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Classical Relativistic Theory of the Longitudinal Ghost Fields of Electromagnetism

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The classical relativistic theory is developed of electric and magnetic fields in terms of boost and rotation generators, respectively, of the Lorentz group of space-time. This development shows that Minkowski geometry requires that there be three *states of polarization of radiation in free space. The magnetic components in a circular basis are right and left circular and longitudinal. The longitudinal component is real and physical, and proportional to one of the three, nonzero, rotation generators of the Lorentz group. The longitudinal electric component is pure imaginary, and proportional to one of the three boost generators. These theoretical arguments conform with experimental data from the Planck radiation law and from magnetic effects of light such as the inverse Faraday effect.*

1. INTRODUCTION

It is well known that the Maxwell equations in free space are relativistically invariant, as was first shown by Lorentz in 1904. Shortly afterwards, Poincaré showed that all the equations of electrodynamics are similarly invariant. These results were proven independently by Einstein in 1905, and shown in the theory of special relativity to be generally valid. Einstein based his theory on two principles. The first asserts that the laws of physics take the same form in all Lorentz frames; the second asserts that the constant c is the same in all Lorentz frames. If the photon is regarded as being without mass, c is the speed of light in vacuo; otherwise, if the photon has mass, its speed varies from frame to frame, giving rise to "tired light."[1,2] Unless otherwise specified, we shall restrict our attention in this paper to the massless photon. However, our results can and will be generalized in further work to the case of finite photon mass, in which the d'Alembert

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equation is replaced by the Proca equation, or variants thereof such as the de Broglie equation or Duffin–Kemmer–Petiau equations.[3–6]

As is well known, the postulated constancy of c allows a connection to be made between Minkowski space-time coordinates in different frames, customarily labeled K and K', the latter moving at v with respect to the former along the Z axis. The pseudo-Euclidean frame of reference is (X, Y, Z, ict) for K and (X', Y', Z', ict') for K', and the assumed linear transformation is based on the relation between frames

$$X'^2 + Y'^2 + Z'^2 - c^2t'^2 = \lambda^2(X^2 + Y^2 + Z^2 - c^2t^2) \tag{1}$$

where λ is a function of v such that $\lambda(0) = 1$. It can be shown[7] that λ is unity for all v. The Lorentz transform is then defined as

$$X' = X, \qquad Y' = Y, \qquad Z' = \gamma(Z - vt), \qquad t' = \gamma\left(t - \beta\frac{Z}{c}\right) \tag{2}$$

where $\beta = v/c$ and $\gamma = (1 - v^2/c^2)^{-1/2}$. This can be written in the 4×4 matrix form

$$\begin{bmatrix} X' \\ Y' \\ Z' \\ ict' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & i\gamma\beta \\ 0 & 0 & -i\gamma\beta & \gamma \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ ict \end{bmatrix} \tag{3}$$

which in tensor notation becomes

$$x'_\mu = a_{\mu\nu} x_\nu \tag{4}$$

The 4×4 matrix in Eq. (3) is the Lorentz transformation matrix and defines the boost generators to be used in this paper, generators which participate in the Lie algebra of the Lorentz group. There are three boost generators, one for each space axis, and each is a pure real 4×4 matrix by definition.[8,9] Additionally, there are three rotation generators in the Lorentz group,[8,9] being 4×4 pure imaginary matrices by definition.

The boost generators, $\hat{K}$, and rotation generators, $\hat{J}$, are based directly on Minkowski geometry, i.e., the pseudo-Euclidean geometry of space-time. In Sec. 2 of this paper it is shown that axial unit vectors in the three dimensions of Euclidean space can be expressed as antisymmetric 3×3 unit tensors which are related directly to the rotation generators of three-dimensional space, and by simple extrapolation, to those of the four-dimensional Lorentz group, generators which are 4×4 matrices in Minkowski space-time. It follows that magnetic fields, which can be expressed in terms of axial vectors in three-dimensional (3-D) space, become rotation generators in the 4-D Lorentz group. By using a suitable circular basis, the $\hat{J}$ matrices are used to define magnetic fields from Maxwell's equations in vacuo. This method leads directly and geometrically to the conclusion that there is a real longitudinal magnetic field from these equations in vacuo. This solution is also related geometrically to the two transverse ones (right and left circular polarization) using the Lie algebra of the rotation generators $\hat{J}$.

In Sec. 3, this method is extended to electric fields, which in 3-D space can be expressed in terms of polar unit vectors. These have no 3×3 matrix equivalents in 3-D, but it is well known that the vector cross product of two polar vectors in 3-D is an axial vector. In 4-D the equivalent is that the commutation of two boost generators is a rotation generator, showing that a polar vector in 4-D is a boost generator of the Lorentz group. With this result, electric fields in space-time also become boost generators, which are expressed in a circular basis.

Section 4 is a summary of the complete Lie algebra of electric and magnetic fields in vacuo, expressed respectively as boost and rotation generators of the Lorentz group of special relativity, using a suitable circular basis to relate these transverse and longitudinal fields to solutions of Maxwell's equations in vacuo. In this way, it is demonstrated geometrically that there exists a real, physically meaningful, longitudinal magnetic field, which in space-time is directly proportional to the rotation generator $\hat{J}^{(3)}$ in a circular basis. The generator $\hat{J}^{(3)}$ is a nonzero 4×4 matrix as a direct result of Minkowski geometry itself, i.e., as a result of the nature of space-time. Similarly, there exists a pure imaginary electric field $i\hat{E}^{(3)}$ which is directly proportional to the nonzero boost generator $\hat{K}^{(3)}$ of the Lorentz group. The electric and magnetic field solutions of Maxwell's equations therefore form the complete Lie algebra of the Lorentz group and part of that of the Poincaré group, longitudinal and transverse components being related by this Lie algebra.

Finally, Sec. 5 gives a discussion of the experimental support for this conclusion, and of experimental consequences in electrodynamics.

2. MAGNETIC FIELDS AS ROTATION GENERATORS

It can be demonstrated in an elementary way[10-12] that there exists a cyclically symmetric relation between the transverse and longitudinal magnetic field solutions of Maxwell's equations in vacuo,

$$\mathbf{B}^{(1)} \times \mathbf{B}^{(2)} = iB^{(0)}\mathbf{B}^{(3)*} \tag{5a}$$

$$\mathbf{B}^{(2)} \times \mathbf{B}^{(3)} = iB^{(0)}\mathbf{B}^{(1)*} \tag{5b}$$

$$\mathbf{B}^{(3)} \times \mathbf{B}^{(1)} = iB^{(0)}\mathbf{B}^{(2)*} \tag{5c}$$

where $B^{(0)}$ is the scalar magnetic flux density magnitude in tesla, and where the fields are defined in vacuo as the following solutions of Maxwell's equations:

$$\mathbf{B}^{(1)} = \frac{B^{(0)}}{\sqrt{2}}(i\mathbf{i} + \mathbf{j})\, e^{i\phi} \tag{6a}$$

$$\mathbf{B}^{(2)} = \frac{B^{(0)}}{\sqrt{2}}(-i\mathbf{i} + \mathbf{j})\, e^{-i\phi} \tag{6b}$$

$$\mathbf{B}^{(3)} = B^{(0)}\mathbf{k} \tag{6c}$$

Here the oscillating transverse component (1) is the complex conjugate of component (2), and is expressed in terms of the phase $\phi = \omega t - \boldsymbol{\kappa}\cdot\mathbf{r}$ of a traveling plane wave in vacuo whose angular frequency is ω at an instant t and whose wave vector is $\boldsymbol{\kappa}$ at a point $\mathbf{r}$ in space. Note that component (3) is phase free, so that $\nabla\cdot\mathbf{B}^{(3)} = 0$.

The magnetic flux densities $\mathbf{B}^{(1)}$, $\mathbf{B}^{(2)}$, and $\mathbf{B}^{(3)}$ are expressed in a natural, circular basis defined[10-12] by unit vectors $\mathbf{e}^{(1)}$, $\mathbf{e}^{(2)}$, and $\mathbf{e}^{(3)}$

$$\mathbf{e}^{(1)} \equiv \frac{1}{\sqrt{2}}(\mathbf{i} - i\mathbf{j}) \tag{7a}$$

$$\mathbf{e}^{(2)} \equiv \frac{1}{\sqrt{2}}(\mathbf{i} + i\mathbf{j}) \tag{7b}$$

$$\mathbf{e}^{(1)} \times \mathbf{e}^{(2)} = i\mathbf{e}^{(3)} \equiv i\mathbf{k} \tag{7c}$$

where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are Cartesian unit vectors in X, Y, and Z respectively, Z being the propagation axis of the plane wave. In Eqs. (5a)–(5c), * denotes "complex conjugate." It is obvious that in this circular basis, which naturally defines right and left circular polarization, $\mathbf{B}^{(2)}$ must be the complex conjugate of $\mathbf{B}^{(1)}$, so that $\mathbf{B}^{(3)}$ must be phase free. The perfectly cyclical symmetry of Eq. (5) is therefore a direct geometrical consequence of the isotropy of 3-D space. If $\mathbf{B}^{(3)} =? \mathbf{0}$, then $\mathbf{B}^{(1)}$ and $\mathbf{B}^{(2)}$ also disappear, showing immediately that the standard approach,[13-16] in which $\mathbf{B}^{(3)}$ is unrelated to $\mathbf{B}^{(1)}$ and $\mathbf{B}^{(2)}$, is *geometrically* unsound in 3-D. The standard approach violates the isotropy of free space and unnaturally reduces it to a plane. The ramifications of this conclusion have been discussed elsewhere.[10-12] The assertion that $\mathbf{B}^{(3)}$ is zero is usually[13-16] based on the Maxwell equation $\nabla \cdot \mathbf{B} = 0$. A longitudinal field in vacuo that depends on the phase of a plane wave[13-16] cannot satisfy this equation, so it is conventionally assumed that $\mathbf{B}^{(3)} =? \mathbf{0}$, or is otherwise "irrelevant" or "unrelated" to the transverse components. The existence of the simple cyclic algebra (5) appears never to have been realized prior to Refs. 10–12. However, from Eq. (5a), the standard $\mathbf{B}^{(3)} = \mathbf{0}$ means that the conjugate product[17-19] $\mathbf{B}^{(1)} \times \mathbf{B}^{(2)}$ is also zero, a result which means, for example, that the inverse Faraday effect[20-26] (phase free, or "static" magnetization by light) disappears. The inverse Faraday effect is, however, well demonstrated experimentally[20-26] and is theoretically[17] *directly proportional* to $\mathbf{B}^{(1)} \times \mathbf{B}^{(2)}$, which is in turn algebraically the same as $iB^{(0)}\mathbf{B}^{(3)}$. This line of reasoning, based on experimental data, exposes a basic paradox in standard electrodynamics, a paradox which is embodied in special relativity,[27] for example, in the well-known fact that the Wigner little group[28, 29] in the conventional approach (with $\mathbf{B}^{(3)} =? \mathbf{0}$) is the unphysical E(2), an Euclidean planar group, and not the natural group of rotations in three dimensions. This in turn leads to the well-known loss of manifest covariance in the vector potential A_μ, something which is fundamentally at odds with the Bohm–Aharonov effect[29, 30] which shows A_μ to be *physically* meaningful. It is illogical in special relativity to assert that a physically meaningful four-vector is not manifestly covariant. In other words, all four components of A_μ must be physically meaningful as for any other four-vector. The usual assertion $\mathbf{B}^{(3)} =? \mathbf{0}$ leads to absurdity therefore in 3-D and 4-D, an absurdity which is habitually accepted in the standard approach to electrodynamics and the U(1) sector of field theory.

With this preamble, the aim of this and the next two sections is to devise the relativistically self-consistent Lie algebra of the electric and magnetic solutions of Maxwell's equations in vacuo without making the usual assertion $\mathbf{B}^{(3)} =? \mathbf{0}$. In this section, it is shown that $\hat{B}^{(1)}$, $\hat{B}^{(2)}$, and $\hat{B}^{(3)}$ are each directly proportional to a standard rotation generator of the Lorentz group. These rotation generators are *all* nonzero by Minkowski geometry.

The starting point of the proof of this result is to express the Cartesian *axial* unit vectors as antisymmetric matrices using the fact that an axial rank-one vector is also a polar antisymmetric rank-two tensor.[31] The three rank-two tensors are

$$\hat{\imath} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad \hat{\jmath} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \hat{k} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{8}$$

which in the circular basis (7) become

$$\hat{e}^{(1)}=\frac{1}{\sqrt{2}}\begin{bmatrix}0 & 0 & i\\ 0 & 0 & 1\\ -i & -1 & 0\end{bmatrix},\qquad \hat{e}^{(2)}=\frac{1}{\sqrt{2}}\begin{bmatrix}0 & 0 & -i\\ 0 & 0 & 1\\ i & -1 & 0\end{bmatrix},\qquad \hat{e}^{(3)}=\hat{k} \tag{9}$$

The latter form a classical commutator algebra which is cyclically symmetric in Euclidean space,

$$\begin{aligned}
[\hat{e}^{(1)}, \hat{e}^{(2)}] &= -i\hat{e}^{(3)*} = -i\hat{e}^{(3)}\\
[\hat{e}^{(2)}, \hat{e}^{(3)}] &= -i\hat{e}^{(1)*} = -i\hat{e}^{(2)}\\
[\hat{e}^{(3)}, \hat{e}^{(1)}] &= -i\hat{e}^{(2)*} = -i\hat{e}^{(1)}
\end{aligned} \tag{10}$$

In these equations, all three commutators are formed from geometrically meaningful component matrices which are all nonzero. Obviously, if one of these is arbitrarily set to zero, the other two vanish, and the geometrical structure is destroyed. This is the geometrical basis for the existence of a real three-dimensional matrix representation of the magnetic part of free-space electromagnetism. Specifically,

$$\begin{aligned}
\hat{B}^{(1)} &= iB^{(0)}\hat{e}^{(1)}e^{i\phi}\\
\hat{B}^{(2)} &= -iB^{(0)}\hat{e}^{(2)}e^{-i\phi}\\
\hat{B}^{(3)} &= B^{(0)}\hat{e}^{(3)}
\end{aligned} \tag{11}$$

from which emerges the classical commutative algebra equivalent to the vectorial algebra (5),

$$\begin{aligned}
[\hat{B}^{(1)}, \hat{B}^{(2)}] &= -iB^{(0)}\hat{B}^{(3)*} = -iB^{(0)}\hat{B}^{(3)}\\
[\hat{B}^{(2)}, \hat{B}^{(3)}] &= -iB^{(0)}\hat{B}^{(1)*} = -iB^{(0)}\hat{B}^{(2)}\\
[\hat{B}^{(3)}, \hat{B}^{(1)}] &= -iB^{(0)}\hat{B}^{(2)*} = -iB^{(0)}\hat{B}^{(1)}
\end{aligned} \tag{12}$$

an algebra which is again cyclically symmetric in Euclidean space. This algebra can now be expressed in terms of the rotation generators, $\hat{J}$, of the O(3) group[29] of three-dimensional space, but using the circular basis (7) instead of the usual[29] Cartesian one,

$$\hat{J}^{(1)}=\frac{\hat{e}^{(1)}}{i},\qquad \hat{J}^{(2)}=\frac{-\hat{e}^{(2)}}{i},\qquad \hat{J}^{(3)}=\frac{\hat{e}^{(3)}}{i} \tag{13}$$

Note that the generator $\hat{J}^{(3)}$ is pure imaginary. The magnetic field matrices and rotation generators are then linked by

$$\begin{aligned}\hat{B}^{(1)} &= -B^{(0)}\hat{J}^{(1)}e^{i\phi}\\ \hat{B}^{(2)} &= -B^{(0)}\hat{J}^{(2)}e^{-i\phi}\\ \hat{B}^{(3)} &= iB^{(0)}\hat{J}^{(3)}\end{aligned} \tag{14}$$

a representation which accounts naturally for the phase, and which is of key importance in recognizing that the commutative algebra of the magnetic part of free-space electromagnetism *is part of the Lie algebra*[28, 29] *of the Lorentz group of Minkowski space-time.* A Lie algebra is that of generators of a group, and therefore, magnetic components of free-space electromagnetism are directly proportional to the *rotation* generators of the Lorentz group. Therefore, commutative relations between magnetic fields are relations between spins, and a magnetic field is a property of space-time itself. Therefore, there are three components of a magnetic field in space because there are three rotation generators. Finally, magnetic field components are interrelated by commutators, in the same way as rotation generators.

The generalization of rotation generators from O(3) to the Lorentz group occurs as follows[29]:

$$\begin{aligned}\hat{J}^{(1)} &= \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & -i & 0\\ -1 & i & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\\ \hat{J}^{(2)} &= \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & i & 0\\ -1 & -i & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\\ \hat{J}^{(3)} &= \begin{bmatrix} 0 & -i & 0 & 0\\ i & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\end{aligned} \tag{15}$$

It follows that magnetic fields in the four coordinates of space-time are also

4×4 matrices. There are three rotation generators in space-time, which obey the classical commutative algebra,

$$\begin{aligned}[\hat{J}^{(1)}, \hat{J}^{(2)}] &= -\hat{J}^{(3)*} = \hat{J}^{(3)} \\ [\hat{J}^{(2)}, \hat{J}^{(3)}] &= -\hat{J}^{(1)*} = -\hat{J}^{(2)} \\ [\hat{J}^{(3)}, \hat{J}^{(1)}] &= -\hat{J}^{(2)*} = -\hat{J}^{(1)}\end{aligned} \tag{16}$$

which becomes the more familiar[29]

$$\begin{aligned}[\hat{J}_X, \hat{J}_Y] &= i\hat{J}_Z \\ [\hat{J}_Y, \hat{J}_Z] &= i\hat{J}_X \\ [\hat{J}_Z, \hat{J}_X] &= i\hat{J}_Y\end{aligned} \tag{17}$$

in a Cartesian basis.

By geometry, therefore, it becomes absurd to assert $\hat{B}^{(3)} =? \hat{0}$, because by Eq. (14) this means $\hat{J}^{(3)} =? \hat{0}$, in direct contradiction with Eq. (15). This vividly and conclusively demonstrates that the standard approach[13-16] unnaturally reduces isotropic space to a plane.

The classical commutative algebra of rotation generators is, within a factor $\hbar$, *the commutator algebra of angular momentum operators in quantum mechanics.* Realizing this immediately leads to the quantization of the magnetic fields of the plane wave in vacuo, giving the result

$$\hat{B}^{(1)} = -B^{(0)} \frac{\hat{J}^{(1)}}{\hbar} e^{i\phi}, \qquad \hat{B}^{(2)} = -B^{(0)} \frac{\hat{J}^{(2)}}{\hbar} e^{-i\phi}, \qquad \hat{B}^{(3)} = iB^{(0)} \frac{\hat{J}^{(3)}}{\hbar} \tag{18}$$

where $\hat{B}^{(i)}$ are now operators in quantum field theory. In particular, the longitudinal operator $\hat{B}^{(3)}$ is the elementary quantum of magnetic flux density in the propagation axis. We refer to this hereinafter as the *photomagneton.* We refer to the expectation value of $\hat{B}^{(3)}$ as *the ghost field,* because $\hat{B}^{(3)}$ has no Planck energy,[10-12] being phase free. In consequence, $\hat{B}^{(3)}$ is not absorbed or emitted by an atom or molecule, and can be detected only by its magnetization of matter in such phenomena as the inverse Faraday effect.[20-26] The ghost field is therefore far more difficult to detect experimentally than the everyday, oscillating $\hat{B}^{(1)}$ and $\hat{B}^{(2)}$. This is probably why $\hat{B}^{(3)}$ has not been considered in electrodynamics. However, it is clear from Eq. (5a), for example, that if $\mathbf{B}^{(3)}$ were zero, then the experimentally observed[20-26] inverse Faraday effect would not exist, because the conjugate product $\mathbf{B}^{(1)} \times \mathbf{B}^{(2)}$ would vanish. Finally, the source of $\mathbf{B}^{(3)}$ is the same as the source of $\mathbf{B}^{(1)}$ and $\mathbf{B}^{(2)}$, and $\hat{B}^{(3)}$ is directly proportional to the angular momentum of the photon through Eq. (18c). The eigenvalues of

$\hat{B}^{(3)}$ are therefore $B^{(0)}$ and $-B^{(0)}$. These are, of course, defined in the *longitudinal* axis (3), i.e., Z of the Cartesian basis. The assertion $\hat{B}^{(3)} =? \hat{0}$ in the quantum field theory is therefore absurd, because it means that the eigenvalues of photon spin are zero, whereas they are well known to be $\pm\hbar$, an irremovable property of the photon.[32]

3. ELECTRIC FIELDS AS BOOST GENERATORS

An electric field is a polar vector in three, Euclidean dimensions, and unlike an axial vector, cannot be put into a 3×3 matrix form such as embodied in Eq. (8). The cross product of two polar vectors is, however, an axial vector in Euclidean space. For example, the product

$$\mathbf{i}\times\mathbf{j}=\mathbf{k} \tag{19}$$

produces the Cartesian, axial, unit vector $\mathbf{k}$, which in the circular basis is $\mathbf{e}^{(3)}$. In Minkowski space-time the axial vector $\mathbf{k}$ is known from the arguments in Sec. 1 to be a 4×4 matrix, related directly to the rotation generator $\hat{J}^{(3)}$, of the Lorentz group. It follows that a rotation generator in space-time is the result of a classical commutation of two matrices which play the role of polar vectors. From the well-established Lie algebra[29] of the generators of the Lorentz group, these matrices are *boost generators*, 4×4 real matrices. The equivalent of Eq. (19) in Minkowski space-time is therefore

$$[\hat{K}_X, \hat{K}_Y] = -i\hat{J}_Z \tag{20}$$

and cyclic permutations. In the circular basis (1), (2), (3) (rather than the Cartesian basis X, Y, Z) this commutator algebra becomes the cyclically symmetric

$$\begin{aligned}
[\hat{K}^{(1)}, \hat{K}^{(2)}] &= -i\hat{e}^{(3)*} = -\hat{J}^{(3)*}\\
[\hat{K}^{(2)}, \hat{K}^{(3)}] &= -i\hat{e}^{(1)*} = -\hat{J}^{(1)*}\\
[\hat{K}^{(3)}, \hat{K}^{(1)}] &= -i\hat{e}^{(2)*} = +\hat{J}^{(2)*}
\end{aligned} \tag{21}$$

Therefore, although polar vectors cannot be put in a matrix form in Euclidean space, they correspond to boost generators, 4×4 matrices, in Minkowski space-time.

This essentially geometrical result leads directly to the conclusion that *electric fields in space-time are proportional to boost generators* because electric fields in Euclidean space are proportional to polar unit vectors.

Therefore the fundamental geometry of Minkowski space-time demands that magnetic fields be composed of rotation generators (imaginary 4×4 matrices) and electric fields be composed of boost generators (real 4×4 matrices). Furthermore, boost and rotation generators are linked by the Lie algebra of the Lorentz group, which is written out in full in the following section. It follows that electric and magnetic fields in space-time also form a Lie algebra of the Lorentz group, in any suitable basis, e.g., Cartesian or circular for the space part of space-time. The circular basis is suited naturally for solutions of Maxwell's equations, because of the fact that there is a right and left circular polarization, mutually orthogonal to the longitudinal polarization of the propagation axis.

In Euclidean space, electric field solutions to Maxwell's equations are conventionally regarded as the transverse, oscillatory counterparts of Eqs. (6a) and (6b),

$$\mathbf{E}^{(1)}=\frac{E^{(0)}}{\sqrt{2}}(\mathbf{i}-i\mathbf{j})\,e^{i\phi} \tag{22a}$$

$$\mathbf{E}^{(2)}=\frac{E^{(0)}}{\sqrt{2}}(\mathbf{i}+i\mathbf{j})\,e^{-i\phi} \tag{22b}$$

which can be written directly in terms of the unit vectors of the circular basis,

$$\begin{aligned}\mathbf{E}^{(1)}&=E^{(0)}\mathbf{e}^{(1)}e^{i\phi}\\ \mathbf{E}^{(2)}&=E^{(0)}\mathbf{e}^{(2)}e^{-i\phi}\end{aligned} \tag{23}$$

In Minkowski space-time, the equivalents are therefore

$$\begin{aligned}\hat{E}^{(1)}&=E^{(0)}\hat{K}^{(1)}e^{i\phi}\\ \hat{E}^{(2)}&=E^{(0)}\hat{K}^{(2)}e^{-i\phi}\end{aligned} \tag{24}$$

(The phase ϕ is a Lorentz invariant,[13] and remains the same in space-time and Euclidean space.) The boost generators appearing in Eq. (24) are written in a circular basis,

$$\hat{K}^{(1)}=\frac{1}{\sqrt{2}}\begin{bmatrix}0&0&0&1\\0&0&0&-i\\0&0&0&0\\-1&i&0&0\end{bmatrix},\qquad \hat{K}^{(2)}=\frac{1}{\sqrt{2}}\begin{bmatrix}0&0&0&1\\0&0&0&i\\0&0&0&0\\-1&-i&0&0\end{bmatrix} \tag{25}$$

and correspond to the complex, polar, unit vectors $\mathbf{e}^{(1)}$ and $\mathbf{e}^{(2)}$ in Euclidean space.

By reference to the Lie commutator algebra (21) it is clear that the commutation of $\hat{K}^{(1)}$ and $\hat{K}^{(2)}$ is $\hat{J}^{(3)} = -\hat{J}^{(3)*}$, a *rotation* generator, directly proportional to a *magnetic* field. The equivalent result in Euclidean space is

$$\hat{e}^{(1)} \times \hat{e}^{(2)} = i\hat{e}^{(3)} \tag{26}$$

It is not possible to form an electric field from the cross product of $\mathbf{E}^{(1)}$ and $\mathbf{E}^{(2)}$ and this conforms with fundamental symmetry.[10–12] The question therefore arises as to what is the electric field proportional to the third boost generator $\hat{K}^{(3)}$ of Minkowski space-time, or equivalently, the *polar* vector $\mathbf{k}$ of Euclidean space. That there must be a longitudinal electric component is clear from both 3-D and 4-D geometry, and from the Maxwell equation in vacuo $\nabla \cdot \mathbf{E} = 0$, this component must be phase free. By writing out the longitudinal rotation and boost generators,

$$\hat{J}^{(3)} = \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \hat{K}^{(3)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \tag{27}$$

it is seen that the former is pure imaginary and the latter is pure real. It follows that there are two possibilities:

(1) that the longitudinal $\hat{B}^{(3)}$ is pure real and the longitudinal $i\hat{E}^{(3)}$ is pure imaginary;

(2) vice versa.

Choice (1) follows, however, from a consideration of the nature of the unit vectors $\mathbf{e}^{(1)}$, $\mathbf{e}^{(2)}$, and $\mathbf{e}^{(3)}$ of the circular basis (7), in which the *axial* $\mathbf{e}^{(3)}$ *is pure real, and equal to the real, Cartesian, axial* $\mathbf{k}$. From Eq. (6c), multiplying this real, axial, unit vector by the amplitude $B^{(0)}$ (a real scalar) gives a *real* $\mathbf{B}^{(3)}$ in Euclidean space, and a real $\hat{B}^{(3)}$ in space-time. This real $\hat{B}^{(3)}$ is therefore defined as

$$\hat{B}^{(3)} \equiv iB^{(0)}\hat{J}^{(3)} \tag{28}$$

in terms of the *imaginary* rotation generator $\hat{J}^{(3)}$. It follows that the *imaginary* $i\hat{E}^{(3)}$ must be defined as

$$i\hat{E}^{(3)} \equiv iE^{(0)}\hat{K}^{(3)} \tag{29}$$

in terms of the *real* boost generator $\hat{K}^{(3)}$.

In the next section, we write out the complete Lie algebra of electric and magnetic fields in the Lorentz group.

4. THE LIE ALGEBRA OF ELECTRIC AND MAGNETIC FIELDS IN THE LORENTZ GROUP

The complete Lie algebra of the boost and rotation generators of the Lorentz group can be written in a Cartesian or circular basis as follows. In a Cartesian basis,[29]

$$\hat{K}_X=\begin{bmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\-1&0&0&0\end{bmatrix},\quad \hat{K}_Y=\begin{bmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&-1&0&0\end{bmatrix},\quad \hat{K}_Z=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&-1&0\end{bmatrix}$$

$$\hat{J}_X=\begin{bmatrix}0&0&0&0\\0&0&-i&0\\0&i&0&0\\0&0&0&0\end{bmatrix},\quad \hat{J}_Y=\begin{bmatrix}0&0&i&0\\0&0&0&0\\-i&0&0&0\\0&0&0&0\end{bmatrix},\quad \hat{J}_Z=\begin{bmatrix}0&-i&0&0\\i&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} \tag{30}$$

so that

$$\begin{aligned}
&[\hat{J}_X, \hat{J}_Y]=i\hat{J}_Z, &&\text{and cyclic permutations}\\
&[\hat{K}_X, \hat{K}_Y]=-i\hat{J}_Z, &&\text{and cyclic permutations}\\
&[\hat{K}_X, \hat{J}_Y]=i\hat{K}_Z, &&\text{and cyclic permutations}\\
&[\hat{K}_X, \hat{J}_X]=0, &&\text{etc.}
\end{aligned} \tag{31}$$

This can be summarized concisely[29] as

$$\hat{J}_{\mu\nu}(\mu, \nu=0,\ldots, 3)\begin{cases}\hat{J}_{ij}=-\hat{J}_{ji}=\varepsilon_{ijk}\hat{J}_k\\ \hat{J}_{i0}=-\hat{J}_{0i}=-\hat{K}_i\end{cases}\qquad (i, j, k=1, 2, 3) \tag{32}$$

and displayed as the 4×4 antisymmetric matrix of matrices,

$$\hat{J}_{\mu\nu}=\begin{bmatrix}0&-\hat{J}_3&\hat{J}_2&-\hat{K}_1\\ \hat{J}_3&0&-\hat{J}_1&-\hat{K}_2\\ -\hat{J}_2&\hat{J}_1&0&-\hat{K}_3\\ \hat{K}_1&\hat{K}_2&\hat{K}_3&0\end{bmatrix} \tag{33}$$

The antisymmetric structure of $\hat{J}_{\mu\nu}$ is reminiscent of the well-known electromagnetic tensor $\hat{F}_{\mu\nu}$ of special relativity (the four-curl of A_μ), which in S.I. units is

$$\hat{F}_{\mu\nu} = \varepsilon_0 \begin{bmatrix} 0 & -cB_3 & cB_2 & -iE_1 \\ cB_3 & 0 & -cB_1 & -iE_2 \\ -cB_2 & cB_1 & 0 & -iE_3 \\ iE_1 & iE_2 & iE_3 & 0 \end{bmatrix} \tag{34}$$

where ε_0 is the permittivity in vacuo. By comparing Eq. (34), which is a matrix of Euclidean space field vector components, with Eq. (33), a matrix of Minkowski boost and rotation generators, it becomes intuitively clear that there is a relation between electric and magnetic fields and boost and rotation generators of the type developed in Secs. 2 and 3. We come to see that the Maxwell equations, two of which[29] can be written as

$$\frac{\partial \hat{F}_{\mu\nu}}{\partial x_\mu} = 0 \tag{35}$$

where $x_\mu \equiv (X, Y, Z, ict)$, *are relations between boost and rotation generators* which can be written formally as

$$\frac{\partial \hat{J}_{\mu\nu}}{\partial x_\mu} = 0 \tag{36}$$

In comparing $\hat{F}_{\mu\nu}$ of standard electrodynamics[29] with $\hat{J}_{\mu\nu}$ of standard special relativity, we see that:

(1) $\hat{F}_{\mu\nu}$ already accounts for the fact that in general, *there is a real longitudinal* cB_3 *and an imaginary longitudinal* iE_3. These are obviously not set to zero in Eq. (34). The $\hat{F}_{\mu\nu}$ matrix, devised by Einstein in 1905,[33, 34] contains components all of which are valid solutions to Maxwell's equations, including the longitudinal solutions cB_3 and iE_3. These components are not zero in vacuo, because they are tied to the transverse components through fundamental space-time geometry.

(2) From a comparison of elements of $\hat{F}_{\mu\nu}$ and $\hat{J}_{\mu\nu}$ it becomes clear that if the electric field components can be converted to boost generators, and the magnetic fields to rotation generators, $\hat{F}_{\mu\nu}$ becomes directly proportional to $\hat{J}_{\mu\nu}$.

In order to demonstrate this proportionality we need to transfer to a circular basis, because plane-wave solutions of Maxwell's equations are written naturally in a circular basis, multiplying a phase factor. It is also necessary to convert the electric and magnetic field components in Eq. (34) from Euclidean space to Minkowski space-time. This is precisely what has been achieved in Secs. 2 and 3. Specifically, the equivalents of Eq. (30) of the Cartesian basis are given in the circular basis of Eqs. (7) by Eqs. (15), (25), and (27). In the circular basis, Eqs. (31) become

$$\begin{aligned}
&[\hat{J}^{(1)}, \hat{J}^{(2)}] = -\hat{J}^{(3)*}, && \text{and cyclic permutations}\\
&[\hat{K}^{(1)}, \hat{K}^{(2)}] = -i\hat{e}^{(3)*}, && \text{and cyclic permutations}\\
&[\hat{K}^{(1)}, \hat{e}^{(2)}] = -i\hat{K}^{(3)*}, && \text{and cyclic permutations}\\
&[\hat{K}^{(1)}, \hat{J}^{(1)}] = 0, && \text{etc.}
\end{aligned} \tag{37}$$

with

$$\begin{aligned}
&i\hat{e}^{(1)*} = \hat{J}^{(1)*}, \quad && -i\hat{e}^{(2)*} = \hat{J}^{(2)*}, \quad && i\hat{e}^{(3)*} = \hat{J}^{(3)*}\\
&i\hat{e}^{(2)} = \hat{J}^{(2)}, \quad && -i\hat{e}^{(1)} = \hat{J}^{(1)}, \quad && i\hat{e}^{(3)} = -\hat{J}^{(3)}
\end{aligned} \tag{38}$$

In terms of electric and magnetic fields, these cyclically symmetric geometrical relations can also be written as a complete algebra, which is by implication a Lie algebra of the Lorentz group,

$$\begin{aligned}
&[\hat{B}^{(1)}, \hat{B}^{(2)}] = iB^{(0)}\hat{B}^{(3)*}, && \text{and cyclic permutations}\\
&[\hat{E}^{(1)}, \hat{E}^{(2)}] = -iE^{(0)2}\hat{e}^{(3)*} = -ic^2B^{(0)}\hat{B}^{(3)*}, && \text{etc.}\\
&[\hat{E}^{(1)}, \hat{B}^{(2)}] = iB^{(0)}(i\hat{E}^{(3)}), && \text{etc.}\\
&[\hat{E}^{(1)}, \hat{B}^{(1)}] = 0, && \text{etc.}
\end{aligned} \tag{39}$$

These relations demonstrate that the assertion[35] $\hat{B}^{(3)} = ?\ \hat{0}$ is relativistically incorrect. Although $i\hat{E}^{(3)}$ is imaginary, it too is not zero.

5. DISCUSSION

In order to obtain the rigorously correct field algebra in 4-D, Eqs. (38), in free space-time, it is essential to use the fundamental geometry, Eq. (37) written in terms of boost and rotation generators. The following relations between fields and generators must then be substituted into Eqs. (37) to obtain Eqs. (39),

$$\begin{aligned}
\hat{B}^{(1)} &= -B^{(0)}\hat{J}^{(1)}e^{i\phi} = iB^{(0)}\hat{e}^{(1)}e^{i\phi}\\
\hat{B}^{(2)} &= -B^{(0)}\hat{J}^{(2)}e^{-i\phi} = -iB^{(0)}\hat{e}^{(2)}e^{-i\phi}\\
\hat{B}^{(3)} &= iB^{(0)}\hat{J}^{(3)} = B^{(0)}\hat{e}^{(3)}\\
\hat{E}^{(1)} &= E^{(0)}\hat{K}^{(1)}e^{i\phi}\\
\hat{E}^{(2)} &= E^{(0)}\hat{K}^{(2)}e^{-i\phi}\\
i\hat{E}^{(3)} &= iE^{(0)}\hat{K}^{(3)}
\end{aligned} \tag{40}$$

These relations emphasize that $i\hat{E}^{(3)}$ is pure imaginary and $\hat{B}^{(3)}$ pure real. In Euclidean space, they reduce to $i\mathbf{E}^{(3)}$ and $\mathbf{B}^{(3)}$, imaginary polar and real axial vectors respectively.

It is critically important to note that the classical electromagnetic energy density in vacuo,[36]

$$U^{(3)} = \varepsilon_0 i\mathbf{E}^{(3)} \cdot i\mathbf{E}^{(3)} + \frac{1}{\mu_0}\mathbf{B}^{(3)} \cdot \mathbf{B}^{(3)} = 0 \tag{41}$$

In other words, the correct *combination* of $i\mathbf{E}^{(3)}$ and $\mathbf{B}^{(3)}$ adds nothing to the Rayleigh–Jeans law, a classical limit of the Planck radiation law. This is another vivid demonstration of how the existence of $\mathbf{B}^{(3)}$ and $i\mathbf{E}^{(3)}$ has been bypassed in electromagnetic theory, so that it has become habitual to neglect them. Recent comments in the literature[35] continue to assert erroneously that $\mathbf{B}^{(3)} =? \mathbf{0}$, an assertion that makes nonsense out of fundamental geometry.

Equation (41) is one illustration of why $\mathbf{B}^{(3)}$ is a ghost field. Its influence cannot be detected through measurements of light intensity leading to the Rayleigh–Jeans law. Following the rule that real fields are physical and imaginary fields are unphysical, it is expected that $\mathbf{B}^{(3)}$ will magnetize material, but that $i\mathbf{E}^{(3)}$ will not produce electric polarization. This conforms with what is available experimentally to date on the magnetizing effects of light. The inverse Faraday effect,[20–26] for instance, can be shown[17] to be directly proportional to $iB^{(0)}\mathbf{B}^{(3)} = \mathbf{B}^{(1)} \times \mathbf{B}^{(2)}$ at second order in $B^{(0)}$, the magnitude of $\mathbf{B}^{(3)}$. This shows *experimentally* that $\mathbf{B}^{(3)}$ is not zero, and that effects due to $\mathbf{B}^{(3)}$ are expected *a priori* at first order.[10–12] If $\mathbf{B}^{(3)}$ were zero, the inverse Faraday effect would not exist experimentally, and fundamental geometry would be invalidated. It remains to be seen experimentally whether $\mathbf{B}^{(3)}$ can act at first order. There appears at present to be no reason why not, and such an effect, proportional to the *square root* of intensity, would appear in the inverse Faraday effect under suitable circumstances, namely in material with a net magnetic dipole moment.[10]

The ghost field $i\mathrm{E}^{(3)}$ is imaginary and unphysical for this reason, and no electric polarization due to $i\mathbf{E}^{(3)}$ can occur. Indeed, to date, no large first-order polarizing effects of this nature have been observed experimentally.

Both $\mathbf{B}^{(3)}$ and $i\mathbf{E}^{(3)}$ are phase free, and in the quantum theory have no Planck energy, being associated with no frequency. They are not therefore absorbed[10–12] by an atom or molecule, and cannot be detected by ordinary techniques of spectroscopy such as infrared or Raman. The field $\mathbf{B}^{(3)}$ can be detected only in a difficult experiment such as the inverse Faraday effect, or in related phenomena such as light shifts,[37] the optical Faraday effect,[38] or light-induced shifts in NMR.[39] The inverse Faraday effect is due to $iB^{(0)}\mathbf{B}^{(3)} = \mathbf{B}^{(1)} \times \mathbf{B}^{(2)}$ at second order, as we have seen. If careful future measurements on the inverse Faraday effect show that $\mathbf{B}^{(3)}$ cannot act at first order, it does not follow that $\mathbf{B}^{(3)}$ is zero, because that would mean that $iB^{(0)}\mathbf{B}^{(3)}$ itself would vanish, contrary to experience. A postulated inability of $\mathbf{B}^{(3)}$ to act at first order might mean, for example, that it takes two modes, (1) and (2), to define it, needing light to act at first order in intensity (second order in the amplitude $B^{(0)}$). However, these matters must be settled by further careful experimentation.

Free-space electromagnetism, by causality, originates in a source infinitely removed from the vacuum. The source is made up of charged particles and currents, for example moving electrons which radiate. The source gives the free-space amplitude $E^{(0)} = cB^{(0)}$ in S.I. units. If the nature of the source is changed so that the sign of $B^{(0)}$ is reversed (e.g., by replacing the electrons by positrons), it is clear from Eqs. (39) that *all six* field components are changed in sign precisely. This process is not equivalent to a phase shift (a change in ϕ) because in a phase shift, the sign of $B^{(0)}$ is not changed. (In other words, the charge conjugation operator $\hat{C}$, *by definition*, changes the sign of $B^{(0)}$ but not of ϕ, because the latter is a spatio-temporal quantity.[40] If it is argued[35] that the change $B^{(0)}$ to $-B^{(0)}$ results in $\mathbf{B}^{(3)} = ?\,\mathbf{0}$, then it must also result in $\mathbf{B}^{(1)} = ?\,\mathbf{0}$ and $\mathbf{B}^{(2)} = ?\,\mathbf{0}$. This argument,[35] which is equivalent to the erroneous assertion that $\mathbf{B}^{(3)}$ violates $\hat{C}$, therefore *leads to the disappearance of electromagnetism.* In the presence of electromagnetism in vacuo, this is obviously not true, and the argument fails. We can see this clearly from Eqs. (5), for example, which conserve $\hat{C}$. (The $\hat{C}$ symmetry on both sides of all three Eqs. (5) is positive.)

On the most fundamental level, electromagnetic theory requires a sign to be allocated to the charge on the electron, i.e., e is negative by convention. In the same way, the equations of electrodynamics implicitly attach a positive sign to $B^{(0)}$ (or to $E^{(0)}$). It is assumed that free-space electromagnetism has been generated by an infinitely distant source in such a way that the sign of $B^{(0)}$ is positive. The equations work equally well with a negative $B^{(0)}$, but by convention, $B^{(0)}$ is positive. At the most basic level this means that evolution in the universe, and in our solar system, has been such that e is negative and $B^{(0)}$ is positive. It is possible to work through all the equations of electrodynamics with a positive e and a negative $B^{(0)}$, but this produces a universe composed of antimatter, and a source of free-space electromagnetism made of antimatter.

It remains true, however, that if a source were available in the laboratory that produced a negative $B^{(0)}$, the sign of $\mathbf{B}^{(3)}$ would be opposite from that produced by a source giving a positive $B^{(0)}$ for the same sense of circular polarization. This clearly requires the use of two different sources, one made up of moving electrons, the other of moving positrons. Any *first order* effect of $\mathbf{B}^{(3)}$ in this case would be opposite in sign for the two different sources. Effects at second order in $B^{(0)}$ would not be changed in sign.

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