat{T}$ SYMMETRIES OF THE FUNDAMENTAL EQUATION (1) OF MCE

We first note that the numerator on the right side of Eq. (1) is the antisymmetric part of the light intensity tensor[16–21] of the standard

literature. It is a nonzero quantity in vacuo whose absolute magnitude is the same as the absolute magnitude of the Stokes parameter S_3 of circularly polarized light.[6] The denominator in Eq. (1) is the product of the scalar amplitude, E_0, of the electric component of the radiation with the speed of light c, and is also nonzero. The quantity $\mathbf{B}^{(3)}$ is therefore nonzero in general, provided that there is some element of circular polarity. $\mathbf{B}^{(3)}$ changes sign with the sense of circular polarity, as does $\mathbf{E}^{(1)} \times \mathbf{E}^{(2)}$ (Refs. 1–5). Since $\mathbf{B}^{(3)}$ is a magnetic field, and is physically meaningful, conventional electrodynamics becomes untenable, because it is no longer sufficient to consider just two polarizations ((1) and (2)) and to arbitrarily discard[10–14] polarization (3) as being "physically meaningless."

It is fundamentally important, therefore, to show that Eq. (1) conserves the symmetries of physics, $\hat{C}$, $\hat{P}$, and $\hat{T}$, and is therefore legitimate in all respects as an equation of electrodynamics in vacuo, because Eq. (1) must mean that conventional electrodynamics is an incomplete description, both in the classical and quantum fields.

A. $\hat{C}$ Symmetry

The charge conjugation operator $\hat{C}$ is defined as[22]

$$\hat{C}(A_\mu) = -A_\mu \tag{6}$$

and in particle physics, the photon, represented by A_μ, is negative to $\hat{C}$, being changed to the antiphoton. By definition, all spacetime quantities are unaffected by $\hat{C}$. Therefore,

$$\hat{C}(\lambda) = \lambda \tag{7}$$

From this, it follows that $\hat{C}$ changes the sign of the scalar amplitudes E_0 and B_0 of the plane wave in vacuo, and therefore changes the sign of the timelike and all spacelike components of the manifestly covariant four-vectors[1, 2] E_μ and B_μ. Thus,

$$\begin{aligned} \hat{C}(\mathbf{B}^{(3)}) &= -\mathbf{B}^{(3)} & \hat{C}(\mathbf{E}^{(1)} \times \mathbf{E}^{(2)}) &= \mathbf{E}^{(1)} \times \mathbf{E}^{(2)} \\ \hat{C}(E_0) &= -E_0 & \hat{C}(ic) &= ic \end{aligned} \tag{8}$$

so that it is clear that Eq. (1) conserves $\hat{C}$ symmetry. (The $\hat{C}$ symmetry of both sides of Eq. (1) is negative.)

B. $\hat{P}$ Symmetry

The $\hat{P}$ operator,[23] parity inversion, is defined as $\hat{P}(\mathbf{r}) = -\mathbf{r}$; $\hat{P}(\mathbf{v}) = -\mathbf{v}$; where $\mathbf{r}$ and $\mathbf{v} = \dot{\mathbf{r}}$ are position and velocity, respectively. It has been shown[23] that

$$\hat{P}(\mathbf{E}^{(1)} \times \mathbf{E}^{(2)}) = \mathbf{E}^{(1)} \times \mathbf{E}^{(2)} \tag{9}$$

The $\hat{P}$ symmetry of magnetic flux density is positive,[23] and since c and $E^{(0)}$ are scalars, Eq. (1) conserves $\hat{P}$ symmetry (both sides are positive).

C. $\hat{T}$ Symmetry

The $\hat{T}$ operator,[23] motion reversal, is defined as $\hat{T}(\mathbf{r}) = \mathbf{r}$; $\hat{T}(\mathbf{v} = \dot{\mathbf{r}}) = -\mathbf{v}$; and reverses all motions in the same frame of reference. It has been shown[23] that

$$\hat{T}(\mathbf{E}^{(1)} \times \mathbf{E}^{(2)}) = -\mathbf{E}^{(1)} \times \mathbf{E}^{(2)} \tag{10}$$

The $\hat{T}$ symmetry of magnetic flux density is negative,[23] and since c and $E^{(0)}$ are scalars, they are positive to $\hat{T}$. Therefore, Eq. (1) conserves $\hat{T}$ symmetry (both sides are negative).

Therefore, Eq. (1) conserves $\hat{C}$, $\hat{P}$, and $\hat{T}$, the fundamental symmetries of physics, and is a legitimate equation of electrodynamics in vacuo. $\mathbf{B}^{(3)}$ has the $\hat{C}$, $\hat{P}$, $\hat{T}$ symmetries, and units, of magnetic flux density in vacuo, and is a solution of Maxwell's equations in vacuo. It is therefore physically meaningful, longitudinal, and phase independent.

III. THE FUNDAMENTAL SYMMETRIES OF THE ELECTROMAGNETIC PLANE WAVE

Since $\mathbf{B}^{(3)}$ (and its electric counterpart $\mathbf{E}^{(3)}$) are physically meaningful solutions of Maxwell's equations, they must be invariant to $\hat{C}$, $\hat{P}$, and $\hat{T}$, in the same way that the well-accepted, oscillating, transverse solutions (1) and (2) are invariant to $\hat{C}$, $\hat{P}$, and $\hat{T}$. The invariance of Eq. (1) is already sufficient proof that $\mathbf{B}^{(3)}$ satisfies these basic symmetry constraints in vacuo. However, a set of self-consistent rules is necessary by which the symmetries of electromagnetic radiation can be identified in terms of its most fundamental variables. We take these to be the helicity λ and the potential four vector A_μ for reasons given already.

The symmetry properties of the electromagnetic wave in vacuo can now be defined as follows:

$$\begin{aligned} [\lambda, A_\mu] &\xrightarrow{\hat{C}} [\lambda, -A_\mu] \\ [\lambda, A, \phi] &\xrightarrow{\hat{P}} [-\lambda, -A, \phi] \\ [\lambda, A, \phi] &\xrightarrow{\hat{T}} [\lambda, -A, \phi] \\ A_\mu &\equiv (A, \mathrm{i}\phi) \end{aligned} \tag{11}$$

Therefore, the $\hat{C}$ operator leaves the spacetime quantity λ unchanged by definition, while changing the sign of A_μ by definition. $\hat{C}$ thus produces a distinct entity which we identify classically as the antiwave and quantum mechanically as the antiphoton, since, by definition, $\hat{C}$ produces the antiparticle from the original particle. The antiwave is defined as the classical electromagnetic entity with the same λ as the original wave but with reversed A_μ and therefore with concommitant electric and magnetic fields of the opposite sign. The spacetime parameter λ of the antiwave is the same as that of the original wave, while the electrodynamic parameter A_μ of the antiwave is opposite in sign. This emphasizes that the antiwave is a distinct entity from the wave.

The $\hat{P}$ operator reverses the sign of λ by definition. The $\hat{P}$ symmetry of the spacelike part of A_μ (the vector potential) is negative, and that of the timelike part (the scalar potential) is positive. $\hat{P}$ again produces a distinct entity, classically the wave with opposite helicity, and quantum mechanically the photon with opposite helicity. The $\hat{T}$ operator does not change the sign of λ, and the $\hat{T}$ symmetry of the spacelike part of A_μ is negative, while that of the timelike part is positive. $\hat{T}$ again produces a distinct entity from the original wave or photon.

Therefore, distinct entities are produced by the application of all three symmetries, $\hat{C}$, $\hat{P}$, and $\hat{T}$, to the two fundamental properties, λ and A_μ of the classical electromagnetic wave or quantized photon. We denote λ and A_μ as symmetry elements of electromagnetic radiation in vacuo. Note that in the above, we have implicitly assumed that the scalar potential is nonzero, and have thus worked in a gauge such as the Lorentz gauge that allows this. In the Coulomb gauge it is assumed that the scalar potential is zero, but this loses manifest covariance unless a zero scalar potential be regarded as physically meaningful. The $\hat{C}$, $\hat{P}$, and $\hat{T}$ symmetries of a zero scalar potential are, however, the same as a nonzero scalar potential, respectively, negative, positive, and positive. In the Coulomb gauge, therefore, the $\hat{C}$, $\hat{P}$, and $\hat{T}$ symmetries are no different from a manifestly covariant gauge such as the Lorentz gauge. Conventionally, gauge invariance means that electric and magnetic fields obtained are invariant to gauge transformation. In MCE, however, it is necessary to regard the scalar potential as physically meaningful, because Eq. (1) implies the existence of four physically meaningful polarizations. Equation (1) automatically satisfies the principle of gauge invariance, because the longitudinal magnetic field $\mathbf{B}^{(3)}$ is formed from the vector product of two transverse electric fields $\mathbf{E}^{(1)}$ and $\mathbf{E}^{(2)}$ which are separately gauge invariant.

IV. DISCUSSION

In Sections II and III it has been demonstrated that Eq. (1), the key equation of manifestly covariant electrodynamics, is invariant to the fundamental symmetries of physics, and defines a quantity $\mathbf{B}^{(3)}$, which is a physically meaningful, gauge invariant, magnetic field. Equation (1) defines $\mathbf{B}^{(3)}$, a solution in vacuo of Maxwell's equations, in terms of other components of the electromagnetic plane wave in vacuo, thus showing that if the transverse components are physically meaningful, then so must be the longitudinal, making the conventional view of electrodynamics untenable, both in the classical and quantum fields.

The fundamental symmetries $\hat{C}$, $\hat{P}$, and $\hat{T}$ have been applied to the symmetry elements λ and A_μ of the electromagnetic field, and it has been shown that $\hat{C}$, $\hat{P}$, and $\hat{T}$ all produce distinct entities, or "distinct situations" by operating on the original entity or situation. The symmetry elements have been defined as λ and A_μ because these parameters are necessary and sufficient to define the spacetime and electrodynamic properties, respectively, of electromagnetic radiation in vacuo. The spacetime symmetry element λ has been chosen because it is the only nonzero Casimir invariant of the Poincaré (inhomogeneous Lorentz) group for electromagnetic radiation; and the symmetry element A_μ has been chosen because it is the only electrodynamic element that appears in d'Alembert's equation.

It is important to find a reasonable (i.e., objective) basis such as this for the definition of symmetry elements for electromagnetic radiation in vacuo. Other, arbitrary, choices of symmetry elements can (i.e., may or may not) lead to erroneous conclusions that conflict with the symmetry invariance of Eq. (1). For example, Barron[15] has recently examined the symmetry of electromagnetic radiation in vacuo using three symmetry elements, which appear to have been chosen subjectively. Since λ and A_μ are sufficient to describe the spacetime and electrodynamic properties of the radiation, Barron has one superfluous element in the three chosen, these being[15] the wave vector $\boldsymbol{\kappa}$, the sense of rotation, and the axial magnetic field $\mathbf{B}^{(3)}$. It is clear that the first two of these elements can be combined into one, the helicity, which can be regarded as a product of the linear and angular momenta of the electromagnetic radiation, and that the third, $\mathbf{B}^{(3)}$, is related to the potential four vector A_μ, and can be expressed in terms of A_μ. Barron asserts that since $\boldsymbol{\kappa}$ is unchanged by $\hat{C}$, and $\mathbf{B}^{(3)}$ is reversed in sign by $\hat{C}$, then $\mathbf{B}^{(3)}$ must be zero. In coming to this conclusion, he asserts that "the photon is its own antiphoton." However, Barron's result conflicts with our explicitly demonstrated symmetry invariance of Eq. (1), and with the fact that the numerator and denominator on the right side of Eq. (1) are both nonzero in general. His assertion that the photon is its own antiphoton conflicts with the symmetry equation

$$[\lambda, A_\mu] \xrightarrow{\hat{C}} [\lambda, -A_\mu] \tag{12}$$

i.e., $\hat{C}$ changes the sign of A_μ while leaving λ unchanged, and thus produces the antiwave (or antiphoton) from the original wave or photon. Barron's choice of three symmetry elements has therefore led to the incorrect conclusion that $\mathbf{B}^{(3)}$ is zero.

In the context of $\hat{T}$ symmetry, Barron,[15] on the other hand, concludes that $\hat{T}$ applied to his three elements does not rule out $\mathbf{B}^{(3)}$. Barron argues that $\hat{T}$ does not produce a distinct situation because the three symmetry elements he uses are all changed in sign by $\hat{T}$, and therefore $\hat{T}$ does not produce a distinct situation. However, we have seen in Section III that the use of the fundamental symmetry elements λ and A_μ produces a distinct situation when operated upon by $\hat{T}$. For this reason, $\hat{T}$ does not rule out the existence of the longitudinal field $\mathbf{B}^{(3)}$. Similarly, $\hat{P}$ acting on λ and A_μ produces a distinct situation that does not rule out $\mathbf{B}^{(3)}$. The choice of symmetry elements is therefore critically important to any argument based on $\hat{C}$, $\hat{P}$, and $\hat{T}$ symmetry that purports to show the existence or nonexistence of electric and/or magnetic fields in vacuo.

Several other arguments may be used to demonstrate why Barron's choice of symmetry elements has led to the erroneous conclusion that $\mathbf{B}^{(3)}$ is zero. These are discussed in detail as follows.

The $\hat{C}$ symmetry of all components of A_μ is negative, so that it follows by Barron's argument that oscillating transverse components such as $\mathbf{E}^{(1)}$ and $\mathbf{E}^{(2)}$ are also zero in the electromagnetic plane wave in vacuo, an erroneous conclusion. Barron's argument also implies, incorrectly, that the scalar amplitude, E_0, of the plane wave is zero for the following reason. $\hat{C}$ is a symmetry that acts on a scalar, such as charge, reversing its sign, and by definition $\hat{C}$ leaves all spacetime quantities unchanged. Any electric or magnetic field can be expressed as the product of scalar amplitude with a vector, e.g.,

$$\mathbf{E} = E_0 \zeta \tag{13}$$

where ζ is a vector, a spacetime quantity. Regardless of whether ζ is transverse or longitudinal, it is unchanged by $\hat{C}$ by definition, and E_0 reverses sign by definition when operated upon by $\hat{C}$. The product

$$\hat{C}(E_0)\hat{C}(\zeta) \tag{14}$$

is therefore always negative, and it cannot be deduced on the grounds of $\hat{C}$ symmetry that in one direction the field is zero, and in orthogonal (or any other) directions nonzero, since direction, by definition, is a spacetime quantity invariant to $\hat{C}$. In manifestly covariant electrodynamics, E_0 is the timelike component of the electric field,[1, 2] a nonzero quantity.

Maxwell's equations in vacuo are invariant to $\hat{C}$, $\hat{P}$, and $\hat{T}$. It follows that all legitimate solutions of Maxwell's equations in vacuo are also invariant to $\hat{C}$, $\hat{P}$, and $\hat{T}$, and Eq. (1) shows that the novel longitudinal solution $\mathbf{B}^{(3)}$ is so. The solution $\mathbf{B}^{(3)}$ cannot violate $\hat{C}$ because the equation to which it is a solution does not violate $\hat{C}$. Similarly, transverse solutions to Maxwell's equations, such as $\mathbf{E}^{(1)}$ and $\mathbf{E}^{(2)}$, conserve $\hat{C}$, $\hat{P}$, and $\hat{T}$ in vacuo. Underpinning Barron's argument is a subjective choice of three symmetry elements, described already, and the assumption that the photon and antiphoton are the same in all respects, i.e., one is not "distinct" from the other. We have argued that the photon and antiphoton are different entities, i.e.:

$$\underset{\text{Photon}}{[\lambda, A_\mu]} \xrightarrow{\hat{C}} \underset{\text{Antiphoton}}{[\lambda, -A_\mu]} \tag{15}$$

and that the choice of λ and A_μ as symmetry elements is rooted in contemporary theory of electromagnetic radiation. Only two elements are needed to define the symmetry of electromagnetic radiation in vacuo, one being an invariant of the Poincaré group, the other being the single variable of the d'Alembert equation.

Barron, therefore, bases his argument[15] on the assumption that the photon and antiphoton are indistinct, so that in an indistinct situation all variables must be relatively the same, so that $\mathbf{B}^{(3)}$ relative to $\boldsymbol{\kappa}$ must not change when both are acted upon by $\hat{C}$. We argue that $\hat{C}$ operates to produce a distinct situation, embodied in the antiwave, or antiphoton, and in a distinct situation, it is no longer reasonable to expect that all variables must be relatively unchanged, so that $\mathbf{B}^{(3)}$ may change sign with respect to

$\boldsymbol{\kappa}$, and so may E_0, $\mathbf{E}^{(1)}$, and $\mathbf{E}^{(2)}$. Even within the framework of his own argument, Barron has shown only that *either* $\mathbf{B}^{(3)}$ or $\boldsymbol{\kappa}$ must be zero, so that on his grounds $\boldsymbol{\kappa}$ may be zero and $\mathbf{B}^{(3)}$ nonzero. It is therefore not possible to assert unequivocally, even within his own argument, that $\mathbf{B}^{(3)}$ is zero.

Barron proceeds to argue, on the basis of his three symmetry elements and on the basis of his assertion that the photon and the antiphoton are distinct, that $\hat{C}$ symmetry does not imply that the inverse Faraday effect,[24] magnetization by a circularly polarized laser, cannot exist. Before commenting on Barron's viewpoint in this context, we note that the inverse Faraday effect is accommodated straightforwardly by Eq. (1). This is because $\mathbf{E}^{(1)} \times \mathbf{E}^{(2)}$ interacts with a $\hat{C}$-negative rank three property tensor to induce a magnetic dipole moment.[25] Similarly, $\mathbf{B}^{(3)}$ interacts with a $\hat{C}$-positive rank two molecular property tensor (the susceptibility) to induce a magnetic dipole moment.[26]

In Barron's viewpoint, on the other hand, $\mathbf{B}^{(3)}$ is zero, so that by Eq. (1), $\mathbf{E}^{(1)} \times \mathbf{E}^{(2)}$ is zero. Despite this, his argument asserts that there is a nonzero inverse Faraday effect, showing conclusively that his viewpoint is illogical. The root error in Barron's approach is that under $\hat{C}$, both $\mathbf{E}^{(1)}$ and $\mathbf{E}^{(2)}$ are separately negative, because they are electric fields of the antiwave, which is distinct from the wave. His assertion that the wave and antiwave are in all respects identical (i.e., "indistinct") implies in his view that $\mathbf{E}^{(1)}$ and $\mathbf{E}^{(2)}$ must be separately zero, and that the product $\mathbf{E}^{(1)} \times \mathbf{E}^{(2)}$ is zero. This means that there is no inverse Faraday effect, in direct conflict with experimental data.[24] It may be argued in Barron's favor[27] that $\mathbf{E}^{(1)} \times \mathbf{E}^{(2)}$ and quantities such as $\mathbf{E}^{(1)} \times \mathbf{B}^{(2)}$, do not change sign with $\hat{C}$, but this is spurious, because neither is an electric or a magnetic field. On these albeit spurious grounds it can be similarly asserted that $\mathbf{B}^{(3)}$ is nonzero because quantities such as $\mathbf{B}^{(3)} \times \mathbf{E}^{(1)}$ and $\mathbf{B}^{(3)} \times \mathbf{B}^{(1)}$ do not change sign under $\hat{C}$, so that $\mathbf{B}^{(3)}$ is nonzero.

It is clear that energy is invariant to $\hat{C}$ and also that, in our viewpoint, the interaction energy of the antiwave with antimatter is identical with the interaction energy of the wave and matter, for example:

$$\begin{aligned} \hat{C}(\boldsymbol{\mu} \cdot \mathbf{E}^{(1)}) &= (-\boldsymbol{\mu}) \cdot (-\mathbf{E}^{(1)}) = \boldsymbol{\mu} \cdot \mathbf{E}^{(1)} \\ \hat{C}(\mathbf{m} \cdot \mathbf{B}^{(3)}) &= (-\mathbf{m}) \cdot (-\mathbf{B}^{(3)}) = \mathbf{m} \cdot \mathbf{B}^{(3)} \end{aligned} \tag{16}$$

where $\boldsymbol{\mu}$ is an electric and $\mathbf{m}$ a magnetic dipole moment. It is therefore quite natural in our argument that the inverse Faraday effect and similar effects may exist in nature, because they are governed by an interaction energy that is invariant under the basic symmetries of physics, $\hat{C}$, $\hat{P}$, and

$\hat{T}$. Thus, in the inverse Faraday effect, for example, $\mathbf{B}^{(3)}$ forms an interaction energy with an electronic magnetic dipole moment, and $\mathbf{E}^{(1)} \times \mathbf{E}^{(2)}$ with an electronic antisymmetric polarizability, both types of interaction energy being invariant under $\hat{C}$, $\hat{P}$, and $\hat{T}$.

In Barron's[15] view, however, the argument is convoluted and obscure, because if the wave be indistinct from the antiwave, as in his view, the interaction energy of the antiwave with antimatter, produced by $\hat{C}$ from matter, is no longer invariant under $\hat{C}$. Obviously, matter must be distinct from antimatter, but if wave be indistinct from antiwave, it follows that the interaction energy of antiwave with antimatter is opposite in sign to the interaction energy of wave with matter, an insupportable conclusion because all forms of energy must be indistinct under the basic symmetries of physics. Thus, if matter be distinct from antimatter, wave must be distinct from antiwave, as in our argument. Barron does not consider interaction energy in his paper,[15] but uses the fact that the induced magnetic dipole moment is $\hat{C}$ negative, which is also naturally accommodated within our argument, and also in that of Woźniak,[25] which Barron does not dispute.

There are several considerations of classical electrodynamics, for example, in the classic text by Jackson,[10] that appear to conflict with the assertion by Barron, that all forms of longitudinal solutions to Maxwell's equations in vacuo are zero. The following examples are mentioned briefly, but there are several more available.[10]

The Maxwell equations in vacuo have spherical solutions,[10] which in general are not transverse, as in plane wave solutions. The most general form of these solutions is given by Jackson's equation (16.35):

$$B = \sum_{l,m} \left[A_{lm}^{(1)} h_l^{(1)}(kr) + A_{lm}^{(2)} h_l^{(2)}(kr) \right] Y_{lm}(\theta, \phi) \tag{17}$$

where A_{lm} are arbitrary constant vectors; where $h_l^{(1)}$ and $h_l^{(2)}$ are Hankel functions, and where Y_{lm} are spherical harmonics. The longitudinal component is found for $\theta = 0$:

$$B_{\theta=0} = \sum_{l} \left[A_l^{(1)} h_l^{(1)}(kr) + A_l^{(2)} h_l^{(2)}(kr) \right] \left(\frac{2l + l}{4\pi} \right)^{1/2} \tag{18}$$

and this is clearly not zero in vacuo, being a special case of the general spherical solution (17) of the vacuum Maxwell equations:

$$(\nabla^2 + \kappa^2)\mathbf{B} = 0$$
$$\nabla \cdot \mathbf{B} = 0 \tag{19}$$

Transverse solutions are also special cases of Eq. (17), cases that are described, for example, in Jackson's equation (16.42).[10] It is impossible to assert on the grounds of $\hat{C}$ symmetry that the fields defined in Eq. (18) are zero, while those in Eq. (16.42) of Ref. 10 are nonzero, because both Eqs (18) and (16.42) of Ref. 10 are special cases of Eq. (17).

These are longitudinal solutions of Maxwell's equations in conducting media, where they are augmented by Ohm's law:

$$\mathbf{J} = \sigma E \tag{20}$$

where $\mathbf{J}$ is a current and σ the conductivity. The effect of $\hat{C}$ on Ohm's law is as follows:

$$-\mathbf{J} = \sigma(-E) \tag{21}$$

i.e., the conductivity is $\hat{C}$-positive. Similarly, electric permittivity and magnetic permeability in a conductor are both $\hat{C}$-positive, so changing conducting matter to conducting antimatter by operating with $\hat{C}$ does not change conductivity, permittivity, and permeability. Thus, Maxwell's equations and Ohm's law in conducting anti matter are the same as in conducting matter, and longitudinal solutions exist in both situations. The interaction of the antiwave with conducting antimatter is therefore the same as that of wave and conducting matter, as in our argument given already for the inverse Faraday effect. In Barron's view there is no antiwave, and the interaction is distinct, an insupportable conclusion.

Longitudinal, but phase dependent, solutions of Maxwell's equations exist in waveguide theory,[10] through equations such as

$$B_Z = B_0 = B_0 \cos\left(\pi \frac{x}{a}\right) e^{i\phi} \tag{22}$$

which are $\hat{C}$-invariant. Here a is a waveguide dimension and x a coordinate in this dimension. The only quantity on the right side of this equation that changes sign with $\hat{C}$ is the magnetic flux density amplitude B_0. All others are spacetime quantities which are invariant to $\hat{C}$ by definition. If we take $a \rightarrow \infty$, then for finite x,

$$B_Z \xrightarrow[a \rightarrow \infty]{} B_0 e^{i\phi} \tag{23}$$

i.e., at a point x inside a waveguide of infinite dimension a, the longitudinal magnetic field B_Z is nonzero. Since all parameters in Eq. (22) are $\hat{C}$-invariant except B_0, then according to Barron's view $\hat{C}$ operating on B_Z

of the waveguide does not produce a distinct situation, and so B_Z must vanish, an erroneous conclusion that conflicts with waveguide theory. Furthermore, if $a \to \infty$ in the waveguide, the wave B_Z is effectively propagating in a container of infinite volume, i.e., free space, and so in this situation B_Z remains nonzero in the free space limit, obtained by setting $a \to \infty$. According to Barron's view, B_Z is zero.

A closely related situation is that of resonant cavities, which again support longitudinal, phase-dependent solutions of Maxwell's equations, as in the example of a right cylindrical cavity, in which the longitudinal electric field is

$$E_Z = E_0 J_0\left(2.405 \frac{\rho}{R}\right) e^{-i\omega t} \tag{24}$$

Here,[10] J_0 is a Bessel function, ρ is a point on the radius R of the cylinder, and E_0 is a scalar electric field strength amplitude. Again, the only quantity on the right side of Eq. (24) that changes sign with $\hat{C}$ is E_0, so that all properties of the cavity are invariant to $\hat{C}$. Therefore, applying $\hat{C}$ in Barron's view produces an indistinct situation in which the wave vector of the field E_Z has not changed, no cavity property has changed, but E_Z has changed. So in Barron's view E_Z is zero, an incorrect conclusion which conflicts with the theory of resonant cavities.

APPENDIX: $\hat{C}\hat{P}\hat{T}$ THEOREM

In the text of the paper, it has been shown that $\mathbf{B}^{(3)}$ does not violate any of the three discrete symmetries. $\hat{C}$, $\hat{P}$, and $\hat{T}$, and by Eq. (1), is nonzero if $\mathbf{E}^{(1)} \times \mathbf{E}^{(2)}$ is nonzero. It follows that the nonobservation of $\mathbf{B}^{(3)}$ would violate $\hat{C}\hat{P}\hat{T}$, striking at the roots of quantum theory. $\hat{C}\hat{P}\hat{T}$ theorem implies that any quantum (and by implication classical) theory of fields that is compatible with special relativity and assumes only local interactions does not violate $\hat{C}\hat{P}\hat{T}$.[8] Therefore, if a physically meaningful magnetic flux density, $\mathbf{B}^{(3)}$, is inviolate of $\hat{C}$, $\hat{P}$, *and* $\hat{T}$ separately, and is nonzero, it must be an observable by the $\hat{C}\hat{P}\hat{T}$ theorem. If it is a nonobservable, the $\hat{C}\hat{P}\hat{T}$ theorem is violated. If $\mathbf{B}^{(3)}$ is observable experimentally, on the other hand, it provides evidence for the manifest covariance of electrodynamics and conservation of $\hat{C}\hat{P}\hat{T}$.

In this context, we note that if $\mathbf{B}^{(3)}$ is an observable, and if it is inviolate of $\hat{P}$ and $\hat{T}$, and therefore of $\hat{P}\hat{T}$, then the $\hat{C}\hat{P}\hat{T}$ theorem shows that it cannot violate $\hat{C}$. Barron's argument[15] is therefore shown to be incorrect, because he has assumed that $\mathbf{B}^{(3)}$ is an observable, and has himself concluded (albeit in a subjective argument) that in consequence $\mathbf{B}^{(3)}$ does not violate $\hat{P}\hat{T}$. It follows that if $\hat{P}\hat{T}$ is conserved, and $\mathbf{B}^{(3)}$ is an observable, as assumed by Barron[15]; then it cannot violate $\hat{C}$. Thus, if $\mathbf{B}^{(3)}$ is an observable, it must conserve $\hat{C}\hat{P}\hat{T}$. Conversely, conservation of $\hat{C}\hat{P}\hat{T}$ means that $\mathbf{B}^{(3)}$ must be an observable.

It follows that the conventional electrodynamical notion that the longitudinal solutions of Maxwell's equations $\mathbf{B}^{(3)}$ and $i\mathbf{E}^{(3)}$ (Ref. 4), be "unphysical" implies $\hat{C}\hat{P}\hat{T}$ violation, thus putting in doubt the fundamentals of quantum field theory applied to the electromagnetic field. Either $\mathbf{B}^{(3)}$ is an observable and $\hat{C}\hat{P}\hat{T}$ is conserved, or $\mathbf{B}^{(3)}$ is a nonobservable and $\hat{C}\hat{P}\hat{T}$ is violated. In other words, the only possible reason why the left side of Eq. (1) is not equal to the right side is if $\mathbf{B}^{(3)}$ violated $\hat{C}\hat{P}\hat{T}$. Therefore, quantum and classical electromagnetic field theory implies that the left and right sides of Eq. (1) must be equal, and in this field theory, since $\mathbf{E}^{(1)} \times \mathbf{E}^{(2)}$ is physically meaningful, then $\mathbf{B}^{(3)}$ must be physically meaningful. Otherwise, electromagnetic field theory is fundamentally flawed.

Acknowledgments

Many interesting discussions are acknowledged with S. Kielich, K. A. Earle, and F. Farahi. L. D. Barron is thanked for a preprint of Ref. 15 and correspondence.

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*THE ELECTROSTATIC AND MAGNETOSTATIC FIELDS GENERATED BY LIGHT IN FREE SPACE**

I. INTRODUCTION

The phenomenological equations of J. C. Maxwell form the basis of the classical understanding of light. The equations were formulated in the mid nineteenth century, before relativity was fully developed, and before the quantum theory came into existence. They were later put on a microscopic basis by H. A. Lorentz in his theory of the electron, and have become the starting point of a vast number of contemporary papers on the nature of light in free space and in materials. In this paper we show that there exist novel electro and magnetostatic fields in the propagation axis of the classical electromagnetic plane wave, fields that propagate in free space and conserve the structure of the well-defined Poynting vector, and therefore do not affect the law of conservation of electromagnetic energy in free space. It is usually assumed that the following are solutions to the free space Maxwell equations for a completely circularly polarized plane wave:

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{\sqrt{2}}E_0(\mathbf{i} + i\mathbf{j})e^{i\phi} \tag{1}$$

$$\mathbf{B}(\mathbf{r},t) = \frac{1}{\sqrt{2}}B_0(\mathbf{j} - i\mathbf{i})e^{i\phi} \tag{2}$$

*R. Gauthier and F. Farahi of UNCC are thanked for suggesting the possibility of $\mathbf{E}_\Pi$.

Here E_0 is the scalar electric field strength amplitude, and B_0 the scalar magnetic flux density amplitude, $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in X and Y of the laboratory frame, and ϕ is the phase of the plane wave. These solutions are oscillatory and time and space dependent through the phase

$$\phi = \omega t - \boldsymbol{\kappa}\cdot\mathbf{r} \tag{3}$$

where ω is the angular frequency of the wave, t the time, $\boldsymbol{\kappa}$ the wave vector, and $\mathbf{r}$ a position vector as usual. A whole literature is available concerning their properties.

However, the equations

$$\mathbf{E}^{G} = \mathbf{E}(\mathbf{r},t) + \mathbf{E}_\Pi \tag{4}$$

$$\mathbf{B}^{G} = \mathbf{B}(\mathbf{r},t) + \mathbf{B}_\Pi \tag{5}$$

are also valid solutions to the free space Maxwell equations. Here $\mathbf{E}_\Pi$ and $\mathbf{B}_\Pi$ are uniform, time-independent, electric and magnetic fields directed in the propagation axis Z of the plane wave. It appears always to have been implicitly assumed that $\mathbf{E}_\Pi$ and $\mathbf{B}_\Pi$ are both zero in free space, and that there is no component in Z of the plane wave in vacuo. There is no mathematical reason for this supposition, however, and as we shall see, the vectors $\mathbf{E}_\Pi$ and $\mathbf{B}_\Pi$ can be related to the well-known $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$. The source of $\mathbf{E}_\Pi$ and $\mathbf{B}_\Pi$ is therefore the same as the source of $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$. If the latter are nonzero, then so are both $\mathbf{E}_\Pi$ and $\mathbf{B}_\Pi$, in general.

Section II introduces $\mathbf{B}_\Pi$ using the imaginary conjugate product,

$$\boldsymbol{\Pi}^{(A)} \equiv E_0 c\,\mathrm{Im}(\mathbf{B}_\Pi) = \mathbf{E}(\mathbf{r},t)\times\mathbf{E}^*(\mathbf{r},t) = -iE_0^2\mathbf{k} \tag{6}$$

of the electromagnetic plane wave,[1-8] where $\mathbf{E}^*(\mathbf{r},t)$ is the complex conjugate of $\mathbf{E}(\mathbf{r},t)$, i.e.,

$$\mathbf{E}^*(\mathbf{r},t) = \frac{1}{\sqrt{2}}E_0(\mathbf{i} - i\mathbf{j})e^{-i\phi} \tag{7}$$

We see in Appendix A that the real and imaginary parts of $\mathbf{B}_\Pi$ are the same.

The law of conservation of energy for a plane wave in free space can be expressed through the continuity equation:

$$\nabla\cdot\mathbf{N} = -\frac{\partial U}{\partial t} \tag{8}$$

where $\mathbf{N}$ is the Poynting vector:

$$\mathbf{N} = \frac{1}{\mu_0}\mathbf{E}(\mathbf{r},t) \times \mathbf{B}(\mathbf{r},t) \tag{9}$$

and U a scalar field. Here μ_0 is the magnetic permeability of free space. The vector $\mathbf{N}$ is the flux of electromagnetic energy of the plane wave, and the scalar U is the electromagnetic field's energy density. Therefore, $\mathbf{N}$ is electromagnetic power per unit area, and U is power per unit volume. The scalar amplitude of the Poynting vector is the light intensity I_0. Therefore, Eq. (8) expresses, in classical electrodynamics, the law of conservation of electromagnetic energy in free space. This idea of field energy has no meaning[9] unless the wave interacts with matter (e.g., an electron). In Section IV, it is shown that the continuity equation (8) is unchanged for nonzero $\mathbf{E}_\Pi$ and $\mathbf{B}_\Pi$ provided they are both complex and

$$\mathbf{B}_\Pi \times \mathbf{E}(\mathbf{r},t) = \mathbf{E}_\Pi \times \mathbf{B}(\mathbf{r},t) \tag{10}$$

In other words, Eq. (10) is the condition for conservation of free space electromagnetic energy given the general solutions (4) and (5) of the Maxwell equations. Equation (10) shows that if $\mathbf{B}_\Pi$ is real, and defined through the conjugate product (6), then $\mathbf{E}_\Pi$ is imaginary. Finally, a discussion is given of the physical meaning of the novel vectors $\mathbf{E}_\Pi$ and $\mathbf{B}_\Pi$, with order of magnitude estimates, and experimental consequences.

II. THE DEFINITION OF $\mathbf{B}_\Pi$ THROUGH THE CONJUGATE PRODUCT

The conjugate product $\mathbf{E} \times \mathbf{E}^*$ appears in the antisymmetric part of Maxwell's stress tensor[10] and is a well-defined property of light. It is an axial vector with magnetic symmetry,[11, 12] i.e., that of angular momentum: positive to parity inversion $\hat{P}$, and negative to motion reversal $\hat{T}$. The vector notation $\mathbf{E} \times \mathbf{E}^*$ is equivalent to the tensor notation

$$\Pi_i^{(A)} = \tfrac{1}{2}\varepsilon_{ijk}(E_j E_k^* - E_k E_j^*) \tag{11}$$

where ε_{ijk} is the Levi-Civita symbol. This shows that the axial vector $\mathbf{E} \times \mathbf{E}^*$ is equivalent to a polar rank two tensor:

$$\Pi_{jk}^{(A)} = \tfrac{1}{2}(E_j E_k^* - E_k E_j^*) \tag{12}$$

which is the antisymmetric part of the tensor $E_j E_k^*$. Therefore, $\mathbf{E} \times \mathbf{E}^*$ is the vector part of light intensity.

The quantity

$$\mathrm{Im}(\mathbf{B}_\Pi) = \frac{\mathrm{i}\mathbf{E} \times \mathbf{E}^*}{E_0 c} \tag{13}$$

is a uniform, divergentless, time-independent, magnetic flux density vector with the required symmetry and units. The magnetic field $\mathbf{B}_\Pi$ exists in free space because $\mathbf{E} \times \mathbf{E}^*$ exists in free space, and is defined in the Z axis:

$$\mathrm{Im}(\mathbf{B}_\Pi) = +\frac{E_0}{c}\mathbf{k} = +B_0\mathbf{k} \tag{14}$$

where $\mathbf{k}$ in axial unit vector. The magnitude of $\mathbf{B}_\Pi$, i.e., $|\mathbf{B}_\Pi|$, is the scalar amplitude B_0 defined in the introduction. A real interaction Hamiltonian is produced from $\mathbf{E} \times \mathbf{E}^*/(E_0 c)$ when it forms a scalar product with the usual imaginary magnetic dipole moment operator, $\mathrm{i}\hat{\mathbf{m}}''$, in quantum mechanics.[13–15] Similarly, the imaginary $\mathbf{E} \times \mathbf{E}^*$ produces a well-defined[1–8] real interaction Hamiltonian when it multiplies the imaginary part of molecular electric polarizability operator, $\mathrm{i}\hat{\boldsymbol{\alpha}}''$. The latter is the vectorial polarizability,[16, 17] which vanishes at zero frequency from time-dependent perturbation theory. Both $\hat{\mathbf{m}}''$ and $\hat{\boldsymbol{\alpha}}''$ are directly proportional (using the Wigner-Eckart theorem, for example,[16, 17] to the net molecular electronic angular momentum operator $\hat{J}$:

$$\hat{\mathbf{m}}'' = \gamma_e \hat{J} \tag{15}$$

$$\hat{\boldsymbol{\alpha}}'' = \gamma_\Pi \hat{J} \tag{16}$$

where γ_e is the gyromagnetic ratio[13–15] and γ_Π is the gyroptic ratio.[18–20] Consequently,

$$\hat{\boldsymbol{\alpha}}'' = \frac{\gamma_\Pi}{\gamma_e}\hat{\mathbf{m}}'' \tag{17}$$

showing that $\hat{m}''$ and $\hat{\alpha}''$ have the same $\hat{T}$-negative, $\hat{P}$-positive symmetry, and are both axial vector operators. The conjugate product $\mathbf{E} \times \mathbf{E}^*$ forms a real Hamiltonian operator when it multiplies $\mathrm{i}\hat{\boldsymbol{\alpha}}''$, and because $\hat{\boldsymbol{\alpha}}''$ is directly proportional to $\hat{J}$ and thus to $\hat{\mathbf{m}}''$, it follows that $\mathbf{E} \times \mathbf{E}^*$ must be proportional to a magnetic field, which we have identified as $\mathbf{B}_\Pi$ in Eq. (13). Clearly, $\hat{\mathbf{m}}''$ can form a real Hamiltonian operator only when multiplied by a magnetic field. The root of Eq. (13) is therefore found in the fact that the molecular property tensors $\hat{\boldsymbol{\alpha}}''$ *and* $\hat{\mathbf{m}}''$ are both axial vectors with magnetic symmetry. This point can be emphasized by assuming that the real part of $\hat{\mathbf{m}}'' \cdot \mathbf{B}_\Pi$ is an interaction Hamiltonian and investigating the logical consequences. To do this, it is convenient to write the interaction Hamiltonian[16, 17] between $\mathrm{i}\hat{\boldsymbol{\alpha}}''$ and $\mathbf{E} \times \mathbf{E}^*$ as

$$\Delta \hat{H} = -\mathrm{i}\hat{\boldsymbol{\alpha}}'' \cdot \mathbf{E} \times \mathbf{E}^* = -\mathrm{i}E_0 c \hat{\boldsymbol{\alpha}}'' \cdot \frac{\mathbf{E} \times \mathbf{E}^*}{E_0 c} \equiv -E_0 c \hat{\boldsymbol{\alpha}}'' \mathrm{Im}(\mathbf{B}_\Pi) \quad (18)$$

where we have defined $\mathbf{B}_\Pi$ in terms of $\mathbf{E} \times \mathbf{E}^*$ as in Eq. (13). Using the proportionality (17) between the magnetic dipole moment and the vectorial polarizability, Eq. (18) becomes

$$\Delta \hat{H} = -E_0 c \frac{\gamma_\Pi}{\gamma_e} \hat{\mathbf{m}}'' \cdot \mathrm{Im}(\mathbf{B}_\Pi) = -\mathrm{i}\hat{\boldsymbol{\alpha}}'' \cdot \mathbf{E} \times \mathbf{E}^* \quad (19)$$

showing that the product $\hat{\mathbf{m}} \cdot \mathbf{B}_\Pi$ is directly proportional to the product $\mathrm{i}\hat{\boldsymbol{\alpha}}'' \cdot \mathbf{E} \times \mathbf{E}^*$ through a nonzero proportionality constant. Therefore, if the energy $\mathrm{i}\hat{\boldsymbol{\alpha}}'' \cdot \mathbf{E} \times \mathbf{E}^*$ is nonzero, then so must the energy $\hat{\mathbf{m}}'' \cdot \mathbf{B}_\Pi$ be nonzero.

Using the Wigner-Eckart theorem, the gyromagnetic and gyroptic ratios can be defined as follows in an atom with net electronic angular momentum, $\hat{J}$, showing that in this case γ_e and γ_Π are nonzero in general:

$$m_0^{1''} = \frac{\langle J \| \hat{m}^{1''} \| J \rangle}{\langle J \| \hat{J}^1 \| J \rangle} J_0^1 = \gamma_e J_0^1 \quad (20)$$

$$\alpha_0^{1''} = \frac{\langle J \| \hat{\alpha}^{1''} \| J \rangle}{\langle J \| \hat{J}^1 \| J \rangle} J_0^1 = \gamma_\Pi J_0^1 \quad (21)$$

$$\langle J \| \hat{J}^1 \| J \rangle = [J(J+1)(2J+1)]^{1/2} \quad (22)$$

III. THE CONSERVATION OF ELECTROMAGNETIC ENERGY

We have assumed that Eqs. (4) and (5) are solutions of the free space Maxwell equations:

$$\boldsymbol{\nabla} \times \mathbf{E}^G = -\frac{\partial \mathbf{E}^G}{\partial t} \quad (23)$$

$$\boldsymbol{\nabla} \times \mathbf{B}^G = \frac{1}{c^2}\frac{\partial \mathbf{E}^G}{\partial t} \quad (24)$$

in SI units. It is clear that if $\mathbf{E}_\Pi$ and $\mathbf{B}_\Pi$ are defined in the Z axis of the plane wave, then

$$\boldsymbol{\nabla} \times \mathbf{E}_\Pi = \frac{\partial \mathbf{B}_\Pi}{\partial t} = 0 \quad (25)$$

$$\boldsymbol{\nabla} \times \mathbf{B}_\Pi = \frac{\partial \mathbf{E}_\Pi}{\partial t} = 0 \quad (26)$$

because these fields are time independent and have no X and Y components. Consider the divergence of $\mathbf{E}^G(\mathbf{r}, t)$ and $\mathbf{B}^G(r, t)$. Using the vector identity,

$$\boldsymbol{\nabla} \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\boldsymbol{\nabla} \times \mathbf{A}) - \mathbf{A} \cdot (\boldsymbol{\nabla} \times \mathbf{B}) \quad (27)$$

it follows that we may expand:

$$\begin{aligned} \boldsymbol{\nabla} \cdot (\mathbf{E}^G \times \mathbf{B}^G) &= \boldsymbol{\nabla} \cdot (\mathbf{E} \times \mathbf{B}) + \boldsymbol{\nabla} \cdot (\mathbf{E} \times \mathbf{B}_\Pi) \\ &\quad + \boldsymbol{\nabla} \cdot (\mathbf{E}_\Pi \times \mathbf{B}) + \boldsymbol{\nabla} \cdot (\mathbf{E}_\Pi \times \mathbf{B}_\Pi) \end{aligned} \quad (28)$$

where $\mathbf{E} \times \mathbf{B}$ is proportional to the Poynting vector of the law of conservation of energy, Eq. (8). From the relations

$$\begin{aligned} \boldsymbol{\nabla} \cdot (\mathbf{E} \times \mathbf{B}_\Pi) &= \mathbf{B}_\Pi \cdot (\boldsymbol{\nabla} \times \mathbf{E}) - \mathbf{E} \cdot (\boldsymbol{\nabla} \times \mathbf{B}_\Pi) \\ \boldsymbol{\nabla} \times \mathbf{B}_\Pi &= 0 \end{aligned} \quad (29)$$

and

$$\mathbf{B}_\Pi \cdot (\boldsymbol{\nabla} \times \mathbf{E}) = -\mathbf{B}_\Pi \cdot \frac{\partial \mathbf{B}}{\partial t} = 0 \quad (30)$$

it follows that

$$\boldsymbol{\nabla} \cdot (\mathbf{E} \times \mathbf{B}_\Pi) = 0 \quad (31)$$

and similarly

$$\boldsymbol{\nabla} \cdot (\mathbf{E}_\Pi \times \mathbf{B}) = 0 \quad (32)$$

Also, the last term in Eq. (28) vanishes because $\mathbf{E}_\Pi$ is parallel to $\mathbf{B}_\Pi$ in Z. It follows, therefore, that

$$\boldsymbol{\nabla} \cdot (\mathbf{E}^G \times \mathbf{B}^G) = \boldsymbol{\nabla} \cdot (\mathbf{E} \times \mathbf{B}) \quad (33)$$

i.e., the continuity equation (8) is unaffected by the presence of $\mathbf{E}_\Pi$ and

$\mathbf{B}_\Pi$, and the relation (Eq. (8)) of the field energy flux density (**N**) to the electromagnetic field energy density (U) is unchanged in the free space electromagnetic plane wave. In other words, the electromagnetic powers per unit area generated by $\nabla \cdot (\mathbf{E} \times \mathbf{B}_\Pi)$ and by $\nabla \cdot (\mathbf{E}_\Pi \times \mathbf{B})$ are both zero, and therefore so are the associated electromagnetic powers per unit volume.

This result is true only if $\mathbf{E}_\Pi$ and $\mathbf{B}_\Pi$ are both in the propagation axis of the plane wave. The argument so far shows that $\mathbf{E}_\Pi$ and $\mathbf{B}_\Pi$ may be separately non zero, or that $\mathbf{E}_\Pi$ may be zero and $\mathbf{B}_\Pi$ nonzero, as defined in Eq. (13).

To obtain a relation between $\mathbf{E}_\Pi$ and $\mathbf{B}_\Pi$ we use the result, from Eq. (8):

$$\nabla \cdot (\mathbf{E}^G \times \mathbf{B}^G) = \nabla \cdot (\mathbf{E} \times \mathbf{B}) = -\frac{\partial U}{\partial t} \tag{34}$$

which implies that the divergence of the product $\mathbf{E}^G \times \mathbf{B}^G$ is nonzero and identical with the divergence of the product $\mathbf{E} \times \mathbf{B}$. This implies that

$$\mathbf{E}^G \times \mathbf{B}^G = \mathbf{E} \times \mathbf{B} + \text{constant} \tag{35}$$

However, we know that

$$\mathbf{E}^G \times \mathbf{B}^G = \mathbf{E} \times \mathbf{B} + \mathbf{E}_\Pi \times \mathbf{B} + \mathbf{E} \times \mathbf{B}_\Pi \tag{36}$$

and from Eqs. (35) and (36) we derive the key result:

$$\mathbf{E}_\Pi \times \mathbf{B} = \mathbf{B}_\Pi \times \mathbf{E} \tag{37}$$

assuming that the constant of integration in Eq. (35) is zero (see Appendix D) and demonstrating that if $\mathbf{B}_\Pi$ is real, then $\mathbf{E}_\Pi$ must be imaginary.

In precise analogy with $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$, the fields $\mathbf{E}_\Pi$ and $\mathbf{B}_\Pi$ take meaning only when there is wave–particle or wave–matter interaction, but these fields propagate through free space (i.e., vacuum). Clearly, electromagnetic waves can be detected only when there is particulate matter with which the waves can interact, otherwise there would be no experimental evidence at all for the existence of electromagnetic fields. The source of $\mathbf{E}_\Pi$ and $\mathbf{B}_\Pi$ is the same as the source of the oscillating fields $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$, because both the static and oscillating components are needed for the complete solution of the free space Maxwell equations and the presence of oscillating components implies through Eqs. (13) and (37) the presence of static components. The static and oscillating components are

both relativistic in nature, because the plane wave propagates at the speed of light. In quantum field theory, there are operator equivalents[18–21] of $\mathbf{E}_\Pi$ and $\mathbf{B}_\Pi$. A fundamentally important difference between the oscillating and static components of the plane wave is that the former vanish upon time averaging and the latter do not. This is the source of several novel physical phenomena when there is wave–matter interaction. Equation (37) conserves $\hat{P}$ and $\hat{T}$ symmetry, and the static components of the solution are related through Eqs. (13) and (37) to the oscillating components with, as we have seen, conservation of electromagnetic energy. The components are therefore completely defined and the definition is self-consistent.

From the properties of the dual transform of special relativity (see Appendix A) the Maxwell equations are invariant to

$$\begin{aligned} c\mathbf{B} &\rightarrow -\mathrm{i}\mathbf{E} \\ c\mathbf{B}_\Pi &\rightarrow -\mathrm{i}\mathbf{E}_\Pi \end{aligned} \tag{38}$$

The dual transform implies immediately that

$$\mathbf{B}_\Pi \times \mathbf{E} = -\mathrm{i}\frac{\mathbf{E}_\Pi}{c} \times \left(-c\frac{\mathbf{B}}{\mathrm{i}}\right) = \mathbf{E}_\Pi \times \mathbf{B} \tag{39}$$

which confirms that the sum

$$\mathbf{E}_\Pi \times \mathbf{B} + \mathbf{E} \times \mathbf{B}_\Pi = 0 \tag{40}$$

and that the general solution of Maxwell's equations must be of the form (see Appendix A)

$$\mathbf{E}^G = \mathbf{E}(\mathbf{r}, t) \pm E_0(\mathrm{i} - 1)\mathbf{k} \tag{41}$$

$$\mathbf{B}^G = \mathbf{B}(\mathbf{r}, t) \pm B_0(\mathrm{i} + 1)\mathbf{k} \tag{42}$$

to be consistent with the theory of special relativity applied to the Maxwell equations.

It is easily checked that Eq. (39) is consistent with Eqs. (1) and (2), with

$$\mathbf{B}_\Pi = \pm B_0(\mathrm{i} + 1)\mathbf{k} \tag{43}$$

$$\mathbf{E}_\Pi = \pm E_0(\mathrm{i} - 1)\mathbf{k} \tag{44}$$

Equation (39) is also consistent with the generalized continuity equation, and with the fact (see appendix A) that

$$\mathbf{F}_\Pi = \mathbf{E}_\Pi + \mathrm{i}c\mathbf{B}_\Pi \tag{45}$$

and

$$F_{\Pi}^2 = E_{\Pi}^2 - c^2 B_{\Pi}^2 + 2ic\mathbf{E}_{\Pi} \cdot \mathbf{B}_{\Pi} \qquad (46)$$

are invariants of the Lorentz transform. From Eq. (40), the net contribution of $\mathbf{E}_{\Pi}$ and $\mathbf{B}_{\Pi}$ to free space electromagnetic energy is zero.

IV. DISCUSSION

The orders of magnitude of $\mathbf{E}_{\Pi}$ and $\mathbf{B}_{\Pi}$ can be estimated directly from the intensity I_0 of the light beam in W m^{-2}, through the free space relations

$$|\mathbf{B}_{\Pi}| = B_0 = \left(\frac{I_0}{\varepsilon_0 c^3}\right)^{1/2} \qquad |\mathbf{E}_{\Pi}| = E_0 = \left(\frac{I_0}{\varepsilon_0 c}\right)^{1/2} \qquad (47)$$

where ε_0 is the electric permittivity in vacuo (8.854×10^{-12} J^{-1} C^{2}m^{-1} in S.I. units). Thus, for a beam of 10 000 W m^{-2}, (1.0 W cm^{-2}), B_0 is about 10^{-8} T and E_0 about 20 V m^{-1}. These are also the scalar amplitudes B_0 and E_0 of the oscillating part of the solution to Maxwell's equations, and the scalar intensity I_0 of the beam is unaffected by the presence of $\mathbf{E}_{\Pi}$ and $\mathbf{B}_{\Pi}$ because I_0 is the magnitude of the Poynting vector. However, $\mathbf{B}_{\Pi}$ is nonzero after time averaging because it is independent of time, and forms a real, nonzero, interaction Hamiltonian with particulate matter. This Hamiltonian leads, therefore, to the prediction of novel physical phenomena, which can be measured as a function of I_0 and of the polarization state of the light beam. If the latter is linearly or incoherently polarized, $\mathbf{E} \times \mathbf{E}^*$ is zero and in consequence, so are $\mathbf{B}_{\Pi}$ and $\mathbf{E}_{\Pi}$; otherwise $\mathbf{E}_{\Pi}$ and $\mathbf{B}_{\Pi}$ are proportional to the square root of I_0. Because $\mathbf{E}_{\Pi}$ and $\mathbf{B}_{\Pi}$ are electrostatic and magnetostatic fields that form part of the general solution of Maxwell's equations in free space, they have the properties of such fields when light interacts with matter. This is the main conclusion of this paper.

On the basis of this conclusion it is easy to see that the various theories of the interaction of conventionally generated electric and magnetic fields can be applied directly to the real field $\mathbf{B}_{\Pi}$, and examples of these applications have been given elsewhere for $\mathbf{B}_{\Pi}$ (Refs. 22–26). These include the inverse Faraday effect, the optical Faraday and Zeeman effects, optically induced shifts in NMR resonances ("optical NMR", recently observed experimentally,[27] the optical Cotton Mouton effect, optical ESR, optical forward backward birefringence, and a reinterpretation of antisymmetric light scattering and related phenomena in terms of

$\mathbf{B}_{\Pi}$. It has also been deduced[18–21] that the quantum field equivalent of $\mathbf{B}_{\Pi}$ is the operator

$$\hat{B}_{\Pi} = B_0 \frac{\hat{J}}{\hbar} \qquad (48)$$

where $\hat{J}$ is the quantized photon angular momentum, and $\hbar$ the reduced Planck constant. It has also been shown,[25] using the properties of the classical Lorentz transformation, that there can be no Faraday induction in free space due to a time derivative of the type $d\mathbf{B}_{\Pi}/dt$, produced, for example, by modulating a laser beam. (Note however, that Faraday induction occurs via the inverse Faraday effect[28] when a circularly polarized laser interacts with matter inside an induction coil.) The reason for this is that the Lorentz transformations do not allow free space X and Y components either of $\mathbf{B}_{\Pi}$ or of $\mathbf{E}_{\Pi}$, and also show that the Z components $\mathbf{E}_{\Pi}$ and $\mathbf{B}_{\Pi}$ must be relativistically invariant.[25]

One of the simplest consequences of the presence of $\mathbf{B}_{\Pi}$ is an optical Zeeman effect, whose semiclassical theory regards $\mathbf{B}_{\Pi}$ as a classical vector.[24] In this approximation the theory of the optical Zeeman effect is the same as that of the conventional Zeeman effect,[29] with the conventional magnetostatic $\mathbf{B}_s$ replaced by $\mathbf{B}_{\Pi}$. In the simplest case, the Zeeman shift is proportional to

$$\Delta f = \hat{m} \cdot \frac{\mathbf{B}_{\Pi}}{h} \qquad (49)$$

and therefore to the square root of the laser intensity $I_0^{1/2}$. This occurs in addition to an optical Zeeman shift caused[30] by the interaction of $\mathbf{E} \times \mathbf{E}^*$ with $\hat{\boldsymbol{\alpha}}''$, a mechanism that is proportional to intensity I_0. There appear to be no experimental investigations to date of the optical Zeeman effect, which requires only a minor modification of optical Stark effect apparatus to circularly polarize the pump laser.

Because $\mathbf{E}_{\Pi}$ is imaginary when $\mathbf{B}_{\Pi}$ is real, no simple physical effects are expected due to $\mathbf{E}_{\Pi}$, and significantly, none appears to have been reported in the literature.

The experimental evidence for the presence of uniform and time-independent components in the free space solutions of Maxwell's equations is available in at least three forms: (1) the inverse Faraday effect (IFE),[28] the optical Faraday effect (OFE),[30] and optically induced frequency shifts in NMR (ONMR).[27] In the IFE, bulk magnetization has been observed when a circularly polarized giant ruby laser pulse was passed through a sample in an inductance coil, thereby producing a measurable voltage that was not present in linear polarization and that changed sign with the sense of circular polarization. These are characteristic properties of $\mathbf{B}_{\Pi}$. In ONMR, a continuous wave argon ion laser was used[27] to shift NMR resonances and the shifts were much larger in circular than in linear polarization of the laser, too large to be explained by mechanisms based on the oscillating **E** and **B**, which time average to zero. For example it may be conjectured that the oscillating $\mathbf{B}(\mathbf{r}, t)$ induces in semiclassical theory a magnetic dipole moment in the electrons of a molecule in ONMR, a dipole moment that sets up a magnetic field at the resonating nucleus. However, this induced magnetic field would produce shifts much smaller than those observed,[27] and would time average to zero and no ONMR shift would be observable. The novel field $\mathbf{B}_{\Pi}$ does not time average to zero, and in principle sets up an interaction Hamiltonian $\hat{\mathbf{m}}_N \cdot \mathrm{Im}(\mathbf{B}_{\Pi})$, where $\hat{\mathbf{m}}_N$ is the nuclear magnetic dipole moment, causing ONMR shifts in circular polarization, as observed,[27] but not in linear polarization. In general, $\hat{\mathbf{B}}_{\Pi}$ is an operator in quantum field theory, and its interaction with the operator $\hat{\mathbf{m}}_N$ must be described properly in terms of quantization both in the applied laser field and in the nucleus. ONMR provides information about the nature of the interaction between $\hat{\mathbf{B}}_{\Pi}$ and the nucleus. However, there are several competing mechanisms in ONMR, and the data cannot yet be interpreted unequivocally. Recently, Frey et al.[30] have reported the optical Faraday effect in magnetic semiconductors, in which the polarization direction and phase of a laser beam are modified by interaction with the material. The authors interpret these changes in terms of nonlinear Faraday processes, but it is interesting to note that a plot of their experimentally observed rotation of the polarization of the laser by the sample against the square root of the laser intensity is a straight line within the experimental uncertainty (Fig. 1). This is the result expected from a mechanism of self-rotation based on the presence of the vector $\mathbf{B}_{\Pi}$ in the laser beam. These results are suggestive, but not unequivocal, because the straight line does not go through the origin in Fig. 1, and because it is not clear from the experimental arrangement of Frey et al.[30] whether the laser they used was circularly polarized. The output beam from the sample was analyzed by these authors with a Soleil compensator to determine its state of polarization, and the fact that a rotation of polarization was observed[30] (Fig. 1) suggests an excess of circular polarization in the output beam. A more critical test for the presence of $\mathbf{B}_{\Pi}$ would consist of a completely circularly polarized pump laser incident on a magnetic semiconductor (showing a giant Zeeman effect) together with a linearly polarized probe laser. The plane of polarization of the latter would be rotated by the $\mathbf{B}_{\Pi}$ vector of the pump, and this rotation should be proportional to the square root of the pump laser's intensity if there are no competing mechanisms.

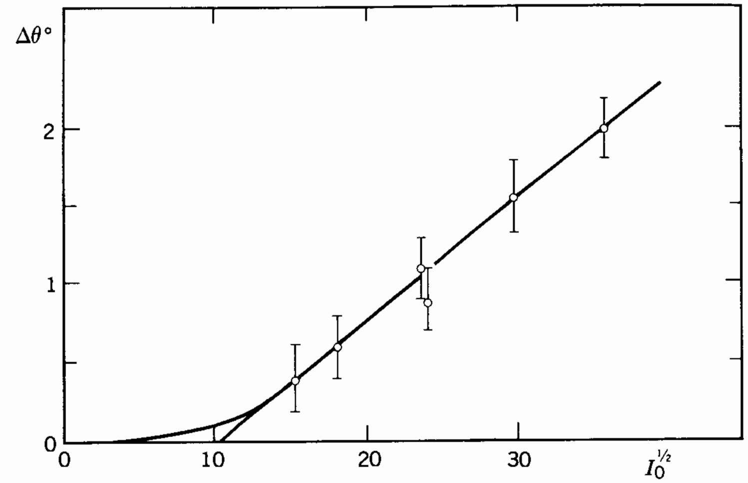

Figure 1 Plot of data from Ref. 30 of the optical Faraday effect, angle of rotation ver the square root of laser intensity. The points and uncertainty bars are those of Ref. 30. — Best fit line of Ref. 30; - - -, linear extrapolation (this work).

The sense of rotation should be reversed on reversing the sense of circul polarization of the pump, and should vanish when the pump is linea polarized. The semiclassical theory of this effect is the same as that[31] the conventional Faraday effect of 1846, with the magnetostatic field of t latter replaced by $\mathbf{B}_{\Pi}$ of the pump laser. The quantum field theory wou treat $\hat{\mathbf{B}}_{\Pi}$ as a quantum operator, Eq. (48), and there is no reason suppose that the results in quantum field theory would be the same as th in semiclassical theory, i.e., there are nonclassical effects, in general, d to $\mathbf{B}_{\Pi}$ treated as quantum field operators.

The available experimental evidence for the existence of $\mathbf{B}_{\Pi}$ is sugge tive, but not unequivocal; therefore, the challenge is to separate t particular influence of $\mathbf{B}_{\Pi}$ from the simultaneous influence of $\mathbf{E}(\mathbf{r}, t)$ a $\mathbf{B}(\mathbf{r}, t)$. It appears that one of the clearest ways of demonstrating t existence of $\mathbf{B}_{\Pi}$ would be through its characteristic square root intens dependence, and through the fact that both vectors change sign with t sense of circular polarization, vanishing in linear polarization. If experimental evidence for $\mathbf{B}_{\Pi}$ were found, such an eventuality would itself be a major challenge to contemporary understanding of the nature light and electromagnetic radiation in general. The reason for this is th the vector $\mathbf{B}_{\Pi}$ is directly proportional (Eq. (13)) to the conjugate produ $\mathbf{E} \times \mathbf{E}^*$, in free space, and if the notion of conjugate product is accepte as is the contemporary practice, then $\mathbf{B}_{\Pi}$ must be accepted, and vice vers

If $\mathbf{B}_\Pi$ is real then E_Π is imaginary, through Eq. (37), and it has been demonstrated in this work that $\mathbf{E}_\Pi$ and $\mathbf{B}_\Pi$ do not affect the law of conservation of electromagnetic energy, the widely accepted continuity equation (8) of the classical theory of fields. The notions of $\mathbf{E} \times \mathbf{E}^*$, $\mathbf{B}_\Pi$, and $\mathbf{E}_\Pi$ are inextricably and ineluctably interrelated, therefore, and experimental evidence for the presence of any one is evidence for all. Conversely, if there is no apparent evidence for one, then all must not exist. The inverse Faraday effect has been interpreted through the notion of $\mathbf{E} \times \mathbf{E}^*$ (Refs. 7 and 8), but this provides an explanation in terms of only one mechanism, proportional to intensity. It has been argued here that there must be another mechanism present, proportional to the square root of intensity (the $\mathbf{B}_\Pi$ mechanism). If these are found experimentally, contemporary understanding would be strengthened. But if evidence for one mechanism (e.g., $\mathbf{E} \times \mathbf{E}^*$) is found and evidence for another (e.g., $\mathbf{B}_\Pi$) is not found, then the theory of electromagnetic fields would be challenged at the most fundamental level.

Clearly, the notion of $\mathbf{E} \times \mathbf{E}^*$ implies that this object is transmitted through free space in an electromagnetic plane wave, and when this wave meets particulate matter, an interaction Hamiltonian is formed between $\mathbf{E} \times \mathbf{E}^*$ and a material property. In atoms and molecules with net electronic angular momentum, this property is the vectorial polarizability vector $\hat{\boldsymbol{\alpha}}''$, well defined and accepted in semiclassical time-dependent perturbation theory, based on the time-dependent Schrödinger equation.[29] Since $\mathbf{B}_\Pi$ is directly proportional to $\mathbf{E} \times \mathbf{E}^*$, it cannot be argued that $\mathbf{E} \times \mathbf{E}^*$ exists and that $\mathbf{B}_\Pi$ does not. The source of $\mathbf{E}_\Pi$ and $\mathbf{B}_\Pi$ is clearly the same as that of $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$. Furthermore, it has been shown that $\mathbf{B}_\Pi$ is part of the general solution of the equations of Maxwell, and is therefore phenomenologically indistinguishable from uniform, magnetostatic flux density, whose symmetry and units it possesses. It cannot therefore be argued that $\mathbf{B}_\Pi$ cannot form an interaction Hamiltonian with the appropriate material property (a magnetic dipole moment), and it has been shown in Eq. (19) that if $i\hat{\boldsymbol{\alpha}}'' \cdot \mathbf{E} \times \mathbf{E}^*$ is accepted as an interaction energy, then $\hat{\mathbf{m}} \cdot \mathrm{Im}(\mathbf{B}_\Pi)$ must also be accepted. Finally, the classical presence of $\mathbf{E}_\Pi$ and $\mathbf{B}_\Pi$ must also have meaning in quantum field theory, where these vector fields become operators (Appendix B).

APPENDIXA: LORENTZ COVARIANCE

The theory of the Lorentz covariance of Maxwell's equations show[10] that the complex quantity

$$\mathbf{F}_\Pi = \mathbf{E}_\Pi + ic\mathbf{B}_\Pi \tag{A.1}$$

is an invariant of the Lorentz transformation of special relativity. Therefore,

$$F_\Pi^2 = E_\Pi^2 - c^2 B_\Pi^2 + 2ic\mathbf{E}_\Pi \cdot \mathbf{B}_\Pi \tag{A.2}$$

is an invariant of $\mathbf{F}_\Pi$ with respect to rotation in $(\mathbf{Z}, t)$ in Minkowski four space. This is equivalent to a rotation in the X, Y plane through an imaginary angle in three dimensions. Thus, $E_\Pi^2 - c^2 B_\Pi^2$ and $\mathbf{E}_\Pi \cdot \mathbf{B}_\Pi$ are the only two independent invariants of the antisymmetric four tensor of the electromagnetic field in the four-dimensional representation. This is the tensor F^{ik}, with

$$\begin{aligned} F_{ik} F^{ik} &= \text{inv.} \\ e^{iklm} F_{ik} F_{lm} &= \text{inv.} \end{aligned} \tag{A.3}$$

and where e^{iklm} denotes the completely antisymmetric unit tensor of rank four.

It is important to note that these invariants are zero only when

$$E_\Pi^2 = c^2 B_\Pi^2 \tag{A.4}$$

and

$$\mathbf{E}_\Pi \cdot \mathbf{B}_\Pi = 0 \tag{A.5}$$

Because, for complex $\mathbf{B}_\Pi$ and $\mathbf{E}_\Pi$,

$$|\mathbf{E}_\Pi| = \sqrt{2} E_0 \quad and \quad |\mathbf{B}_\Pi| = \sqrt{2} E_0 \qquad E_0 = cB_0,$$

we obtain

$$E_\Pi^2 - c^2 B_\Pi^2 = 0 \tag{A.6}$$

However, because $\mathbf{E}_\Pi$ and $\mathbf{B}_\Pi$ are parallel in Z, the propagation axis,

$$\mathbf{E}_\Pi \cdot \mathbf{B}_\Pi \neq 0 \tag{A.7}$$

If $F_\Pi^2 \neq 0$, it is known that $\mathbf{F}_\Pi = a\mathbf{n}$ (Ref. 10), where $\mathbf{n}$ is a complex unit vector ($nn^* = 1$). Using a complex rotation it is always possible to direct $\mathbf{n}$ along a coordinate axis,[10] and $\mathbf{n}$ becomes real, determining the directions of $\mathbf{E}_\Pi$ and $\mathbf{B}_\Pi$:

$$\mathbf{F}_\Pi = (E_\Pi + icB_\Pi)\mathbf{n} \tag{A.8}$$

so $\mathbf{E}_\Pi$ is parallel to $\mathbf{B}_\Pi$. Therefore, in any frame of reference, $\mathbf{E}_\Pi$ must be parallel to $\mathbf{B}_\Pi$; and there can be no components of either $\mathbf{E}_\Pi$ and $\mathbf{B}_\Pi$ perpendicular to the direction of the plane wave.

It is concluded that $\mathbf{E}_\Pi$ and $\mathbf{B}_\Pi$ form a nonzero invariant, Eq. (A.7), of the Lorentz transformation of special relativity. The oscillating components $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$ form zero invariants, because they are mutually perpendicular and $E_0 = cB_0$.

Furthermore, the dual transformation of special relativity corresponds to[10]

$$\begin{aligned} c\mathbf{B}^G &\to -\mathrm{i}\mathbf{E}^G \\ -\mathrm{i}\mathbf{E}^G &\to c\mathbf{B}^G \end{aligned} \tag{A.9}$$

which leaves the Maxwell equations invariant in vacuo. The dual transformation therefore corresponds to

$$\begin{aligned} c(\mathbf{B}(\mathbf{r}, t) + \mathrm{i}\mathbf{B}_\Pi) &\to -\mathrm{i}\mathbf{E}(\mathbf{r}, t) + \mathbf{E}_\Pi \\ -\mathrm{i}(\mathbf{E}(\mathbf{r}, t) + \mathbf{E}_\Pi) &\to c(\mathbf{B}(\mathbf{r}, t) + \mathbf{B}_\Pi) \end{aligned} \tag{A.10}$$

Therefore, the Maxwell equations are invariant in vacuo to the transformations $c\mathbf{B}_\Pi \to -\mathrm{i}\mathbf{E}_\Pi$ and $-\mathrm{i}\mathbf{E}_\Pi \to c\mathbf{B}_\Pi$. Exchanging $c\mathbf{B}_\Pi$ and $-\mathrm{i}\mathbf{E}_\Pi$ everywhere, or vice versa, the Maxwell equations are the same in any frame of reference, which is consistent with the fact that $\mathbf{E}_\Pi$ is parallel to $\mathbf{B}_\Pi$ in Z, and if $\mathbf{B}_\Pi$ is real, then $\mathbf{E}_\Pi$ is imaginary. A gauge transformation leaves the Lorentz relation unchanged, and leaves $\mathbf{E}_\Pi$ and $\mathbf{B}_\Pi$ unaltered. Thus, gauge invariance means that $\mathbf{E}_\Pi$ and $\mathbf{B}_\Pi$ are unaltered in any valid gauge.

It is important to note that the invariant in Eq. (A.1) is a complex quantity, and that $\mathbf{E}_\Pi$ and $\mathbf{B}_\Pi$ also appear in Eqs. (4) and (5) as complex. Otherwise, the invariant in Eq. (A.6) would not vanish, and $\mathbf{E}_\Pi$ and $\mathbf{B}_\Pi$ would contribute to the electromagnetic energy density and flux density. This is essentially the result of the special theory of relativity, which uses four-dimensional Minkowski space:

$$m = (X, Y, Z, \mathrm{i}ct)$$

The Lorentz transformation describes rotations in this four space, and it follows that $\mathbf{E}_\Pi$ and $\mathbf{B}_\Pi$ are unaffected by rotations in spacetime. The fourth coordinate of Minkowski space is $\mathrm{i}ct$, i.e., imaginary. The reason is that this gives a space in which the four-dimensional Pythagorean theorem has the same form as the usual three-dimensional theorem. This is taken as a fundamental criterion of a Cartesian system. Maxwell's equations can be written in tensor form in Minkowski space, leading to the Lorentz transformation.

It is concluded that $\mathbf{E}_\Pi$ and $\mathbf{B}_\Pi$ are entirely consistent with the Lorentz covariance of the Maxwell equations, and form invariants of the Lorentz transformation. The fields $\mathbf{E}_\Pi$ and $\mathbf{B}_\Pi$ are therefore physically meaningful in the classical theory of electromagnetic radiation. The free space vector

$$\mathbf{F}_\Pi = \mathbf{E}_\Pi + \mathrm{i}c\mathbf{B}_\Pi \tag{A.11}$$

is a nonzero invariant of the Lorentz transformation in the special theory of relativity. The existence of $\mathbf{B}_\Pi$ implies that Maxwell's equations in free space support the nonlinear solution:

$$\mathbf{B}^G(\mathbf{r}, t) = \mathbf{B}(\mathbf{r}, t) + \frac{1}{\mathrm{i}E_0 c}\mathbf{E}(\mathbf{r}, t) \times \mathbf{E}^*(\mathbf{r}, t) \tag{A.12}$$

Finally, the dual transformations show that the Maxwell equations are invariant to

$$\begin{aligned} E_0(\mathbf{i} + \mathrm{i}\mathbf{j})e^{\mathrm{i}\phi} &\to B_0(\mathbf{i} + \mathrm{i}\mathbf{j})e^{\mathrm{i}\phi} \\ B_0(\mathbf{j} - \mathrm{i}\mathbf{i})e^{\mathrm{i}\phi} &\to E_0(\mathbf{j} - \mathrm{i}\mathbf{i})e^{\mathrm{i}\phi} \\ -E_0\mathbf{k} &\to \mathrm{i}E_0\mathbf{k} \\ \mathrm{i}B_0\mathbf{k} &\to B_0\mathbf{k} \end{aligned} \tag{A.13}$$

so Eq. (40) is also satisfied by

$$\begin{aligned} \mathbf{B}^G &= \mathbf{B} - \mathrm{i}B_0\mathbf{k} \\ \mathbf{E}^G &= \mathbf{E} - E_0\mathbf{k} \end{aligned} \tag{A.14}$$

Therefore (A.14) is also a valid solution of Maxwell's equations, and it is therefore possible to obtain valid solutions of Maxwell's equations in which $\mathbf{E}_\Pi$ is real and $\mathbf{B}_\Pi$ is imaginary, or in which $\mathbf{E}_\Pi$ is imaginary and $\mathbf{B}_\Pi$ is real. Since $\mathbf{B}_\Pi$ from eqn. (18) is real, $\mathbf{E}_\Pi$ is imaginary.

It may also be verified that solutions of the type

$$\begin{aligned} \mathbf{E}^G &= \mathbf{E}(\mathbf{r}, t) \pm E_0(\mathrm{i} - 1)\mathbf{k} \\ \mathbf{B}^G &= \mathbf{B}(\mathbf{r}, t) \pm B_0(\mathrm{i} + 1)\mathbf{k} \end{aligned} \tag{A.15}$$

satisfy Maxwell's equations in vacuo and also relation (40) of the text. The most general solution of Maxwell's equations is therefore in this case:

$$\mathbf{E}^{G} = \frac{E_0}{\sqrt{2}}\left[(\mathbf{i} \pm \mathrm{i}\mathbf{j})\mathrm{e}^{\mathrm{i}\phi} \pm \sqrt{2}\,(\mathrm{i} - 1)\mathbf{k}\right]$$
$$\mathbf{B}^{G} = \frac{B_0}{\sqrt{2}}\left[(\mathbf{j} \mp \mathrm{i}\mathbf{i})\mathrm{e}^{\mathrm{i}\phi} \pm \sqrt{2}\,(\mathrm{i} + 1)\mathbf{k}\right] \qquad (A.16)$$

where the normalisation factor $1/\sqrt{2}$ has been used, as is the standard practice.

Solutions (A.16) show that, in general, Maxwell's equations in vacuo support components in **i**, **j**, and **k** unit vectors in X, Y, and Z, respectively. In this case, the fields $\mathbf{E}_\Pi$ and $\mathbf{B}_\Pi$ are both complex:

$$\mathbf{B}_\Pi = B_0(1 + \mathrm{i})\mathbf{k}$$
$$\mathbf{E}_\Pi = E_0(-1 + \mathrm{i})\mathbf{k} \qquad (A.17)$$

APPENDIX B: QUANTIZATION OF $\mathbf{B}_\Pi$ AND $\mathbf{E}_\Pi$

Defining the annihilation and creation operators

$$\hat{a}(t) = \hat{a}(0)\mathrm{e}^{-\mathrm{i}\omega t}$$
$$\hat{a}^{+}(t) = \hat{a}^{+}(0)\mathrm{e}^{\mathrm{i}\omega t} \qquad (B.1)$$

the quantized equivalent of the oscillating electric field

$$\mathbf{E}(t) = \frac{E_0}{\sqrt{2}}\mathrm{e}^{\mathrm{i}\boldsymbol{\kappa}\cdot\mathbf{r}}(\mathbf{i}\mathrm{e}^{-\mathrm{i}\omega t} + \mathrm{i}\mathbf{j}\mathrm{e}^{-\mathrm{i}\omega t}) \qquad (B.2)$$

becomes the operator

$$\hat{\mathbf{E}}(t) = \frac{E_0}{\sqrt{2}}\mathrm{e}^{\mathrm{i}\boldsymbol{\kappa}\cdot r}(\hat{a}_X(t)\mathbf{i} + \mathrm{i}\hat{\mathbf{a}}_Y(t)\mathbf{j}) \qquad (B.3)$$

Similarly,

$$\hat{\mathbf{E}}^{*}(t) = \frac{E_0}{\sqrt{2}}\mathrm{e}^{-\mathrm{i}\boldsymbol{\kappa}\cdot\mathbf{r}}(\hat{a}_X^{+}(t)\mathbf{i} - \mathrm{i}\hat{a}_Y^{+}(t)\mathbf{j}) \qquad (B.4)$$

and

$$\mathbf{E} \times \mathbf{E}^{*} = -\mathrm{i}\frac{E_0^2}{2}\left(\hat{a}_X\hat{a}_Y^{+} - \hat{a}_Y\hat{a}_X^{+}\right)\mathbf{k} \qquad (B.5)$$

where

$$\hat{a}_X\hat{a}_Y^{+} \equiv \hat{a}_X(t)\hat{a}_Y^{+}(t) = \hat{a}_X(0)\hat{a}_Y^{+}(0) \qquad (B.6)$$

and

$$E_0^2 = \frac{2\hbar\omega}{\varepsilon_0 L_0^3}$$

Here L_0^3 is the volume of quantization[9] and ε_0 the vacuum permittivity.

The expectation value of the operator in Eq. (B.5) is the classical conjugate product $\mathbf{E} \times \mathbf{E}^{*}$:

$$\langle n|\hat{\mathbf{E}} \times \hat{\mathbf{E}}^{*}|n\rangle = -\mathrm{i}E_0^2\mathbf{k} \qquad (B.7)$$

implying that

$$\langle n|\hat{a}_X\hat{a}_Y^{+} - \hat{a}_Y\hat{a}_X^{+}|n\rangle = 2 \qquad (B.8)$$

It follows from Eq. (B.5) that

$$\hat{B}_\Pi = \tfrac{1}{2}B_0\left(\hat{a}_X\hat{a}_Y^{+} - \hat{a}_Y\hat{a}_X^{+}\right)\mathbf{k} \qquad (B.9)$$

where

$$B_0 = \left(\frac{2\mu_0\omega}{L_0^3}\right)^{1/2} \qquad (B.10)$$

with μ_0 denoting the vacuum permeability. As is the standard practice in quantum field theory, $|n\rangle$ is an eigenfunction whose eigenvalues are nonnegative integers n, i.e.,

$$\hat{N}|n\rangle = n|n\rangle \qquad (B.11)$$

where $\hat{N}$ is the photon number operator, whose eigenstates are photon number states of the quantized field. The photon is a quantum of the field, with energy $\hbar\omega$. Thus, $\hat{a}^{+}$ operates on $|n\rangle$ to produce a field increment of energy $\hbar\omega$, i.e., to produce a photon, and $\hat{a}$ operates conversely:

$$\begin{aligned} \hat{a}^{+}|n\rangle &= (n+1)^{1/2}|n+1\rangle \\ \hat{a}^{-}|n\rangle &= n^{1/2}|n-1\rangle \end{aligned} \tag{B.12}$$

the normalization factor being chosen so that[9]

$$\langle n|n'\rangle = \delta_{nn'} \tag{B.13}$$

which is the orthonormality condition.

It is instructive to note that Eq. (B.9) for the operator $\hat{B}_{\Pi}$ can be derived independently of the conjugate operator $\hat{\mathbf{E}} \times \hat{\mathbf{E}}^*$, showing, inter alia, that $\hat{\mathbf{B}}_{\Pi}$ *and* $\hat{\mathbf{E}} \times \hat{\mathbf{E}}^*$ are rigorously proportional to each other in quantum field theory. The independent derivation proceeds from a direct quantization of the classical magnetostatic field

$$\mathbf{B}_{\Pi} = B_0\mathbf{k} \tag{B.14}$$

where we have made no assumptions concerning the origin of this field. Here $\mathbf{k}$ is a axial unit vector in the Z axis of the frame (X, Y, Z):

$$\mathbf{k} = \mathbf{i} \times \mathbf{j} \tag{B.15}$$

where $\mathbf{i}$ is a polar unit vector in X and $\mathbf{j}$ a polar unit vector in Y. Thus, B_0 is a scalar magnetic flux density amplitude (tesla).

It is always possible to write Eq. (B.14) as

$$\begin{aligned} \mathbf{B}_{\Pi} &= B_0\mathbf{k} \equiv \tfrac{1}{2}B_0 e^{i\boldsymbol{\kappa}\cdot\mathbf{r}}e^{-i\boldsymbol{\kappa}\cdot\mathbf{r}} \\ &\quad\times\left(e^{-i\omega t}\mathbf{i} \times e^{i\omega t}\mathbf{j} - e^{-i\omega t}\mathbf{j} \times e^{i\omega t}\mathbf{i}\right) \\ &= \tfrac{1}{2}B_0 e^{-i\omega t}e^{i\omega t} \\ &\quad\times\left(\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} - \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}\right) \end{aligned} \tag{B.16}$$

a purely mathematical identity, which leads to the well-known conclusion of tensor algebra that an axial vector is equivalent to a second-rank polar tensor.

If we now assume that ω is the angular frequency of the classical electromagnetic plane wave, then the quantized form of any magnetic field operator of the form (B.14) carried by such a wave pattern in quantum

field theory must be

$$\hat{\mathbf{B}}_{\Pi} = \frac{B_0}{2}\left(\hat{a}_X\hat{a}_Y^{+} - \hat{a}_Y\hat{a}_X^{+}\right)\mathbf{k} \tag{B.17}$$

which, except for sign, is the same as Eq. (B.9). The sign of $\hat{\mathbf{B}}_{\Pi}$ i switched[1–7] by switching from left to right circular polarization, and ir consequence $\hat{\mathbf{B}}_{\Pi}$ can always be defined as plus or minus. Equation (B.17 has been derived directly from Eq. (B.16) using the definitions (B.1), anc using no other assumption. This shows that the magnetic field operator $\hat{\mathbf{B}}_{\Pi}$ and the conjugate product operator $\hat{\mathbf{E}} \times \hat{\mathbf{E}}^*$ are rigorously proportional ir quantum field theory. Both are described by the operator $(\hat{a}_X\hat{a}_Y^{+} - \hat{a}_Y\hat{a}_X^{+}$ whose expectation values between states is always 2.

Further insight to the physical interpretation of $\hat{\mathbf{B}}_{\Pi}$ *and* $\hat{\mathbf{E}} \times \hat{\mathbf{E}}^*$ can be gained by using the quantum field definition[9] of the Stokes parameter S_3

$$\hat{S}_3 = -\frac{E_0^2}{2}\left(\hat{a}_X\hat{a}_Y^{+} - \hat{a}_Y\hat{a}_X^{+}\right) \tag{B.18}$$

so that

$$\hat{\mathbf{B}}_{\Pi} = -\frac{\hat{S}_3}{E_0 c}\mathbf{k} \tag{B.19}$$

showing that $\hat{\mathbf{B}}_{\Pi}$ *and* $\hat{\mathbf{E}} \times \hat{\mathbf{E}}^*$ are both directly proportional to the scalar operator $\hat{S}_3$, the third Stokes operator of quantum field theory.

Furthermore, the Stokes operators $\hat{S}_1$, $\hat{S}_2$, *and* $\hat{S}_3$ obey the commutator equations of angular momentum in quantum field theory,[9] showing that $\hat{\mathbf{B}}_{\Pi}$ has the properties of quantized angular momentum of the electromagnetic wave. Standard theory[9] shows that

$$\langle n|\hat{S}_3|n\rangle = \langle\hat{\sigma}_Z\rangle = \frac{\langle\hat{J}_Z\rangle}{\hbar} \tag{B.20}$$

where $\hat{\sigma}_Z$ is the Pauli matrix operator:

$$\hat{\sigma}_Z \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{B.21}$$

so that

$$\boldsymbol{\psi} = \frac{E_0}{\sqrt{2}}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \tag{B.22}$$

is a spinor. The operator $\hat{J}_Z$ is defined by

$$\langle n|\hat{J}_Z|n\rangle = \hbar S_3 \tag{B.23}$$

Using Eq. (B.21), we obtain

$$\hat{\mathbf{B}}_{\Pi} = B_0 \frac{\hat{J}_Z}{\hbar}\mathbf{k} = B_0 \frac{\hat{J}}{\hbar} \tag{B.24}$$

with B_0 given by Eq. (B.10) in SI units.

It is seen from Eq. (B.23) that the expectation value of $\hat{J}_Z$ between eigenfunctions $|n\rangle$ is S_3, the classical third Stokes parameter, multiplied by $\hbar$, the unit angular momentum in quantum theory. The expectation value of $\hat{J}_Z$ is S_3 units of quantized angular momentum for any eigenstate $|n\rangle$ of the quantized field. Classically, the third Stokes operator is

$$S_3 = -\frac{i}{2}(E_X iE_Y^* + iE_Y E_X^*) = \tfrac{1}{2}(E_X E_Y^* + E_Y E_X^*) \equiv E_L^2 \tag{B.25}$$

a real quantity. The intensity of the beam is classically

$$I_0 = \varepsilon_0 c E_L^2 \tag{B.26}$$

showing that the expectation value of the angular momentum operator $\hat{\mathbf{J}}$ is

$$\frac{1}{\hbar}\langle n|\hat{\mathbf{J}}|n\rangle = \frac{I_L}{\varepsilon_0 c} \tag{B.27}$$

which is directly proportional to the intensity of a beam that is fully left circularly polarized. In a beam that has elements of both left and right circular polarization,

$$\frac{1}{\hbar}\langle n|\hat{\mathbf{J}}|n\rangle = \frac{1}{\varepsilon_0 c}(I_L - I_R) \tag{B.28}$$

If the beam were to consist of a wave pattern corresponding to one photon of energy $\hbar\omega$, then $|n\rangle = |1\rangle$ and $\langle 1|\hat{\mathbf{J}}|1\rangle$ is the expectation value of $\hat{\mathbf{J}}$ for one photon. In this case, $\hat{\mathbf{B}}_{\Pi}$ is the magnetic flux density operator of one photon, whose scalar magnitude we denote $B_0^{(1)}$:

$$B_0^{(1)} = \left(2\mu_0 \frac{\hbar\omega}{L_0^3}\right)^{1/2} \tag{B.29}$$

which is proportional to the square root of $\hbar\omega$, the energy of the single photon under consideration. The energy of n photons is describable by the expectation value[9]

$$\langle n|\hat{H}^{R}|n\rangle = (n + \tfrac{1}{2})\hbar\omega \tag{B.30}$$

i.e., by an integer number n of energy quanta $\hbar\omega$, plus a "background" $\hbar\omega/2$ independent of n. Therefore,

$$nB_0^{(1)2} = \frac{2n\omega_0}{L_0^3}\hbar\omega \tag{B.31}$$

and the energy $n\hbar\omega$ is proportional to $nB_0^{(1)2}$. Using $E_0 = cB_0$ and Eq. (B.26), it becomes clear that the intensity of a beam of n photons is proportional to $nB_0^{(1)2}$, so that $B_0^{(1)}$ is an elementary quantum of magnetostatic flux density associated with one photon. This is in analogy with the fact that the quantum of energy associated with the photon is $\hbar\omega$. From Eq. (B.29), $B_0^{(1)}$ vanishes if $\omega = 0$, i.e., if the frequency of the wave is zero. In this case the energy $\hbar\omega$ is zero, and there are no photons. Alternatively, if $L_0^3 \to \infty$, i.e., if the quantization volume tends to infinity, then $B_0^{(1)}$ tends to zero, even for finite ω.

Under all other conditions, $B_0^{(1)}$ is nonzero, and produces finite and measurable physical effects, as described in the text.

The quantization of the imaginary $\mathbf{E}_{\Pi}$ proceeds similarly using the classical dual transformation

$$\mathbf{B}_{\Pi} \to -\frac{i}{c}\mathbf{E}_{\Pi} \tag{B.32}$$

And Eq. (B.9)

$$i\hat{\mathbf{E}}_{\Pi} = \tfrac{1}{2}E_0 i\left(\hat{a}_X \hat{a}_Y^+ - \hat{a}_Y \hat{a}_X^+\right)\mathbf{k} \tag{B.33}$$

whose expectation value is

$$\mathbf{E}_{\Pi} = \langle n|\hat{\mathbf{E}}_{\Pi}|n\rangle = -E_0\mathbf{k} \tag{B.34}$$

is an acceptable definition (see Appendix A) of $\mathbf{E}_{\Pi}$. Here $\mathbf{k}$ is of course a polar unit vector.

It is seen that both $\hat{\mathbf{B}}_{\Pi}$ *and* $i\hat{\mathbf{E}}_{\Pi}$ are defined in terms of the operator $\hat{a}_X \hat{a}_Y^+ - \hat{a}_Y \hat{a}_X^+$, which operates on any number state $|n\rangle$ to give the constant expectation value of 2. This expectation value is independent of the number state $|n\rangle$ of the photons, and generates the third Stokes parameter S_3 of the classical field.

Classically, the $\mathbf{B}_\Pi$ is an axial vector, $\hat{P}$-positive and $\hat{T}$-negative, and proportional to angular momentum. The field $\mathbf{E}_\Pi$ is a polar vector, $\hat{P}$-negative and $\hat{T}$-positive, and cannot therefore be proportional to an angular momentum. It is essential to note, therefore, that $\mathbf{k}$ in Eq. (B.9) is an axial unit vector ($\hat{T}$-negative and $\hat{P}$-positive) and that $\mathbf{k}$ in Eq. (B.33) is a polar unit vector ($\hat{T}$-positive and $\hat{P}$-negative).

Finally, we note that $\hat{a}_X\hat{a}_Y^* - \hat{a}_Y\hat{a}_X^*$ operates to give an eigenvalue of 2, and this does not change the energy $n\hbar\omega$ of n photons. This is in agreement with the classical theory (see text), which shows that $\mathbf{B}_\Pi$ and $\mathbf{E}_\Pi$ do not contribute to the field energy.

APPENDIX C: DEFINITION OF $\mathbf{B}_\Pi$ AND $\mathbf{E}_\Pi$ IN TERMS OF THE VECTOR POTENTIAL IN FREE SPACE

In free space, the oscillating fields **E** and **B** of the plane wave are defined in terms of the vector potential **A**. Using the Coulomb gauge:

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi; \qquad \mathbf{B} = \nabla \times \mathbf{A} \tag{C.1}$$

In free space the scalar part, $\nabla\phi$ is zero and the Lorentz condition and Coulomb gauge are both describable by

$$\nabla \cdot \mathbf{A} = 0 \tag{C.2}$$

From the definition in the text,

$$\mathbf{B}_\Pi = \frac{1}{E_0 c}\left(\frac{\partial \mathbf{A}}{\partial t} \times \frac{\partial \mathbf{A}^*}{\partial t}\right) \equiv B_0\mathbf{k} \tag{C.3}$$

where

$$\mathbf{E}^* = -\frac{\partial \mathbf{A}^*}{\partial t} - \nabla\phi^* \tag{C.4}$$

Using the condition for conservation of energy,

$$\mathbf{E}_\Pi \times \mathbf{B} = \mathbf{B}_\Pi \times \mathbf{E} \tag{C.5}$$

with the definition (C.3) implies

$$\mathbf{E}_\Pi = \mathrm{i}E_0\mathbf{k} \tag{C.6}$$

and

$$\mathbf{B} \times \mathbf{E}_\Pi = \mathrm{i}B_0\mathbf{E} \tag{C.7}$$

From (C.1) in (C.7),

$$(\nabla \times \mathbf{A}) \times \mathbf{E}_\Pi = \mathrm{i}B_0\mathbf{E} = -\mathrm{i}B_0\left(\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi\right) \tag{C.8}$$

which defines $\mathbf{E}_\Pi$ in terms of **A**.

Equation (C.8) can be simplified to

$$\nabla(\mathbf{E}_\Pi \cdot \mathbf{A}) = \mathrm{i}B_0\left(\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi\right) \tag{C.9}$$

using

$$(\nabla \times \mathbf{A}) \times \mathbf{E}_\Pi = \mathbf{A}(\mathbf{E}_\Pi \cdot \nabla) - \nabla(\mathbf{E}_\Pi \cdot \mathbf{A}) \tag{C.10}$$

and

$$\mathbf{E}_\Pi \cdot \nabla = \nabla \cdot \mathbf{E}_\Pi = 0 \tag{C.11}$$

Note that (C.9) is a type of continuity equation which defines the imaginary $\mathbf{E}_\Pi$ in terms of **A** of the oscillating components **E** and **B** of the plane wave.

In texts on the electromagnetic plane wave it is usually asserted that **E** and **B** are transverse plane waves, with no components in the direction of propagation. Equations (C.1) and (C.2) are usually taken as justification for this conclusion. Most texts assert that electromagnetic plane waves in vacuo are necessarily time-varying, because the solutions for constant E and B from Maxwell's equations in the absence of charge and current are zero. While this is true for linear solutions, we can form a nonlinear solution, Eq. (C.3), for $\mathbf{B}_\Pi$, which is well defined as in this paper, and which is a product of time-varying solutions. With the condition (C.5), derived in the text of this paper, the field $\mathbf{E}_\Pi$ is also well defined in terms of **A** as in Eq. (C.9), a novel continuity equation.

Nonlinear solutions of Maxwell's equations therefore support the existence of $\mathbf{E}_\Pi$ and $\mathbf{B}_\Pi$ in the axis of propagation of the plane wave in vacuo.

APPENDIX D: THE CONSTANT OF INTEGRATION IN EQUATION (35)

Most generally, from Eq. (35),

$$\mathbf{E}_\Pi \times \mathbf{B} = \mathbf{B}_\Pi \times \mathbf{E} + \text{constant} \tag{D.1}$$

The dual transformation of special relativity means that $\mathbf{E}_\Pi$ and $-(c/\mathrm{i})\mathbf{B}_\Pi$, for example, are indistinguishable solutions of Maxwell's equations; i.e., it is possible to replace $\mathbf{E}_\Pi$ everywhere by $-(c/\mathrm{i})\mathbf{B}_\Pi$ without changing the Maxwell equations, and therefore without changing the solutions to the equations. The dual transformation, however, does not affect the constant in Eq. (D.1), which is independent of $\mathbf{E}$, $\mathbf{B}$, $\mathbf{E}_\Pi$, and $\mathbf{B}_\Pi$. Thus, applying the dual transform,

$$\mathbf{B}_\Pi \times \mathbf{E} = \mathbf{E}_\Pi \times \mathbf{B} + \text{constant} \tag{D.2}$$

Adding Eqs. (D.1) and (D.2) yields

$$\text{Constant} = 0$$

Acknowledgments

Support for this work is acknowledged from Cornell Theory Center, the Materials Research Laboratory of Penn State University, and Optical Ventures Inc. Many interesting discussions are acknowledged with colleagues at UNCC and elsewhere.

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ON LONGITUDINAL FREE SPACETIME ELECTRIC AND MAGNETIC FIELDS IN THE EINSTEIN-DE BROGLIE THEORY OF LIGHT

I. INTRODUCTION

It is usually concluded in electrodynamical literature[1-16] that the photon is massless and that the range of the electromagnetic field is infinite. This conclusion is not, however, supported by experimental data. To the contrary, Vigier[17] has recently reviewed a substantial amount of evidence that leads to the conclusion of finite photon rest mass. These data include, to take two of many examples, the direction-dependent anisotropy of the frequency of light in cosmology and frequently observed anomalous red shifts.

In papers and correspondence circa 1916 to 1919,[17] Einstein[18] proposed a photon rest mass[19] that can be estimated from the Hubble constant to be about 10^{-68} kg. An immediate consequence is that the d'Alembert equation is replaced by the Einstein-de Broglie-Proca (EBP) equation, which can be expressed[20, 21] in the form

$$\Box A_\mu = -\xi^2 A_\mu \tag{1}$$

where

$$\xi = \frac{m_0 c}{\hbar}$$

Here m_0 is the photon rest mass, c the speed of light, the universal constant of special relativity, and $\hbar$ the reduced Planck constant. The potential four vector A_μ of the de Broglie-Proca field is manifestly covariant, and has four, physically meaningful, components, one timelike ((0)) and three spacelike, of which two are transverse ((1) and (2)) and one is longitudinal ((3)). From Eq. (2), the range ξ^{-1} of the field becomes 10^{26} m, cosmic in dimensions, but finite. Equation (1) is an expression of the Einstein-de Broglie theory of light[17] and implies that gauge transformations of the first and second kind[20, 21] can no longer be interpreted as implying zero photon rest mass. It is well known that the EBP equation implies mathematically[20, 21] the Lorentz condition

$$\frac{\partial A_\mu}{\partial x_\mu} = 0 \tag{2}$$

for the massive boson. If the photon has rest mass, it is always described by the Lorentz condition. Experimental evidence[17] for finite photon rest mass implies that gauge invariance must be reinterpreted fundamentally, and this is part of the purpose of this paper, in which it is shown that finite m_0 is consistent with gauge invariance of the first and second kind if and only if

$$A_\mu A_\mu = 0, \quad m_0 \neq 0 \tag{3a}$$

a condition that implies

$$\phi = c|\mathbf{A}| \tag{3b}$$

where

$$A_\mu = \left(\mathbf{A}, \frac{i}{c}\phi\right) \tag{3c}$$

and ϕ is the scalar potential and $\mathbf{A}$ the vector potential of the de Broglie-Proca field. Condition (3a) is consistent with the Lorentz condition (2), but is inconsistent with a massless gauge such as the traditional Coulomb gauge.[1–16]

Furthermore, the notion of zero photon rest mass leads to considerable physical obscurity, for example, in the quantization of the Maxwellian electromagnetic field.[20, 21] The traditional theory abandons the longitudinal and timelike field polarizations as being "unphysical," and in so doing inevitably loses manifest covariance. Another traditional difficulty[20] is that the little group of the Poincaré group[20] for the massless photon becomes the Euclidean E(2), which is physically obscure. The Lie algebra for the Maxwellian electromagnetic field on the other hand is that of the Lorentz group. These difficulties are accepted because it is traditionally thought that special relativity implies zero photon mass, and that gauge invariance of the first and second kind can be interpreted only in terms of zero photon rest mass. In this paper it is shown that both of these traditional viewpoints are flawed, and that in consequence, the Einstein-de Broglie theory of light is consistent with both special relativity and gauge transformation. We recall for reference that the massless electromagnetic field is summarized in the d'Alembert equation:

$$\Box A_\mu = 0 \tag{4}$$

Quantization[20] of Eq. (1) is straightforward, but that of Eq. (4) is beset with considerable difficulty. From quantization of Eq. (1), for the massive boson, the conclusion is reached that the massive boson is a particle (the photon) with finite mass and three physically meaningful spacelike polarizations, (1), (2), and (3). Quantization[20] of Eq. (4) traditionally proceeds in the Coulomb or Lorentz gauge. To quote from Ryder,[20] "Quantisation of the electromagnetic field suffers from difficulties posed by gauge invariance. The quantisation procedure is outlined in both the radiation (Coulomb) gauge, in which there appear only the two physical (transverse) polarisation states, and in the Lorentz gauge, in which all four polarisation states appear, the formalism being Lorentz covariant. The resulting difficulties are resolved by the method of Gupta and Bleuler." The reader is referred to Ryder[20] for an excellent account of these difficulties. The Coulomb gauge is inconsistent, furthermore, with a nonzero photon rest mass, so that, conversely, finite m_0 implies immediately that the notion of there being only two physically meaningful photon polarization states must be abandoned. One is led ineluctably to the conclusion that there are four physically meaningful photon polarizations ((0) to (3)).

Lorentz gauge quantization[20] in the limit $m_0 \to 0$ is possible only with the Gupta-Bleuler condition,[22] which leads to the conclusion that admixtures of timelike and longitudinal spacelike photon polarizations are physical states.[20] In a diametrically self-contradictory procedure, the traditional theory abandons these physical states as unphysical.

This procedure is logically untenable, and recently[23-28] this has become clear through the discovery of a simple relation between longitudinal and transverse solutions of Maxwell's equations in vacuo:

$$\mathbf{B}^{(3)} = \frac{\mathbf{E}^{(1)} \times \mathbf{E}^{(2)}}{E_0 c \mathrm{i}} = \frac{\mathbf{B}^{(1)} \times \mathbf{B}^{(2)}}{B_0 \mathrm{i}} = B_0 \mathbf{k} \tag{5}$$

Equation (5) comes directly from the original Maxwell equations, without the introduction of scalar and vector potentials, and is an entirely novel relation between physically meaningful electric and magnetic components of the electromagnetic field in vacuo. It can be derived without reference to gauge theory, but is consistent with gauge invariance. Here $\mathbf{E}^{(1)}$ and $\mathbf{E}^{(2)}$ are the oscillating transverse components of the electric field, taken to be a plane wave in vacuo. The vector product in Eq. (5) is defined by the Stokes parameter S_3:

$$\mathbf{E}^{(1)} \times \mathbf{E}^{(2)} = -S_3 \mathbf{k} \tag{6}$$

In a light beam in which there is some degree of circular polarization, therefore, S_3 is always nonzero, implying that the longitudinal magnetic field $\mathbf{B}^{(3)}$ is nonzero in vacuo. The transverse components in Eq. (5) are the usual vacuum plane wave solutions of Maxwell's equations:

$$\begin{aligned} \mathbf{E}^{(1)} &= \frac{E_0}{\sqrt{2}}(\mathbf{i} - \mathrm{i}\mathbf{j})e^{\mathrm{i}\phi} \qquad & \mathbf{E}^{(2)} &= \frac{E_0}{\sqrt{2}}(\mathbf{i} + \mathrm{i}\mathbf{j})e^{-\mathrm{i}\phi} \\ \mathbf{B}^{(1)} &= \frac{B_0}{\sqrt{2}}(\mathrm{i}\mathbf{i} + \mathbf{j})e^{\mathrm{i}\phi} \qquad & \mathbf{B}^{(2)} &= \frac{B_0}{\sqrt{2}}(-\mathrm{i}\mathbf{i} + \mathbf{j})e^{-\mathrm{i}\phi} \end{aligned} \tag{7}$$

where the phase is

$$\phi = \omega t - \boldsymbol{\kappa} \cdot \mathbf{r}$$

Here ω is the angular frequency at an instant t, and $\boldsymbol{\kappa}$ is the wave vector

at a point $\mathbf{r}$. It can also be shown[23] that the concommitant longitudinal electric field $\mathbf{E}^{(3)}$ exists in vacuo, and is related to $\mathbf{B}^{(3)}$ by

$$\mathbf{E}^{(3)} \times \mathbf{B}^{(2)} = \mathbf{B}^{(3)} \times \mathbf{E}^{(2)} \tag{8}$$

so that $\mathbf{E}^{(3)}$ is nonzero if $\mathbf{B}^{(3)}$ is nonzero. The imaginary $\mathrm{i}\mathbf{E}^{(3)}$ is expressible as:

$$\mathrm{i}\mathbf{E}^{(3)} \propto \mathrm{i}E_0 \mathbf{k} \tag{9}$$

It is worth demonstrating explicitly that $\mathbf{B}^{(3)}$ and $\mathbf{E}^{(3)}$ are solutions in vacuo of the Maxwell equations, because

$$\begin{aligned} \nabla \times \mathbf{E}^{(3)} &= 0 \qquad & -\frac{\partial \mathbf{B}^{(3)}}{\partial t} &= 0 \\ \nabla \times \mathbf{B}^{(3)} &= 0 \qquad & \frac{1}{c^2}\frac{\partial \mathbf{E}^{(3)}}{\partial t} &= 0 \\ \nabla \cdot \mathbf{E}^{(3)} &= 0 \qquad & \nabla \cdot \mathbf{B}^{(3)} &= 0 \end{aligned} \tag{10}$$

These relations follow from Eqs. (5) and (9); i.e., $\mathbf{B}^{(3)}$ and $\mathbf{E}^{(3)}$ are solenoidal and phase independent.

In this paper we show that $\mathbf{B}^{(3)}$ and $\mathbf{E}^{(3)}$ are natural consequences of the Einstein-de Broglie theory of light, and are physically meaningful magnetic and electric fields. Experiments to detect them would support the theory of Einstein and de Broglie. Equations (5) and (9) are therefore relations between longitudinal and transverse field components in the massless limit of the Einstein-de Broglie theory. This conclusion is consistent with the recent development[24] by the present author of manifestly covariant electrodynamics, using electric and magnetic four vectors. This development is equivalent to the Einstein-de Broglie theory in the massless limit ($m_0 \to 0$), and is a direct consequence of the existence of $\mathbf{B}^{(3)}$ and $\mathbf{E}^{(3)}$ defined by Eqs. (5) and (9), respectively. It is impossible to reconcile the existence of Eqs. (5) and (9) with traditional thinking, in which $\mathbf{B}^{(3)}$ and $\mathbf{E}^{(3)}$ are abandoned as unphysical. Clearly, $\mathbf{B}^{(3)}$ and $\mathbf{E}^{(3)}$ are formed from physical quantities such as the Stokes parameter S_3. In the Einstein-de Broglie theory, on the other hand, $\mathbf{B}^{(3)}$ and $\mathbf{E}^{(3)}$ are physical fields, components of the four vectors E_μ and B_μ in vacuo. A longitudinal solution of Eq. (1) for $\mathbf{B}^{(3)}$ is given in Section II, where it is shown that $\mathbf{B}^{(3)}$ is an exponentially decaying function of ξ in the propagation axis Z of the light beam. The divergence of $\mathbf{B}^{(3)}$ is nonzero for finite m_0, and is given by $-\xi B^{(0)}$, a magnetic monopole in vacuo. The numerical value of ξ (10^{-26} m^{-1}) is so small that for all practical purposes, and for laboratory dimensions, $\mathbf{B}^{(3)}$ is a constant magnetic field, independent of distance and time. Section III derives general solutions of Eq. (1) for the transverse and longitudinal fields of the electromagnetic plane wave in vacuo. A discussion follows of the role of $\mathbf{B}^{(3)}$ and $\mathbf{E}^{(3)}$ in various experimental tests of the Einstein-de Broglie theory of light, taking into account experimental evidence[17] for finite photon mass.

II. LONGITUDINAL SOLUTIONS OF THE EBP EQUATION IN VACUO

In quantum optics interpreted by Einstein and de Broglie[17] light is constituted by real Maxwellian waves which coexist in spacetime with moving particles—photons. In the Copenhagen interpretation of Bohr, Schrödinger, Pauli, Glauber, and others, on the other hand, light is made up of waves of probability, which cannot coexist in spacetime with photons. In the interpretation of the Einstein-de Broglie school, the photon is massive; in that of the Copenhagen school, it is not necessarily so. The basic electrodynamical equations are therefore (1) and (4), respectively. Although it is frequently asserted[1–16, 20, 21] that the photon is massless in its rest frame, there is no supporting experimental evidence. Indeed, it appears to be impossible to test the hypothesis of zero m_0, because it is impossible to test the implication that the range of electromagnetic radiation is infinite. On the other hand, finite m_0 leads[17] to such observable implications as anomalous red shifts, reported on numerous occasions, and tired light phenomena. Einstein,[18] some years after his theory of special relativity (1905), and during his development of general relativity, proposed that the photon's rest mass is finite, i.e., that the mass of the photon is finite in a frame of reference moving at the speed of light. This leads[17–19] to Eq. (1). It is clear therefore that Einstein saw no contradiction with special relativity in his proposal; i.e., Eq. (1) is Lorentz covariant, even though the photon rest mass, m_0, is nonzero. Several conclusions flow immediately from this proposal.

Firstly, the notion that the photon is massless in the frame of the observer (laboratory frame) because it travels at the speed of light is incorrect if the photon rest mass m_0 is nonzero. In the contemporary description[20] of special relativity, the reason for this is that the quantity

$$C = P_\mu P_\mu \tag{11}$$

is the first (or "mass") Casimir invariant of the Poincaré (inhomogeneous Lorentz) group. Here P_μ is the generator of spacetime translations, first introduced by Wigner in 1939.[29] A spacetime translation is defined by the operation

$$x'_\mu = x_\mu + a_\mu \tag{12}$$

where x_μ is the distance/time four vector of Minkowski spacetime. P_μ does not appear in the homogeneous Lorentz group,[20] i.e., in a group made up only of boost transformations and Lorentz rotations. The quantity m_0^2 (the square of the rest mass) is therefore invariant to Lorentz transformations, i.e., is the same in the rest frame of the photon (which travels at the speed of light) and in the observer frame. The invariant m_0^2 appears in Eq. (1), which is Lorentz covariant, i.e., fully consistent with special relativity. The latter theory does not imply, therefore, that the photon rest mass is zero. It is clear that Einstein himself[17, 18] saw no inconsistency with special relativity in his proposal of finite m_0, and contemporary theory also shows that m_0^2 is an invariant of the Poincaré group. The Einstein-de Broglie theory of light is therefore consistent with special relativity. This means that the rest frame momentum of the photon (a massive boson) is timelike, not lightlike, and that the photon has rest energy m_0c^2, i.e., that the energy of the photon in its own frame of reference, which moves at the speed of light, is m_0c^2, about 10^{-57} J. The spacelike momentum of the photon in its own rest frame is zero, because it does not move relative to this rest frame. In its rest frame, the photon is thus described by a four vector:

$$\begin{aligned} q_\mu &= (0,0,0,\mathrm{i}m_0c) \\ &= \left(0,0,0,\mathrm{i}\frac{En_0}{c}\right) \end{aligned} \tag{13}$$

in Minkowski spacetime. In the laboratory frame of the observer, however, the photon's momentum is finite, and the vector (13) is transformed into

$$p_\mu = L_{\mu\nu} q_\nu \tag{14}$$

where $L_{\mu\nu}$ denotes a Lorentz transformation[20] which transforms q_ν into p_μ. Clearly, the latter is observed in the laboratory. Wigner[29] showed that this transformation can be described from a knowledge of the rotation group, and that the little group for q_μ is a rotation group.

As discussed by Vigier[17] the consequences are that photons slow down in the laboratory frame of an observer, although the rest frame must move at the speed of light, which is a universal constant of special relativity.

Photons in the frame of the observer behave like relativistic nonzero mass particles, with rest mass $m_0 \doteq 10^{-68}$ kg. The energy momentum four vector in the observer frame is p_μ, with components[17]

$$p_\mu \equiv \left(\mathbf{p}, \mathrm{i}\frac{En}{c}\right)$$
$$En = h\nu = m_0 c^2\left(1 - \frac{v^2}{c^2}\right)^{-1/2} \tag{15}$$
$$|\mathbf{p}| \doteq \frac{h\nu}{v} \doteq \frac{h\nu}{c}$$

The velocity of the photon in the observer frame is therefore not c, but v, defined from the Guiding Theorem of de Broglie, the basis of wave mechanics:

$$En_0 = h\nu_0 = m_0 c^2 \tag{16}$$

In other words, the energy of the photon in the rest frame is

$$En_0 = m_0 c^2 = h\nu_0 \tag{17}$$

and its energy in the observer frame is

$$En = m_0 c^2\left(1 - \frac{v^2}{c^2}\right)^{-1/2} = h\nu \tag{18}$$

so that there is a change in the frequency of light from one frame to the other. This is the origin of *observed* distance proportional shifts,[17] the "tired light" of Hubble and Tolman. There are photons, therefore, that move at low velocities and contribute to the mass of the universe. Clearly, this is a consequence of the fact that the field has a finite range, of about 10^{26} m, as discussed in the introduction. This conclusion does not contradict the principle of conservation of energy, because in special relativity, the quantity $P_\mu P_\mu$ is invariant to Lorentz transformation. Therefore, special relativity does not imply that the rest mass of the photon is zero, as in the traditional interpretations.[1-16, 20, 21]

Secondly, if m_0 is not zero, the traditional interpretation[20, 21] of gauge transformations must be revised fundamentally, because it leads to the conclusion that the photon rest mass m_0 is zero and therefore contradicts the Einstein-de Broglie theory and experimental evidence[17] for finite photon mass. Traditional considerations of gauge transformation also lead to the principle of gauge invariance (eicheninvarianz prinzip), which holds if and only if the photon mass is identically zero. For these reasons, we consider carefully the basic Lagrangian formalism of gauge theory, and modify its interpretation to make it consistent with finite m_0. The result of our considerations is Eq. (3a) of the introduction.

Geometrically, a gauge transformation of the first kind[20, 21] is a rotation in the (1, 2) plane of the "vector" field

$$\boldsymbol{\phi} = \phi_1 \mathbf{i} + \phi_2 \mathbf{j} \tag{19}$$

through an angle Λ. Under such a rotation, Noether's theorem leads to conserved charge Q in a volume V

$$Q = \mathrm{i}\int\left(\phi^*\frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^*}{\partial t}\right)\mathrm{d}V \tag{20}$$

and a conserved current

$$J_\mu = \mathrm{i}\left(\phi^*\frac{\partial\phi}{\partial x_\mu} - \phi\frac{\partial\phi^*}{\partial x_\mu}\right) \tag{21}$$

The existence of Q and J_μ is based on the invariance of action. When the action is real, the Lagrangian is[20]

$$\mathscr{L} = \left(\frac{\partial\phi}{\partial x_\mu}\right)\left(\frac{\partial\phi^*}{\partial x_\mu}\right) - m^2\phi^*\phi \tag{22}$$

where m is a mass associated with the complex field ϕ, defined by

$$\phi = \frac{\phi_1 + \mathrm{i}\phi_2}{\sqrt{2}}, \qquad \phi^* = \frac{\phi_1 - \mathrm{i}\phi_2}{\sqrt{2}} \tag{23}$$

Since Λ is a constant (an angle in (1, 2)) the gauge transformation of the first kind, which can be expressed[20] as

$$\phi \rightarrow e^{-i\Lambda}\phi \qquad \phi^* \rightarrow e^{i\Lambda}\phi \tag{24}$$

is the same at all points in spacetime, so that at an instant t the same rotation occurs for all points in space. This contradicts special relativity[20] whose universal constant is the speed of light, and which implies that action at a distance is impossible. Electrodynamics cannot, therefore, be invariant to a gauge transformation of the first kind. To comply with special relativity, Λ is made an arbitrary function of spacetime:

$$\Lambda \equiv \Lambda(x_\mu) \tag{25}$$

so defining a gauge transformation of the second kind. For $\Lambda \ll 1$, electrodynamics is invariant to the gauge transformation of the second kind:

$$\phi \rightarrow \phi - i\Lambda(x_\mu)\phi \tag{26}$$

Condition (25) implies,[20] however, that $\partial\phi/\partial x_\mu$ does not transform in the same way as ϕ, i.e., does not transform covariantly, so that the action is no longer invariant[20, 21]:

$$\delta\mathscr{L} = J_\mu \frac{\partial\Lambda}{\partial x_\mu} \neq 0 \tag{27}$$

with $\mathscr{L}$ defined by Eq. (22). To preserve the invariance of action under (26), the potential four vector A_μ is introduced through

$$\mathscr{L}_1 = -eJ_\mu A_\mu \tag{28}$$

This implies the need for two more conditions[20]:

$$A_\mu \rightarrow A_\mu + \frac{1}{e}\frac{\partial\Lambda}{\partial x_\mu} \tag{29}$$

and

$$\mathscr{L}_2 \equiv e^2 A_\mu A_\mu \phi^* \phi \tag{30}$$

Equations (28) to (30) imply[20]

$$\delta\mathscr{L} + \delta\mathscr{L}_1 + \delta\mathscr{L}_2 = 0 \tag{31}$$

In fundamental gauge theory, therefore, A_μ of the conventional d'Alembert equation is introduced to produce Eq. (31) in association with the extra term (30). So far, nothing has been said about the need for zero mass. We note that if $A_\mu A_\mu = 0$, $\mathscr{L}_2$ is automatically zero.

The field A_μ itself makes a contribution to the Lagrangian, implying the need for an additional $\mathscr{L}_3$ to maintain a zero overall action[20]:

$$\mathscr{L}_3 \equiv -\tfrac{1}{4}F_{\mu\nu}F_{\mu\nu} \tag{32}$$

where

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} \tag{33}$$

the four curl of A_μ, is the electromagnetic field four tensor,[20] an invariant under (29). The complete Lagrangian is therefore

$$\mathscr{L}_{\text{tot}} = \mathscr{L} + \mathscr{L}_1 + \mathscr{L}_2 + \mathscr{L}_3 \tag{34}$$

If the mass, m_0, associated with the electromagnetic field is not zero, then the form of the Lagrangian is changed from (32) to

$$\mathscr{L}_4 = -\tfrac{1}{4}F_{\mu\nu}F_{\mu\nu} + \tfrac{1}{2}m_0^2 A_\mu A_\mu \tag{35}$$

and this is invariant to Eq. (29) if and only if

$$m_0^2 A_\mu A_\mu = 0 \tag{36}$$

If $m_0 \neq 0$, then

$$A_\mu A_\mu = 0 \qquad A_\mu \neq 0 \tag{37}$$

is the only alternative possible, as described in the introduction. Conventionally, it is asserted[20, 21] that the invariance of $\mathscr{L}_4$ under (29) means that $m_0 = 0$. However, in the Einstein-de Broglie theory, Eq. (37) is consistent with Eq. (29), and m_0 is quantized as the photon rest mass. Equation (37) is also consistent with Eq. (31) of fundamental gauge theory, because[20] $\mathscr{L}_2 = 0$ if $A_\mu A_\mu = 0$. This implies that

$$\delta\mathscr{L}_2 = 2eA_\mu\left(\frac{\partial\Lambda}{\partial x_\mu}\right)\phi^*\phi = 0 \tag{38}$$

so that

$$\begin{aligned}\delta\mathscr{L} + \delta\mathscr{L}_1 &= -\delta\mathscr{L}_2 \\ &= -2eA_\mu\left(\frac{\partial\Lambda}{\partial x_\mu}\right)\phi^*\phi \\ &= 0\end{aligned} \tag{39}$$

i.e., the action is conserved as in Eq. (31). The condition (37) for finite m_0 is one in which the EBP equation is invariant to the gauge transformation (29), which is implied by the need to conserve action under the gauge transformation of the second kind, Eq. (26). We therefore conclude that gauge theory does not imply that photon rest mass is zero.

If $m_0 \neq 0$, the quantity $A_\mu A_\mu$ vanishes, implying that $\phi = c|\mathbf{A}|$ where $A_\mu \equiv (\mathbf{A}, \mathrm{i}\phi/c)$. This condition is furthermore consistent with Eqs. (1) and (2), which is

$$\nabla\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t} = 0 \tag{40}$$

that is,

$$\nabla\cdot\mathbf{A} = -\frac{1}{c}\frac{\partial A}{\partial t} \tag{41}$$

Additionally, using the Lorentz condition, Eq. (40),

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A} = 0 \tag{42a}$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi = 0 \tag{42b}$$

whose solutions are the Liénard-Wiechert potentials.[1-16, 20] Clearly, Eqs. (42a) and (42b) become the same if $\phi = c|\mathbf{A}|$ in S.I. units.

Fundamental gauge theory does not imply that the photon rest mass must be zero, contrary to much of the current literature.[1-16, 20, 21] Secondly, special relativity, as pointed out by Einstein[17, 18] also does not imply zero photon rest mass. Thirdly, there is experimental evidence,[17] for nonzero m_0, and none for zero photon rest mass. Fourthly, the transverse, radiation, or Coulomb gauge[1-16, 20] is inconsistent with $\phi = c|\mathbf{A}|$, because in that gauge $\phi = 0$, $\mathbf{A} \neq 0$. The Lorentz gauge and Dirac gauge[17] are, on the other hand, consistent with $m_0 \neq 0$.

Having argued in some detail in this way, it becomes easy to see that much of the obscurity in the current thought on electromagnetism is due to the notion that the photon is massless and travels at the speed of light. Both statements contradict experimental evidence.[17] These notions result in "too much gauge freedom," in that Eqs. (26) and (29) can be satisfied with $m_0 = 0$ by the Coulomb, Lorentz, and other gauges. For $m_0 \neq 0$, as in the Einstein-de Broglie theory, the Coulomb gauge is invalidated, but the Lorentz gauge is a direct mathematical consequence of the EBP equation (1). The excess gauge freedom for $m_0 = 0$ results in severe[20, 21] difficulties of quantization of the electromagnetic field, whereas quantization of the EBP equation (a wave equation) is straightforward,[20] leading to longitudinal, physically meaningful, spacelike photon polarization, as well as the two transverse spacelike polarizations. It is natural to expect that a particle, the photon, should have three spacelike polarizations in three physical dimensions, X, Y, and Z.

The $m_0 = 0$ assertion is conventionally associated with the notion that the electromagnetic field is a massless gauge field with two independent components, customarily identified with left and right circular polarization. However, even in the limit $m_0 = 0$, the same Maxwellian field is covariantly described by the four components of A_μ. The Bohm-Aharonov effect[20] shows that A_μ is physically meaningful. Recent work,[23-28] leading to Eq. (5), shows conclusively that there is a well-defined relation between the transverse ((1) and (2)) and longitudinal ((3)) components of solutions of Maxwell's field equations in vacuo. It is straightforward to show that the three magnetic field components form a classical cyclic permutation in the circular basis, (1), (2), and (3): with $B^{(0)} = B_0$

$$\mathbf{B}^{(1)} \times \mathbf{B}^{(2)} = \mathrm{i}B^{(0)}\mathbf{B}^{(3)*} \qquad \mathbf{B}^{(3)*} = \mathbf{B}^{(3)} \tag{43a}$$

$$\mathbf{B}^{(2)} \times \mathbf{B}^{(3)} = \mathrm{i}B^{(0)}\mathbf{B}^{(1)*} \qquad \mathbf{B}^{(1)*} = \mathbf{B}^{(2)} \tag{43b}$$

$$\mathbf{B}^{(3)} \times \mathbf{B}^{(1)} = \mathrm{i}B^{(0)}\mathbf{B}^{(2)*} \qquad \mathbf{B}^{(2)*} = \mathbf{B}^{(1)} \tag{43c}$$

Furthermore, there exist classical permutations involving $\mathbf{E}^{(3)}$. If we assert $\mathbf{E}^{(3)} \equiv E^{(0)}\mathbf{k}$, these are, algebraically,

$$\mathbf{E}^{(1)} \times \mathbf{E}^{(2)} = \mathrm{i}\mathbf{E}^{(0)}c\mathbf{B}^{(3)*} \tag{44}$$

$$\begin{aligned} \mathbf{E}^{(2)} \times \mathbf{E}^{(3)} &= -\mathbf{E}^{(0)}c\mathbf{B}^{(1)*} \\ \mathbf{E}^{(3)} \times \mathbf{E}^{(1)} &= \mathbf{E}^{(0)}c\mathbf{B}^{(2)*} \\ \mathbf{E}^{(1)} \times \mathbf{B}^{(2)} &= \mathbf{B}^{(0)}\mathbf{E}^{(3)*} \\ \mathbf{E}^{(2)} \times \mathbf{B}^{(3)} &= \mathrm{i}\mathbf{B}^{(0)}\mathbf{E}^{(1)*} \\ \mathbf{E}^{(3)} \times \mathbf{B}^{(1)} &= -\mathbf{B}^{(0)}\mathbf{E}^{(2)*} \end{aligned} \tag{45}$$

$$\begin{aligned} \mathbf{E}^{(1)} \times \mathbf{B}^{(1)} &= 0 \\ \mathbf{E}^{(2)} \times \mathbf{B}^{(2)} &= 0 \\ \mathbf{E}^{(3)} \times \mathbf{B}^{(3)} &= 0 \end{aligned} \tag{46}$$

and are reminiscent of the Lie algebra of the Lorentz group,[20] a classical commutator algebra that is built up with boost and rotation generators defined in Minkowski spacetime. However, all the eqns. (45) violate $\hat{T}$ symmetry, which is a consequence of the fact that $\mathbf{E}^{(3)}$ is imaginary and cannot be derived from transverse solutions of Maxwell's equations. Eqns. (45) are not valid equations of electrodynamics while eqn. (44) is valid and identical with eqn. (43a). This does not mean that $\mathbf{E}^{(3)}$ itself violates $\hat{T}$ symmetry.

Before proceeding to the derivation of $\mathbf{B}^{(3)}$ for nonzero m_0, the purpose of this section, it is demonstrated that Lie algebra also applies to the electric and magnetic components of electromagnetic radiation in vacuo (the Maxwellian field) provided that these components are defined as classical field operators directly proportional respectively to the boost and rotation generators of the Lorentz transformation. This is a mathematical demonstration of the fact that if the longitudinal spacelike components of these fields are unphysical (i.e., zero), then the Lie algebraic structure of the Lorentz group is contradicted. This means that the Lorentz transformation itself is incorrectly defined, in that the longitudinal (Z) boost and rotation generator components are incorrectly asserted to be zero. This is equivalent to destroying the geometrical structure of Minkowski spacetime. Even in the Maxwellian limit $m_0 \to 0$, therefore, the assertion that $\mathbf{B}^{(3)}$ and $\mathbf{E}^{(3)}$ are zero results in a mathematical reductio ad absurdum.

That the Maxwell equations in vacuo are the Lorentz covariant equations[20, 21]:

$$\frac{\partial F_{\mu\nu}}{\partial x_\mu} = 0 \qquad \frac{\partial \tilde{F}_{\mu\nu}}{\partial x_\mu} = 0 \tag{47}$$

where $\tilde{F}_{\mu\nu}$ is the dual of $F_{\mu\nu}$, the electromagnetic field four tensor. The latter is antisymmetric under Lorentz transformation and its structure can be displayed as

$$F_{\mu\nu} \equiv \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -cB_3 & cB_2 \\ E_2 & cB_3 & 0 & -cB_1 \\ E_3 & -cB_2 & cB_1 & 0 \end{bmatrix} \tag{48}$$

We note that this structure is identical with that of the Lie algebra of the Lorentz group, defined[20] by the dimensionless, boost generator $\hat{K}_i$, an operator, and the rotation generator $\hat{J}_i$, also a dimensionless operator. The Lie algebra of the Lorentz group can be displayed as

$$\hat{J}_{\mu\nu} = \begin{bmatrix} 0 & \hat{K}_1 & \hat{K}_2 & \hat{K}_3 \\ -\hat{K}_1 & 0 & \hat{J}_3 & -\hat{J}_2 \\ -\hat{K}_2 & -\hat{J}_3 & 0 & \hat{J}_1 \\ -\hat{K}_3 & \hat{J}_2 & -\hat{J}_1 & 0 \end{bmatrix} \tag{49}$$

i.e., as

$$\hat{J}_{\mu\nu}(\mu, \nu = 0, \ldots, 3) \begin{bmatrix} \hat{J}_{ij} = -\hat{J}_{ji} = \varepsilon_{ijk}, \hat{J}_k \\ \hat{J}_{i0} = -\hat{J}_{0i} = -\hat{K}_i \end{bmatrix} \tag{50}$$
$$(i, j, k = 1, 2, 3)$$

Equations (49) and (50) are condensed representations of the classical commutator (Lie) algebra of the Lorentz group[20]:

$$\left[\hat{J}_X, \hat{J}_Y\right] = \mathrm{i}\hat{J}_Z \text{ and cyclic permutations} \tag{51a}$$

$$\left[\hat{K}_X, \hat{K}_Y\right] = -\mathrm{i}\hat{J}_Z \text{ and cyclic permutations} \tag{51b}$$

$$\left[\hat{K}_X, \hat{J}_Y\right] = \mathrm{i}\hat{K}_Z \text{ and cyclic permutations} \tag{51c}$$

$$\left[\hat{K}_X, \hat{J}_X\right] = 0 \text{ etc.} \tag{51d}$$

The geometrical equivalence of (48) and (49) means that

$$\begin{aligned} \hat{E}_i &= E_0\hat{K}_i \\ \hat{B}_i &= B_0\hat{J}_i \end{aligned} \tag{52}$$

where $\hat{E}_i$ and $\hat{B}_i$ are classical electric and magnetic field operators, a result that is implied by the proportionality of the classical operator matrices $\tilde{F}_{\mu\nu}$ and $\hat{J}_{\mu\nu}$. In the Cartesian basis (X, Y, Z),

$$\left[\hat{B}_X, \hat{B}_Y\right] = \mathrm{i} B_0 \hat{B}_Z \text{ and cyclic permutations} \tag{53a}$$

$$\left[\hat{E}_X, \hat{E}_Y\right] = -\mathrm{i} c B_0 \hat{B}_Z \text{ and cyclic permutations} \tag{53b}$$

$$\left[\hat{E}_X, \hat{B}_Y\right] = \mathrm{i} B_0 \hat{E}_Z \text{ and cyclic permutations} \tag{53c}$$

$$\left[\hat{E}_X, \hat{B}_X\right] = 0 \text{ etc.} \tag{53d}$$

Equations (53) represent a classical operator equivalent of the vector products in Eqs. (36), where the Maxwellian fields are vectors in space, and not operators defined in spacetime.

The ansatz (52) is based on the fundamental Lie algebraic structure of Minkowski spacetime, and implies the following:

1. The classical electric field operator $\hat{E}_i$ is proportional to a boost generator, and the classical magnetic field operator $\hat{B}_i$ is proportional to a rotation generator in Minkowski spacetime.
2. If the longitudinal component operators $\hat{B}_Z$ and $\hat{E}_Z$ are asserted to be zero, or unphysical, the structure of the Lie algebra is destroyed in Eqs. (53). For example, if $\hat{B}_Z = \hat{E}_Z = \hat{0}$, $[\hat{B}_X, \hat{B}_Y] \doteq \hat{0}$ and from the structure of $\hat{J}_i$ in Eq. (52), this is mathematically incorrect. Explicitly,

$$\left[\hat{B}_X, \hat{B}_Y\right] = B_0^2\left[\hat{J}_X, \hat{J}_Y\right] = B_0^2 \hat{J}_Z \neq 0 \tag{54}$$

because[20]

$$\hat{J}_X \equiv -\mathrm{i}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{55a}$$

$$\hat{J}_Y \equiv -\mathrm{i}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{55b}$$

$$\hat{J}_Z \equiv -\mathrm{i}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{55c}$$

3. The Maxwell equations (47) are seen to be relations between boost and rotation generators defined in spacetime:

$$\frac{\partial J_{\mu\nu}}{\partial x_\mu} = 0 \qquad \frac{\partial \tilde{J}_{\mu\nu}}{\partial x_\mu} = 0 \tag{56}$$

and are thus given a precise geometrical interpretation. In this light, it is seen that the d'Alembert equation (4) is also geometrical in nature:

$$\Box \hat{L}_\mu = 0 \tag{57}$$

where $\hat{J}_{\mu\nu}$ is the four curl of $\hat{L}_\mu$:

$$\hat{J}_{\mu\nu} = \frac{\partial \hat{L}_\nu}{\partial x_\mu} - \frac{\partial \hat{L}_\mu}{\partial x_\nu} \tag{58}$$

4. It may be seen precisely that the conventional notion that the Maxwellian $\hat{B}_Z$ (and $\hat{E}_Z$) is unphysical is equivalent to the geometrically incorrect assertion

$$\hat{J}_Z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{59}$$

which by implication habitually[1-16, 20, 21] replaces the correct rotation generator (55c).

There is of course no experimental evidence for Eq. (59), even in the massless limit $m_0 \to 0$ conventionally associated with the Maxwellian field.

Our geometrical interpretation of the Maxwell field equations is a direct logical consequence of the geometry of Minkowski spacetime itself and of the theory of special relativity. This is consistent with the fact that Einstein's considerations of the Maxwell equations led to his formulation of special relativity. If it is asserted that longitudinal solutions of Maxwell's equations be unphysical, special relativity is contradicted and the structure of the Lorentz group and its associated Lie algebra is destroyed. There is no experimental evidence whatsoever that the longitudinal solutions of Maxwell's equations in vacuo are unphysical, and there is no evidence for $m_0 = 0$.

The commutator relations (51a) and (53a) lead to a method of quantization of the Maxwellian field simply by noting the ordinary angular momentum commutator relations of quantum mechanics. In Cartesian terms,

$$\left[\hat{J}_X, \hat{J}_Y\right] = i\hbar\hat{J}_Z \tag{60}$$

are structurally identical with Eq. (51a) except for $\hbar$ (which has the units of angular momentum). In quantum mechanics, the $\hat{J}$ operators in Eq. (60) are angular momentum operators. Quantized angular momentum is therefore a consequence of the classical rotation generator,[20] as is well known. The quantized equivalent of Eq. (53a) must therefore be

$$\left[\hat{B}_X, \hat{B}_Y\right] = i\hbar\left(\frac{B_0}{\hbar}\right)\hat{B}_Z \tag{61}$$

to balance units, symmetries, and dimensions on the left and right sides. This implies

$$\hat{B} = B^{(0)}\frac{\hat{J}}{\hbar} \tag{62}$$

which is identical with the result obtained recently by the present author[25] using an independent method of derivation. Therefore,

$$\hat{B}_Z = B^{(0)}\frac{\hat{J}_Z}{\hbar} \tag{63}$$

is the elementary longitudinal component of the quantized Maxwellian magnetic field in vacuo. In the same way that $\hbar$ is the archetypical elementary quantum of angular momentum, $B^{(0)}$ is the elementary quantum of magnetic flux density of the Maxwellian field in vacuo.

The eigenvalues of $\hat{J}_Z$ in Eq. (63) may be identified with those of a massless boson (the "conventional" photon), i.e., $\hbar M_J$, where $M_J = \pm 1$, so that the classical limit of Eq. (63) is

$$\mathbf{B}_Z = B^{(0)}\mathbf{k} \tag{64}$$

which is Eq. (5) in Cartesian terms instead of a circular basis. Equation (5) is therefore geometrically consistent with the Lie algebra of the Lorentz group. The generalization of our development to $m_0 \neq 0$ is now straightforward.

Having considered in some detail the geometrical structure of the Lorentz group, we revert to a simpler development of the EBP equation (1), solving it as a classical eigenvalue equation with the differential operator:

$$\Box \equiv -\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \tag{65}$$

The order of magnitude of ξ is such that

$$\Box A_\mu \doteq 10^{-52}A_\mu \tag{66}$$

which closely approximates the d'Alembert equation (4). It is clear, therefore, that the classical interpretation of the EBP field closely approximates the Maxwellian field. However, in the EBP field, the Coulomb gauge is inconsistent with Eq. (66), which must be written in terms of the spacelike **A** as

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A} = \xi^2\mathbf{A} \tag{67}$$

with $\phi = c|\mathbf{A}|$. In the Galilean limit this equation becomes

$$\nabla^2\mathbf{A} = \xi^2\mathbf{A} \tag{68}$$

Using the relation

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{69}$$

it can be seen that the equation

$$\nabla^2\mathbf{B} = \xi^2\mathbf{B} \tag{70}$$

is the same as Eq. (68), because

$$\nabla^2(\nabla \times \mathbf{A}) = \xi^2\nabla \times \mathbf{A} \tag{71a}$$

$$\nabla \times \nabla^2\mathbf{A} = \nabla \times \xi^2\mathbf{A} \tag{71b}$$

In considering the Galilean limit, we have removed the time dependence in the solution for **B** of Eq. (70). Furthermore, since

$$\nabla^2\mathbf{B} = 10^{-52}\mathbf{B} \sim \mathbf{0} \tag{72}$$

describes the magnetic component in vacuo of an electromagnetic field closely resembling the Maxwellian field, we know that the time-independent solution to Eq. (70) must be the longitudinal component, defined in the propagation axis Z. The solution to Eq. (70) in Cartesian terms is therefore

$$\mathbf{B} = B^{(0)} \exp(-\xi Z)\mathbf{k}$$
$$|\mathbf{B}| = B_Z \tag{73}$$

and since $\xi \sim 10^{-26}\ \mathrm{m}^{-1}$, this is for all practical purposes identical with Eq. (64) of the Maxwellian field. Several physical consequences follow from Eqs. (64) and (73):

1. The longitudinal solution for **B** of the EBP field, Eq. (73), is for all practical purposes identical with the corresponding Maxwellian solution, Eq. (64). By the caveat "for all practical purposes" we imply laboratory dimensions and time scales. On a cosmic scale, in which $Z \sim 1/\xi$, Eq. (73) is different from Eq. (64) in general. In the "tired light" terminology of Hubble,[17] **B** becomes a "tired field" if Z is big enough (ca. 10^{26} m).
2. Physically meaningful, practically identical, and longitudinal solutions exist for **B** from the EBP and Maxwell equations, the former being considered as a classical wave equation. To assert $\mathbf{B} = \mathbf{0}$ in Eq. (64) is mathematically incorrect in the Maxwellian field, because it corresponds to the assertion (59) in spacetime. For all practical purposes, therefore, this assertion is incorrect in the EBP field. Quantization of the EBP field[20] confirms this conclusion, leading to a physically meaningful longitudinal photon polarization.
3. Since the EBP and Maxwellian fields are practically (i.e., in the laboratory) identical, the EBP field obeys the various commutator relations of this paper for all practical purposes, and the transverse EBP solutions are practically those of Eqs. (7). In the cosmology of light from distant sources, however, this simple classical interpretation is no longer tenable.

Quantization of the EBP field is straightforward,[20] whereas that of the Maxwellian field is obscure. Although the rest mass m_0 of the photon is very small, it is essential that it be rigorously nonzero to maintain a logical and self-consistent, physically meaningful, structure for the quantized electromagnetic field in vacuo. If this is done, quantization results in a consistent particle interpretation[20] in terms of a massive boson, with eigenvalues $M_J\hbar$, $M_J = -1, 0, +1$. The three polarization vectors of the

quantized EBP field are orthonormal and spacelike; i.e., there are physically meaningful longitudinal and transverse components. The little group of Wigner[29] is a physically meaningful rotation group, utilizing the three dimensions of space. If $m_0 = 0$, on the other hand, the constraint $A_\mu A_\mu = 0$ is conventionally lost, resulting in "too much gauge freedom." The two Casimir invariants[20] of the Poincaré group vanish for $m_0 = 0$, meaning that physical quantities that are invariant under the most general type of Lorentz transformation must vanish identically for the massless gauge field. This implies $A_\mu A_\mu = 0$, if $A_\mu A_\mu$ is to be an invariant of the Poincaré group, diametrically contradicting the conventional use of gauge freedom for a massless particle, i.e., contradicting the conventional assertion that $A_\mu A_\mu \not\equiv 0$ for $m_0 = 0$. Thus, the conventional assertion $A_\mu A_\mu \not\equiv 0$ for $m_0 = 0$ is geometrically unsound, i.e., contradicts the geometry of Minkowski spacetime, a geometry that requires $A_\mu A_\mu$ to be an invariant of the Poincaré (inhomogeneous Lorentz) group. We are forced to conclude that the widespread use of the Coulomb gauge, in which $A_\mu A_\mu = 0$, is relativistically incorrect. The conventional assertion that m_0 must be zero because $A_\mu A_\mu$ is nonzero is also basically incorrect, because $A_\mu A_\mu$ is always zero in vacuo.

It is the habitual use of the Coulomb (or "transverse") gauge that more than any other factor leads to the conventional assertion that the electromagnetic field can have no longitudinal solution that is physically meaningful. The Coulomb gauge is relativistically incorrect, and is inconsistent with finite photon rest mass, for which there is experimental evidence.[17] The widespread use of the Coulomb gauge[1–16, 20, 21] should therefore be viewed with caution. It is obvious that quantization in the Coulomb gauge cannot be consistent with special relativity, because its use is equivalent to the incorrect assertion (59). These difficulties are frequently compounded in the literature by a series of misstatements, traceable to the relativistically incorrect assertion $A_\mu A_\mu \not\equiv 0$. For example, it is frequently asserted that the Lorentz gauge does not define A_μ uniquely. This is true if and only if $A_\mu A_\mu \not\equiv 0$. If $A_\mu A_\mu = 0$, then the Lorentz condition defines A_μ uniquely. Quantization in the Coulomb gauge is therefore a mathematically incorrect procedure, and we discard its results as meaningless. In other words it is meaningless to assert that the Maxwellian field has only two transverse polarizations.

Quantization of the Maxwellian field in the Lorentz gauge[20] retains manifest covariance, but is physically obscure. It also relies on the notion that the gauge field is massless, so that quantization of the field must lead to a massless photon. In consequence, the internally inconsistent notion $A_\mu A_\mu \not\equiv 0$ is habitually retained in the Lorentz gauge. This immediately leads to the difficulty that the Lagrangian has to be modified with a gauge fixing term, a procedure that leads to a non-Maxwellian equation of motion.[20] Even with this artifice, the conjugate momentum field $\mathbf{\Pi}^0$ vanishes,[20] and the traditional method is forced to assert that the Lorentz condition, within whose framework the method is developed, cannot hold as an operator identity. This difficulty is habitually resolved by the method of Gupta and Bleuler,[20] a method that results in the conclusion that admixtures of timelike and longitudinal spacelike photon polarizations are physical states.[20] Despite this conclusion, these states are abandoned as unphysical in order to comply with the results of Coulomb gauge quantization, which, as we have just seen, are incorrect. Quantization of the Maxwellian field, regarded as a massless gauge field, is therefore inconsistent and physically obscure.

In considerations of the Poincaré group, the notion of a massless gauge field, habitually associated with the Maxwellian field, leads to the little group[20, 29] E(2), the Euclidean group of rotations and translations in a plane. The physical significance of this little group is obscure.[20] Its Lie algebra does not correspond to that of a rotation group, but it is the group that is needed to maintain a lightlike vector invariant under the most general Lorentz transformation. This suggests that the notion of a massless field is physically meaningless. The traditional line of reasoning, however, considers a massless particle traveling in the propagation axis (Z) described by a lightlike four vector k_μ. Invariance of k_μ under the most general type of Lorentz transformation leads to the Lie algebra:

$$\begin{aligned} \left[\hat{L}_1, \hat{L}_2\right] &= 0 \\ \left[\hat{J}_3, \hat{J}_1\right] &= i\hat{L}_3 \\ \left[\hat{L}_2, \hat{J}_3\right] &= i\hat{L}_1 \end{aligned} \tag{74}$$

where

$$\hat{L}_1 \equiv \hat{K}_1 - \hat{J}_2$$

$$\hat{L}_2 \equiv \hat{K}_2 + \hat{J}_1$$

Thus,

$$\begin{aligned} \left[\hat{L}_1, \hat{L}_2\right] &= \left[\hat{K}_1, \hat{K}_2\right] + \left[\hat{K}_1, \hat{J}_1\right] \\ &\quad - \left[\hat{J}_2, \hat{K}_2\right] - \left[\hat{J}_2, \hat{J}_1\right] \\ &= 0 \end{aligned} \tag{75}$$

In Cartesian terms, $X = 1, Y = 2, Z = 3$ and if we attempt to apply to Eq. (75) the Lorentz group algebra of Eqs. (51),

$$\left[\hat{K}_1, \hat{K}_2\right] = \left[\hat{K}_X, \hat{K}_Y\right] = -iJ_Z \tag{76a}$$

$$\left[\hat{J}_2, \hat{J}_1\right] = -\left[\hat{J}_X, \hat{J}_Y\right] = -i\hat{J}_Z \tag{76b}$$

$$\left[\hat{K}_1, \hat{J}_1\right] = \left[\hat{K}_X, \hat{J}_X\right] = 0 \tag{76c}$$

$$\left[\hat{J}_2, \hat{K}_2\right] = \left[\hat{J}_Y, \hat{K}_Y\right] = 0 \tag{76d}$$

we obtain

$$\left[\hat{L}_1, \hat{L}_2\right] = 2i\hat{J}_Z \neq \hat{0} \tag{77}$$

Since in the Lorentz group

$$\hat{J}_Z \neq \hat{0} \tag{78}$$

in general, Eq. (77) contradicts Eq. (75).

Therefore, the most general Lorentz transformation that leaves the lightlike momentum vector $\mathbf{k}_\mu$ invariant cannot be described by the Lie algebra of the Lorentz group. This implies that the notion of lightlike momentum (a massless particle traveling at the speed of light), is not relativistically self-consistent. This is another way of demonstrating that the quantization of a massless field into a massless particle is beset with obscurity; i.e., we are led to the conclusion that the Maxwellian field has no meaning in quantum theory. Attempts to impose a meaning lead into physical obscurity as we have described. In the Einstein-de Broglie theory of light, the quantization of the EBP field leads directly and without difficulty[20] to a particle interpretation of light in terms of a massive boson. Quantum/classical equivalence in the EBP field is therefore clear. The only physically meaningful and consistent interpretation is to accept the photon as a massive boson whose classical field is described by the classical limit of the EBP equation. The mathematical limit of this field for zero mass is the Maxwellian field. Direct quantization of the Maxwellian field, regarded as a classical massless gauge field, is physically obscure. The quantized Maxwellian field must therefore be defined as being for all practical purposes the quantized EBP field, with which it is practically identical because photon mass is numerically very small.

III. TRANSVERSE SOLUTIONS IN VACUO FOR FINITE PHOTON MASS

The EBP equation can be written in terms of the tensor $F_{\mu\nu}$ defined in Eq. (43) as

$$\frac{\partial F_{\mu\nu}}{\partial x_\mu} = -\xi^2 A_\nu \sim 0 \tag{79}$$

and, as we have seen, the Lie algebra associated with $F_{\mu\nu}$ is given by Eqs. (43)–(46). Therefore, transverse solutions of the EBP equation in its classical limit obey the classical cross products in Eqs. (43)–(46). Using Eq. (43a) with the longitudinal solution of the EBP equation (73), we obtain

$$\mathbf{B}^{(1)} = \frac{B_0}{\sqrt{2}}(\mathbf{i}i + \mathbf{j})e^{i\phi}e^{-\xi Z/2}$$
$$\mathbf{B}^{(2)} = \mathbf{B}^{(1)*} \tag{80}$$

and its complex conjugate. The difference between this solution and the equivalent Eq. (7c) for the Maxwell equations can be expressed by replacing the wave vector of the Maxwell equations by

$$\kappa_Z \to \kappa_Z - \frac{i\xi}{2} \text{ for polarization 1}$$
$$\kappa_Z \to \kappa_Z + \frac{i\xi}{2} \text{ for polarization 2}$$

At visible frequencies, the order of magnitude of the Maxwellian κ in vacuo is given by

$$\kappa_Z = \frac{\omega}{c} \sim \frac{10^{15}}{10^8} \sim 10^7 \text{ m}^{-1} \tag{81}$$

so that at these frequencies κ_Z is about 33 orders of magnitude greater than ξ. For all practical purposes, therefore, the transverse solutions of the classical limit of the EBP equation are identical with those of the Maxwell equations.

This is a simple demonstration in the classical limit that the fields associated with the EBP and Maxwell equations contain physically meaningful longitudinal as well as transverse components in vacuo. In the next section we discuss several experimental consequences of physically meaningful longitudinal fields when electromagnetic radiation interacts with matter. Firstly, however, we review the available experimental evidence for finite photon mass, following a recent account by Vigier.[17]

IV. DISCUSSION

There is available an increasing amount of evidence for finite photon rest mass, upon which is based the theory of Einstein and de Broglie. A recent experiment by Mizobuchi and Ohtake[17] has demonstrated for single photons the simultaneity of classical wave and particle behavior in light. This has demonstrated for the first time that the Copenhagen interpretation cannot be valid, but supports the Einstein-de Broglie interpretation as reviewed recently by Vigier,[17] an interpretation that implies, for example, that photons are emitted from a source in quanta of energy with well-defined directionality. The wave associated with a single photon has a physical reality. Light is constituted by massive bosons (photons) controlled or piloted[17] by real surrounding spin one fields. The motion of the photon is thus controlled by a quantum potential. The photons are the only directly observable elements of light and behave in Minkowski spacetime as relativistic particles with finite mass. Light is also constituted in the Einstein-de Broglie theory by physically meaningful fields (waves), which, as we have seen, obey Maxwell's equations for all practical purposes, essentially because the photon rest mass is finite (10^{-68} kg) but small. These fields are described by complex vector waves, which also describe photon motion. Thus, if there is a longitudinal photon polarization, there must be a longitudinal field polarization, as described already. Longitudinal field solutions of the EBP equation were first derived by Schrödinger and de Broglie and, in general, the EBP equation has longitudinal and transverse WAVE solutions.[17] Since these are also wave solutions of Maxwell's equations for all practical purposes, it becomes clear that Maxwell's equations must have physically meaningful longitudinal solutions. The relation of these to the transverse solutions has only recently become clear,[23–28] as described in Sections II and III of this paper.

Following Vigier's recent description[17] there are several consequences of finite photon mass. The r dependence of the Coulomb potential is replaced by that of the Yukawa potential:

$$V_Y \propto \frac{\exp(-\xi Z)}{Z} \tag{82}$$

There exist low velocity photons (i.e., photons traveling at considerably less than the speed of light c), whose small but finite mass contributes to that of the universe. There is, thirdly, a red shift proportional to $\exp(-\xi Z)$, which can be applied to explain recent astronomical observations of anomalous red shifts from several distant sources, such as quasars. These "tired light" phenomena originate in the EBP equation and may account for observed anomalies in double-star motions, galaxy clusters, observed variations of the Hubble "constant," and other evidence reviewed in the literature.[17] A photon with finite rest mass behaves relativistically in the frame of observation, leading to the expectation[17] of a direction dependent anisotropy in the frequency of light in the observer frame. Such an anisotropy has been observed experimentally by Hall et al.[30] in the direction of the apex of the 2.7 K background of microwave radiation. These Boulder experiments are currently being repeated in Copenhagen by Poulsen and coworkers.[17] Experimental evidence for the Einstein-de Broglie theory of light has also been reviewed by Vigier[17] in the following areas:

1. Super-luminal action at a distance, a facet of Einstein's interpretation of light
2. The question of locality or nonlocality of the quantum potential
3. Direct experimental testing of Heisenberg's uncertainty principle using single photons
4. Experimental testing for the existence of particle trajectories in light (einweg/welcherweg)
5. Testing the existence of physically meaningful waves without the presence of particles, for example, the recent experimental observation by Bartlett and Corle[31] of the Maxwell displacement current in vacuo
6. Testing directly the existence of the quantum potential with intersecting laser beams and laser-induced fringe patterns

There is, therefore, a considerable amount of experimentation in progress concerning the existence of finite photon mass, and it is no longer tenable to assert[1–16, 20, 21] that the photon mass is zero.

Similarly, it is not reasonable to assert that $\mathbf{B}^{(3)}$ and $\mathbf{E}^{(3)}$ must be zero, "irrelevant," "unphysical," or similar, as in much of the contemporary literature. It is in fact implied, but not specifically stated, in the work of de Broglie and Schrödinger[17] that $\mathbf{B}^{(3)}$ and $\mathbf{E}^{(3)}$ must exist. They exist, as we have seen, both for finite photon mass and in the Maxwellian limit, but finite photon rest mass is essential for a natural quantization of the electromagnetic field. For all intents and purposes, therefore, evidence for $\mathbf{B}^{(3)}$ and $\mathbf{E}^{(3)}$ is evidence for finite photon mass, and corroboration for

other sources of evidence quoted already. The present author has proposed a number of different magneto-optic experiments[23–28] that would test for $\mathbf{B}^{(3)}$ through its interaction with matter, using its characteristic square root dependence on light intensity I_0 (W m^{-2}). In free space, fundamental electrodynamics leads to[23–28]

$$|\mathbf{B}^{(3)}| \sim 10^{-7} I_0^{1/2} \tag{83}$$

and assuming that $\mathbf{B}^{(3)}$ acts as a magnetic field whose time average is nonzero, it is to be expected[23–28] that there exist the following effects (collected details in Ref. 26) proportional to the square root of laser intensity, provided that the laser is circularly polarized: (1) inverse Faraday effect (magnetization due to $\mathbf{B}^{(3)}$), (2) optical Faraday effect (azimuth rotation due to $\mathbf{B}^{(3)}$), (3) effects of $\mathbf{B}^{(3)}$ in NMR (preliminary observations reported in Ref. 32) and ESR spectroscopy, (4) Cotton-Mouton effect due to $\mathbf{B}^{(3)}$, (5) forward–backward birefringence due to $\mathbf{B}^{(3)}$, and (6) reinterpretation of antisymmetric light scattering and similar phenomena in terms of $\mathbf{B}^{(3)}$.

Finally, we propose the Bohm-Aharonov effect due to $\mathbf{B}^{(3)}$ of a circularly polarized laser, which replaces the solenoid, or iron whisker[20] of the conventional Bohm–Aharonov effect. The Bohm-Aharonov effect[20] indicates that the vector potential in quantum mechanics is physically meaningful, and that the vacuum has a nontrivial topology. It is therefore one of the most incisive effects in contemporary electrodynamics. The experiment has been repeated independently several times and consists of placing a small solenoid between two slits, which are used to generate interference fringes due to electron beams. The magnetic flux density $\mathbf{B}$ (tesla) is confined within the solenoid, and is inaccessible to the interfering electrons passing through the two slits. Despite this, the solenoid is observed experimentally[20] to produce a shift in the interference pattern (or fringes) set up by the electrons. This shift is due to the curl of the vector potential $\mathbf{A}$ set up outside the solenoid. Essentially, $\mathbf{A}$ changes the electron wave function

$$\psi = |\psi| \exp\left(i\frac{\mathbf{p}\cdot\mathbf{r}}{\hbar}\right) \tag{84}$$

because $\mathbf{p}$, the electron momentum, is changed to $\mathbf{p} - e\mathbf{A}$, where e is the electronic charge. This does not occur in classical mechanics, but in quantum theory, the electronic wave function, and thus the electron, is influenced by $\mathbf{A}$ even though it travels in regions where magnetic flux density $\mathbf{B}$ is zero. This means that there is nonlocality in the integral

$\oint$**A** · d**r** (Ref. 20). The Bohm-Aharonov effect is therefore evidence for this type of nonlocality.

The shift is given in meters by

$$\Delta x = \frac{L\lambda}{d}\frac{e}{h}\Phi \tag{85}$$

where λ is the wavelength of the electron beam entering the two slits, L is the distance between the screen containing the two slits and the detector plane, d is the distance between the two slits, and

$$\Phi = \int \mathbf{B} \cdot d\mathbf{S} = \oint \mathbf{A} \cdot d\mathbf{r} \tag{86}$$

is a surface integral.

It is clear that if the solenoid is replaced by a thin, circularly polarized, laser beam, there should be a Bohm-Aharonov effect due to $\mathbf{B}^{(3)}$ in which this field shifts the interference pattern of the electrons, with **B** of Eq. (85) replaced by $\mathbf{B}^{(3)}$. This shift should be proportional to the square root of the laser intensity, reverse with the sense of circular polarization of the laser (because $\mathbf{B}^{(3)}$ changes sign), and disappear if the laser is linearly polarized or incoherently polarized. This laser-induced fringe displacement would be a particularly interesting investigation of the nature of $\mathbf{B}^{(3)}$, and of its concommitant $\mathbf{A}^{(3)}$. Presumably $\mathbf{B}^{(3)}$ is confined to the radius of the laser beam, and $\mathbf{A}^{(3)}$ exists outside this beam, as in a solenoid generating a conventional, longitudinal, magnetostatic field. The experiment would prove both the existence and the nonlocality of $\mathbf{A}^{(3)}$.

Acknowledgments

J. P. Vigier is thanked for a preprint of Ref. 17, and for his suggestion in a letter of 4 January 1993 that $\mathbf{B}^{(3)}$ and $\mathbf{E}^{(3)}$ can be accommodated naturally within the Einstein-de Broglie theory of light.

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