

NOETHER'S THEOREM AND THE FIELD $\mathbf{B}^{(3)}$

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It is shown that the longitudinal, magnetic flux density, $\mathbf{B}^{(3)}$, of vacuum electromagnetic radiation can be accommodated rigorously within Noether's theorem, which relates fundamental spacetime symmetries to fundamental conservation laws. This demonstration links $\mathbf{B}^{(3)}$ to the canonical energy-momentum tensor $T_{\mu\nu}$ that appears in Einstein's field equations of general relativity. Thus, $\mathbf{B}^{(3)}$ provides a link between electromagnetism and gravitation which might eventually lead to an unified understanding of field theory.

Key words: $\mathbf{B}^{(3)}$ field, Noether theorem.

1. INTRODUCTION

The conventional understanding of vacuum electromagnetism [1-3] has recently been advanced considerably by the emergence [4-12] of the longitudinal field $\mathbf{B}^{(3)}$. The consequences of $\mathbf{B}^{(3)}$ have been developed systematically [6] in several branches of field theory, and the conditions for its experimental measurement defined precisely [6,14]. These involve magnetization of an electron plasma by intense microwave pulses. The magnetization is expected to be proportional to the square root of the beam power density (W m^{-2}), an inference which is obtained directly from the classical equation of motion of one electron in circularly polarized electromagnetic radiation -- the Hamilton-Jacobi

equation [13]. In this Letter it is demonstrated that $\mathbf{B}^{(3)}$ is compatible with the most fundamental known law of physics, Noether's theorem [14,15].

In Sec. 2, $\mathbf{B}^{(3)}$, and its concomitant plane waves $\mathbf{B}^{(1)} = \mathbf{B}^{(2)*}$ [4-12] are related to classical rotation generators of the $O(3)$ and Lorentz groups [5], which are within a factor \hbar (the Dirac constant) angular momentum operators of quantum mechanics. Thus, $\mathbf{B}^{(1)}$, $\mathbf{B}^{(2)}$ and $\mathbf{B}^{(3)}$ are directly proportional to, and form the same commutators as, angular momentum operators [6]. In Sec. 3, the three field components $\mathbf{B}^{(1)}$, $\mathbf{B}^{(2)}$ and $\mathbf{B}^{(3)}$ are accommodated within a single angular momentum conservation law, which is shown to be rigorously compatible with Noether's theorem. In Sec. 4, the field $\mathbf{B}^{(3)}$ is linked, using Noether's theorem, to the canonical momentum-energy tensor of Einstein's field equations for gravitation, the well known field equations of general relativity which show that gravitation is the result of curvature in spacetime. Finally we discuss to what extent does $\mathbf{B}^{(3)}$ allow a unification of electromagnetic field theory and general relativity in the vacuum.

2. $\mathbf{B}^{(1)}$, $\mathbf{B}^{(2)}$ AND $\mathbf{B}^{(3)}$: ROTATION GENERATORS, AND ANGULAR MOMENTUM OPERATORS

The spin field $\mathbf{B}^{(3)}$ is linked to the plane waves $\mathbf{B}^{(1)}$ and $\mathbf{B}^{(2)}$ in a circular basis (1), (2) and (3) through a Lie algebra [4-6] which is also the algebra of the rotation generators of the Poincaré group. Since $O(3)$, the rotation group in three dimensional space, is a sub-group of the Poincaré group [14]. $\mathbf{B}^{(1)}$, $\mathbf{B}^{(2)}$ and $\mathbf{B}^{(3)}$ also form a Lie algebra in the $O(3)$ group. This algebra can be expressed in terms of axial vectors as [6]:

$$\begin{aligned} \mathbf{B}^{(1)} \times \mathbf{B}^{(2)} &= iB^{(0)} \mathbf{B}^{(3)*}, & \mathbf{B}^{(2)} \times \mathbf{B}^{(3)} &= iB^{(0)} \mathbf{B}^{(1)*}, \\ \mathbf{B}^{(3)} \times \mathbf{B}^{(1)} &= iB^{(0)} \mathbf{B}^{(2)*}, \end{aligned} \quad (1)$$

where $B^{(0)}$ is the magnetic flux density amplitude of the electromagnetic wave. The Lie algebra (1) is non-Abelian in nature, and therefore cannot be described by the group, $O(2)$, of rotations in a plane. This alone is enough to show that $\mathbf{B}^{(3)}$ makes a profound difference to the conventional view [1-3] of vacuum electrodynamics, which is based on the Abelian group $O(2)$ (identified with $U(1)$). There are many

fundamental consequences [4-12] of the belated [4] realization that $\mathbf{B}^{(3)}$ exists in the vacuum, and can be experimentally measured [8] with precision. One example is that the little group of the Poincaré group becomes $O(3)$, and not the obscure, physically meaningless, $E(2)$ [16]. Since $E(2)$ is a Wigner little group [17] if and only if the photon mass is identically zero, its abandonment in favor of $O(3)$ *must* mean that the photon has mass if it is to be viewed as a particle. The alternative is unphysical, because it implies that $\mathbf{B}^{(3)}$ is identically zero, whereupon $\mathbf{B}^{(1)}$ and $\mathbf{B}^{(2)}$ also vanish from Eqs. (1). The assertion that $\mathbf{B}^{(3)}$ is zero means that all electromagnetism vanishes, a reduction to absurdity.

Since $\mathbf{B}^{(1)}$, $\mathbf{B}^{(2)}$ and $\mathbf{B}^{(3)}$ form the Lie algebra of an $O(3)$ group, they must each be infinitesimal angular momentum generators of the group. Specifically [6] it can be shown that

$$\begin{aligned}\hat{B}^{(1)} &= -B^{(0)} \hat{J}^{(1)} e^{i\phi}, & \hat{B}^{(2)} &= -B^{(0)} \hat{J}^{(2)} e^{-i\phi}, \\ \hat{B}^{(3)} &= iB^{(0)} \hat{J}^{(3)},\end{aligned}\tag{2}$$

where $\hat{B}^{(1)}$, $\hat{B}^{(2)}$, and $\hat{B}^{(3)}$ are matrix forms of the axial vectors appearing in Eq. (1) and where $\hat{J}^{(1)}$, $\hat{J}^{(2)}$, and $\hat{J}^{(3)}$ are the $O(3)$ infinitesimal rotation generators [14] in the circular basis,

$$\mathbf{e}^{(1)} = \mathbf{e}^{(2)*} = \frac{1}{\sqrt{2}}(\mathbf{i} - i\mathbf{j}), \quad \mathbf{e}^{(3)} = \mathbf{k}.\tag{3}$$

Here ϕ is the electromagnetic phase $\phi = \omega t - \boldsymbol{\kappa} \cdot \mathbf{r}$, where ω and $\boldsymbol{\kappa}$ are the usual angular frequency and wave vector of the beam at an instant t and position \mathbf{r} in space. The rotation generators form a commutative Lie algebra [6],

$$[\hat{J}^{(1)}, \hat{J}^{(2)}] = -\hat{J}^{(3)*}, \text{ et cyclicum,}\tag{4}$$

which becomes the more familiar [14]

$$[\hat{J}_x, \hat{J}_y] = i\hat{J}_z, \text{ et cyclicum,}\tag{5}$$

in the ordinary Cartesian basis for $O(3)$. Within a factor \hbar [14], the algebra (4) is that of the fundamental quantum mechanical angular momentum operators [18]. The quantum mechanical operators of which $\mathbf{B}^{(1)}$, $\mathbf{B}^{(2)}$ and $\mathbf{B}^{(3)}$ are expectation values (classical equivalents) are therefore

$$\begin{aligned}\hat{B}^{(1)} &= -B^{(0)} \frac{\hat{J}^{(1)}}{\hbar} e^{i\phi}, & \hat{B}^{(2)} &= -B^{(0)} \frac{\hat{J}^{(2)}}{\hbar} e^{-i\phi}, \\ \hat{B}^{(3)} &= iB^{(0)} \frac{\hat{J}^{(3)}}{\hbar}.\end{aligned}\tag{6}$$

In particular the longitudinal *photomagneton* [6] $\hat{B}^{(3)}$ is the quantum mechanical version of $B^{(3)}$ in the classical theory of electromagnetism. The existence of $\hat{B}^{(3)}$ was deduced using an independent method in Ref. [4], and its expectation value is the experimentally observable $B^{(3)}$.

3. CONSERVATION OF ANGULAR MOMENTUM, NOETHER'S THEOREM, AND $B^{(3)}$

Noether's theorem [14] is fundamental to physics, because it links fundamental symmetries to fundamental conservation laws. It is therefore necessary to show that $B^{(3)}$ is rigorously compatible with Noether's theorem. In view of the link between $\hat{B}^{(3)}$ and $\hat{J}^{(3)}$ in Eq. (6), this can be proven through the fact that Noether's theorem implies conservation of angular momentum as a consequence of fundamental spacetime symmetries [14]. Specifically, the rotation generators appearing in Eqs. (6) are, within a factor \hbar , the three space-space components of the quantized version of the angular momentum four-tensor $M_{\mu\nu}$. Noether's theorem states that this is a conserved classical quantity,

$$\frac{dM_{\mu\nu}}{dt} = 0.\tag{7}$$

If x_μ is defined in Minkowski notation by (X, Y, Z, ict), then $M_{\mu\nu}$ is given by an integral over the canonical energy-momentum tensor $T_{\mu\nu}$,

$$M_{\mu\nu} = \int (T_{0\mu}x_\nu - T_{0\nu}x_\mu) d^3x,\tag{8}$$

and by virtue of conservation of angular momentum, $T_{\mu\nu}$ must be symmetric. It is important to note that a symmetric $T_{\mu\nu}$ is also necessary in Einstein's field equations for gravitation, a non-Abelian field [14], and the non-Abelian nature of the algebra (1) forges a link between $B^{(3)}$, $M_{\mu\nu}$ and $T_{\mu\nu}$, one which is missing in the conventional view of electromagnetism

in which $\mathbf{B}^{(3)}$ does not occur. This link is exploited in the next section, and may well form the basis for unification of electromagnetism and gravitation.

The proof of compatibility of $\mathbf{B}^{(3)}$ with Noether's theorem follows simply from the fact that the eigenvalues of $\hat{B}^{(3)}$ are those of $\hat{J}^{(3)}$, the Z axis angular momentum operator component of quantum mechanics. This is the usual specified (observable) component with quantum numbers $M_J = -J, \dots, J$. The theory of angular momentum in quantum mechanics is of course well understood [18], and asserts that if $\hat{J}^{(3)}$ is specified, then the orthogonal components are unspecified. The only non-zero eigenvalues of the electromagnetic beam's $\hat{B}^{(1)}$, $\hat{B}^{(2)}$ and $\hat{B}^{(3)}$ components are therefore those of $\hat{B}^{(3)}$, and this is a constant of motion that commutes with the Hamiltonian [18]. In other words its classical expectation value is conserved. Thus $\hat{B}^{(3)}$ is compatible with Noether's theorem, Q.E.D. In ordinary electrodynamics, in which $\hat{B}^{(3)}$ does not appear, the expectation values of $\hat{B}^{(1)}$ and $\hat{B}^{(2)}$ (the only components considered) vanish, and there is no magnetization at first order in $B^{(0)}$ due to circularly polarized radiation. Therefore there is no $I_0^{1/2}$ magnetization profile (introduction), which is a specific signature of $\hat{B}^{(3)}$, and of $\hat{B}^{(3)}$ only. The $I_0^{1/2}$ profile therefore isolates $\hat{B}^{(3)}$ experimentally to any required degree of precision. Noether's theorem is satisfied in conventional electrodynamics because the fields $\hat{B}^{(1)}$ and $\hat{B}^{(2)}$ average to zero and zero is a conserved quantity. In a self-consistent view [6], in which photon mass is non-zero, we obtain an exponentially decaying $\mathbf{B}^{(3)}$,

$$\mathbf{B}^{(3)} = B^{(0)} e^{-\xi z} \mathbf{e}^{(3)}, \quad (9)$$

where ξ is a rest wave-number [6]. This is again compatible with Noether's theorem because for each Z,

$$\frac{d\mathbf{B}^{(3)}}{dt} = 0. \quad (10)$$

4. $\mathbf{B}^{(3)}$: THE LINK BETWEEN ELECTROMAGNETISM AND GRAVITATION

Having demonstrated the compatibility of $\mathbf{B}^{(3)}$ with Noether's theorem through conservation of the angular

momentum of the free electromagnetic field, it becomes possible to establish several interesting links between the non-Abelian Lie algebra (1) and the non-Abelian theory of gravitation [14] in the vacuum. The Lie algebra (1) implies that for self-consistency [6], the gauge group of electromagnetism must be enlarged to that of the rotation group itself, because there is an observable field, $\mathbf{B}^{(3)}$, in the axis orthogonal to the plane of definition of $O(2)$. As in all non-Abelian gauge structures [14], the field itself may emit gauge particles, and significantly, in general relativity, the gravitational field itself carries energy, which is equivalent to mass, and is itself [14] a source of gravitation. Thus, in the Einstein field equations, both sides are tensors whose covariant divergence vanishes. The covariant derivative in general relativity has a geometrical origin, and is due to the fact that spacetime itself becomes non-Euclidean in the presence of a gravitating object. The ordinary divergence of the Einstein tensor $G_{\mu\nu}$ is not zero, even in the absence of matter, and this is an expression of the fact that the field couples to itself.

With the belated recognition of the vacuum $\mathbf{B}^{(3)}$ in electrodynamics [4-12], it is now possible to construct close analogies in electrodynamics to these well-accepted and experimentally verified properties in general relativity. An $O(3)$ gauge symmetry for vacuum electrodynamics results [6] in the replacement of the ordinary $O(2)$ four-tensor [6, 14] $F_{\mu\nu}$ by a tensor $G_{\mu\nu}$ which is also a vector in the circular basis (1), (2) and (3). As for the Einstein tensor the covariant derivative of $G_{\mu\nu}$ vanishes in the vacuum (i.e., in the absence of a matter source) but its ordinary derivative does not [6]. As shown elsewhere [6] these properties succeed in incorporating $\mathbf{B}^{(3)}$ self-consistently into the Maxwell equations, which are thereby generalized to $O(3)$ gauge symmetry. It can be shown [19] that each of the equations (1) are Biot-Savart-Ampere laws in which an electron with intrinsic spin propagates at the speed of light, in which case its radiated fields become indistinguishable from those concomitant with the uncharged photon, as first shown by Jackson [3]. Therefore, the conjugate product [7] $\mathbf{B}^{(1)} \times \mathbf{B}^{(2)}$ can be thought of as the source of $\mathbf{B}^{(3)}$ in the vacuum, and so the electromagnetic field becomes self-propagating in the non-Abelian sense, i.e., the field $\mathbf{B}^{(1)}$ vector multiplies $\mathbf{B}^{(2)}$ to produce $\mathbf{B}^{(3)}$; and so on in cyclic symmetry. This process is of course absent in the conventional view of electrodynamics, whose structure has hitherto thought to have been Abelian (gauge group $U(1) = O(2)$).

As in general relativity, the covariant derivative in $O(3)$ vacuum electrodynamics also has a geometrical origin [6]; and the four-potentials $A_\mu^{(1)}$, $A_\mu^{(2)}$ and $A_\mu^{(3)}$ [6] can be identified with connection coefficients in general relativity. The latter are denoted $\Gamma_{\lambda\nu}^\mu$ in covariant-contravariant notation [14]. The quantity analogous to $G_{\mu\nu}$ of non-Abelian electrodynamics is the Riemann-Christoffel curvature tensor, $R_{\lambda\mu\nu}^\kappa$, which indicates that spacetime is non-Euclidean and that there is a gravitational field present. If it is possible to express $G_{\mu\nu}$ in the same tensorial structure as $R_{\lambda\mu\nu}^\kappa$, then spacetime becomes non-Euclidean (i.e., curved in loose terminology) if there is a non-Abelian electromagnetic field present. This would be a step towards the unification of general relativity and electrodynamics in the vacuum using non-Abelian structures for both. Recall that the experimental observability of $\mathbf{B}^{(3)}$ is the basis of this unified theory. Continuing the analogy, the Bianchi identity of general relativity becomes synonymous with the homogeneous Maxwell equations [6] in the $O(3)$ gauge group.

Finally, it can be shown that in $O(3)$ vacuum electrodynamics, there occurs a charge quantization condition which can be written as

$$\hbar\kappa = eA^{(0)}, \quad (11)$$

The quantized momentum of the photon, $\hbar\kappa$, is therefore identified with the product $eA^{(0)}$, where e is electron charge and $A^{(0)}$ is the magnitude of the four-vector $A_\mu^{(1)}$ or $A_\mu^{(2)}$ (these being complex conjugates). The physical meaning of Eq. (11) has been demonstrated [20] by identifying the energy of an electron accelerated to the speed of light with the photon, the quantum of light energy $\hbar\omega$. Since eA_μ is an energy momentum vector, it is analogous with the $T_{\mu\nu}$ in the Einstein field equations, and the charge quantization condition (11) becomes analogous with the Einstein field equation itself,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\frac{8\pi G}{c^2}T_{\mu\nu}, \quad (12)$$

so that p_μ is analogous with the Einstein tensor $G_{\mu\nu}$. Again, the charge quantization condition does not occur in vacuum electrodynamics described by the Abelian group $O(2)$, but nevertheless emerges directly from the Hamilton-Jacobi

equation of one electron in a classical, circularly polarized, electromagnetic field. It emerges independently [6,20] as a direct result of the $O(3)$ gauge geometry itself.

Therefore, there are several rich analogies between $O(3)$ vacuum electrodynamics, which depends on the existence of $\mathbf{B}^{(3)}$ as an experimental observable [19], and vacuum gravitation in general relativity.

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