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REPLY TO COMMENT BY E. COMAY ON THE LONGITUDINAL MAGNETIC FIELD OF CIRCULARLY POLARIZED ELECTROMAGNETIC WAVES

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The argument presented by E. Comay in Ref. 1 is in error precisely at the point where he uses the Cartesian form of Stokes' theorem. His Comment is therefore erroneous and inconsequential.

Key words: Stokes' theorem, $\boldsymbol{B}^{(3)}$ theory, reply to Comay.

1. COUNTER ARGUMENT

Comay's definition [1] of $\boldsymbol{B}^{(3)}$ for a plane wave in his Eq. (1) uses the fact that the well-known optical conjugate product $\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)}$ is empirically irrotational. Therefore, for a plane wave in vacuo [2–6],

$$\nabla \times \boldsymbol{B}^{(3)} = 0. \tag{1}$$

Stokes' theorem implies that for the vacuum plane wave, any path integral over $\boldsymbol{B}^{(3)}$ is zero. This result is supported empirically [7,8]

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by the fact that there is no observable induction from $\nabla \times \boldsymbol{B}^{(3)}$ when a modulated laser beam is passed through a vacuum within an induction coil of high precision. Comay's claim is in clear contradiction to experience.

In his Comment, he attempts to evaluate a line integral of Stokes' theorem for the elliptically polarized spherical wave in vacuo and concludes that, on average, the complete field

$$\boldsymbol{B} = \boldsymbol{B}^{(1)} + \boldsymbol{B}^{(2)} + \boldsymbol{B}^{(3)} \tag{2}$$

is not irrotational. In other words, the curl of $\boldsymbol{B}^{(3)}$ is asserted to be non-zero. In Sec. 5.4 of Ref. 2, however, it has already been shown that $\boldsymbol{B}^{(3)}$ is irrotational for the general spherical wave in vacuo. The $\boldsymbol{B}^{(3)}$ field is given by Eq. (196) of Ref. 2 in terms of the vector spherical harmonics, which are irreducible compound tensors, representations of the full rotation group. In operator form [2],

$$\hat{B}^{(3)} Y_{lm} = m B^{(0)} Y)lm, \tag{3}$$

where $m = -1, \ldots, 1$.

Thus $\boldsymbol{B}^{(3)}$ is directed in Z and is irrotational, its curl is zero, and any line integral over $\boldsymbol{B}^{(3)}$ is zero by Stokes' theorem. Similarly, the angular momentum $\boldsymbol{M}^{(3)}$ of the spherical wave is Z-directed, as discussed by Jackson [9] following his Eq. (16.64). Thus $\boldsymbol{B}^{(3)}$ for the spherical wave is irrotational for all multipole components and all zones. A more general form of this result is given in Sec. 5.4 of Ref. 2, which has been available to Comay for two years.

So where did Comay go wrong?

The precise point at which Comay's argument fails occurs in his sentence

> Along the radial segment $\overline{PQ}$ (on the z-axis) the radiation is circularly polarized and due to assertion (A), the longitudinal $\boldsymbol{B}^{(3)}$ is independent of ω and makes a non-zero contribution to the integral, while the ordinary transverse (rotating) magnetic field makes no contribution because it is perpendicular to the line segment $\overline{PQ}$.

According to Comay's *own* definition, it is clear that the partial derivatives of with $\boldsymbol{B}_Z^{(3)}$ respect to Y and X are both zero. He states specifically that $\boldsymbol{B}^{(3)}$ makes no contribution other than on the Z axis. If so, the Cartesian form of Stokes' theorem [10] concludes that

$$\oint B_Z^{(3)} dZ = \int\int \left(\frac{\partial B^{(3)}}{\partial Y} dY dZ - \frac{\partial B^{(3)}}{\partial X} dZ dX \right) = 0, \tag{4}$$

for all paths, including Comay's path, *q.e.d.* Comay further says,

> Both arcs ($\{QR\}$ and $\{SP\}$) are perpendicular to the direction of propagation, and hence the longitudinal field, $B^{(3)}$, makes no contribution regardless of the wave's polarization (which varies along each arc from being circular at P and Q, to linear at R and S).

Therefore Comay defines $B^{(3)}$ along the Z axis only, and so defines it to be irrotational.

Consider now Comay's comment, "... while the ordinary transverse (rotating) magnetic field makes no contribution because it is perpendicular to the line segment $\overline{PQ}$." This is also incorrect because the Stokes theorem for the transverse components reads

$$\oint_C B_X dX + \oint_C B_Y dY = \int_S \int \frac{\partial B_X}{\partial Z} dZ\, dX - \int_S \int \frac{\partial B_Y}{\partial Z} dY\, dZ, \quad (5)$$

and it is seen that the line integrals depend on the fact that B_X and B_Y are functions of Z (through the electromagnetic phase). This leads to the ordinary Ampere-Maxwell law in vacuo,

$$\nabla \times B^{(1)} = \frac{1}{C^2}\frac{\partial E^{(1)}}{\partial t}, \quad (6)$$

and to a non-zero curl for the transverse $B^{(1)} = B^{(2)*}$. This can be illustrated by a simple example of the line integral over the transverse X component of a plane wave,

$$B := B_I := i\frac{B^{(0)}}{\sqrt{2}} \exp\left(i(\omega t - \kappa Z)\right) \mathbf{i}. \quad (7)$$

Let the path be defined by $ABCDA$, where,

> from A to $B, Z = 0, X$ goes from 0 to b,
> from B to $C, X = b, Z$ goes from 0 to a,
> from C to $D, Z = a, X$ goes from b to 0,
> from D to $A, X = 0, Z$ goes from a to 0.

The line integral is

$$\begin{aligned}\oint B \cdot dr = &\int_0^b B_X(Z=0)dX + \int_0^a B_X(X=b)dZ \\ &+ \int_b^0 B_X(Z=a)dX + \int_a^0 B_X(X=0)dZ,\end{aligned} \quad (8)$$

with

$$B_X(X=b)=B_X(X=0)=B_X, \quad \text{(no specific } X \text{ dependence)} \tag{9}$$

and

$$B_X(Z=0)=i\frac{B^{(0)}}{\sqrt{2}}e^{i\omega t}, \quad B_X(Z=a)=i\frac{B^{(0)}}{\sqrt{2}}e^{i(\omega-t\kappa a)}. \tag{10}$$

The result is

$$\oint \boldsymbol{B}\cdot d\boldsymbol{x}=i\frac{B^{(0)}}{\sqrt{2}}be^{i\omega t}(1-e^{\kappa a}) \tag{11}$$

and thus non-zero. The curl of the transverse $\boldsymbol{B}^{(1)}=\boldsymbol{B}^{(2)*}$ is non-zero in vacuo if a and b are zero. If $a=0$ and $b=0$ there is no closed loop, or path. Furthermore, the line integral is non-zero because of the difference in the contributions of segments AB and CD, which are in the X or transverse, direction. There is no net contribution in the Z direction only because the segments BC and DA cancel each other, and *not* because of Comay's assertion that there is no contribution in DA and BC taken separately. To work out the line integral it is vital to take into consideration the Z dependence of B_X. Consequently Comay's following assertion is erroneous: "... since the arc $\{QR\}$ is traversed in the opposite direction to $\{SP\}$, it follows that the contribution from $\{QR\}$ is cancelled by the contribution from $\{SP\}$." As seen above, however, the AB and CD contributions do not cancel, because the curl of $\boldsymbol{B}^{(1)}$ is *not* zero in the vacuum by Maxwell's equations. Comay concludes that curl $\boldsymbol{B}^{(3)}$ is zero, whereas it is not zero; and he concludes that curl is not zero, whereas it is zero.

Through Eq. (13.73), page 452 of Ref. 11, it is easily seen that the magnetic component of dipole radiation is linearly polarized, not elliptically polarized as asserted by Comay. This component is given by [11] in the radiation zone by

$$\boldsymbol{B}=\frac{-\mu_0 c p_0}{4\pi r\left(\frac{\lambda}{2\pi}\right)^2}\sin\theta e^{i\phi_0}\boldsymbol{i}:=\frac{B^{(0)}}{\sqrt{2}}\sin\theta e^{i\phi_0}\boldsymbol{i}, \tag{12}$$

where μ_0 is the vacuum permeability, c the speed of light, ϕ_0 the electromagnetic phase, p_0 the dipole magnitude, r the distance from the mid-point of the dipole to point $P(r,\phi,\theta)$ [11] in spherical polar coordinates, and where $\lambda/(2\pi)$ is the wavelength divided by 2π. Integrating over the surface of a sphere, we find that the total magnetic flux radiated by the oscillating (or rotating) dipole is

$$\begin{aligned}\Phi &= \frac{B^{(0)}}{\sqrt{2}} e^{i\phi_0} \boldsymbol{i} \int_0^{2\pi} \int_0^{\pi} r^2 \sin^2\theta d\theta d\phi \\ &= \frac{B^{(0)}}{\sqrt{2}} \pi^2 r^2 e^{i\phi_0} \boldsymbol{i},\end{aligned} \tag{13}$$

and so the integrated flux density is,

$$\boldsymbol{B} := \frac{\Phi}{\pi^2 r^2} = \frac{B^{(0)}}{\sqrt{2}} e^{i\phi_0} \boldsymbol{i}. \tag{14}$$

If we now resolve the component $\boldsymbol{B}$ in Eq. (14) into a sum of right- and left- circularly polarized components

$$\boldsymbol{B} = \boldsymbol{B}^{(1)}(\text{right}) + \boldsymbol{B}^{(1)}(\text{left}) = \frac{B^{(0)}}{\sqrt{2}} (\boldsymbol{i} - i\boldsymbol{j} + i\boldsymbol{j}) e^{i\phi_0}, \tag{15}$$

it is found that

$$\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)}(\text{right}) = -\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)}(\text{left}) = iB^{(0)} \boldsymbol{B}^{(2)*}, \tag{16}$$

and that

$$\nabla \times \boldsymbol{B}^{(3)} = 0. \tag{17}$$

This is another way of demonstrating that the $\boldsymbol{B}^{(3)}$ field from dipole radiation is irrotational. As for the Poynting vector [11], the $\boldsymbol{B}^{(3)}$ vector is everywhere purely radial. Elliptical polarization can be described in Eq. (15) by adding a factor $ia\boldsymbol{j}$ and subtracting a factor $ib\boldsymbol{j}$, where a and b are dimensionless constants. This makes no difference to that fact that everywhere purely radial. In the vacuum, the curl of the Poynting vector for dipole radiation is also zero. According to Comay's argument, the curl would be non-zero. The functional dependence of $\sin\theta$ on Z is given by

$$\sin\theta = \left(1 - \frac{Z^2}{r^2}\right)^{1/2}, \tag{18}$$

and, if we prefer not to integrate over a spherical surface, the curl of the unintegrated $\boldsymbol{B}^{(3)}$ is also zero. From Eqs. (12), (16), and (18), we obtain, in this case,

$$\boldsymbol{B}^{(3)} = B^{(0)}\boldsymbol{k} = -\frac{\sqrt{2}\mu_0 c p_0}{4\pi r \left(\frac{\lambda}{2\pi}\right)^2} \left(1 - \frac{Z^2}{r_2}\right)^{1/2} \boldsymbol{k} \tag{19}$$

and

$$\nabla \times B^{(3)} = 0. \tag{20}$$

2. DISCUSSION

Due to elementary errors, Comay's Comment is inconsequential. He also confuses the theory of $B^{(3)}$ of in his claims (A) to (C). The actual theory is available in the literature. Comay's claims (A) to (C) distort the fact that $B^{(3)}$ is the classical field analogue of the fundamental spin of the photon, i.e., the analogue of an irreducible representation of the rotation sub-group of the Poincaré group, as shown by Wigner [2–6] in his classic 1939 paper. The $B^{(3)}$ theory develops classical electrodynamics from the first principles of classical special relativity, whose group is the Poincaré group of spacetime. In linear polarization, there is a 50-50 mixture of circular components, for which the sign of $B^{(3)}$ reverses. For each photon, $B^{(3)}$ is always non-zero because it is the irreducible field representation of the particulate photon's fundamental spin. This representation has been developed in terms of the Pauli-Lyuban'ski vectors for the classical field. In vacuo, the only component of these vectors is $B^{(3)}$. The cyclic theorems upon which $B^{(3)}$ theory is developed show the existence of $-iE^{(3)}/c$ in vacuo. Additionally, there is always present a Coulombic field in vacuo. The theory has been independently corroborated by several groups and is reviewed in Ref. 5.

For multipole radiation in all zones, the operator form of the $B^{(3)}$ theory is given in Ref. 12. The eigenfunctions of $B^{(1)}$, $B^{(2)}$ and $B^{(3)}$ are the spherical harmonics, and the photomagneton operator is defined by Eq. (3). For all multipole fields in all zones it is directed in the Z axis. It is shown on p. 82 of Ref. 2 that the $B^{(1)}, B^{(2)}, B^{(3)}$ operators form irreducible representations of the complete rotation group, and $B^{(3)}$ is the irreducible component in Z. It is well known that the spherical harmonics are the eigenfunctions of angular momentum for all l and m, i.e., for all multipoles. The irreducible field representations $B^{(3)}$ is always proportional directly to the irreducible unit vector representation $e^{(3)}$ of the full rotation group. The unit vector $e^{(3)}$ is the unit vector k in the Z axis, and in no other axis. Therefore $\nabla \times B^{(3)}$ is always zero in the full rotation group for all multipole field components, including Comay's dipole component.

Therefore, Comay's Comment is erroneous, and indeed trivially so.

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