BELTRAMI VECTOR FIELDS IN NON-ABELIAN ELECTRODYNAMICS

ABSTRACT

It is shown that the three magnetic components of electromagnetic radiation in O(3) electrodynamics are Beltrami vector fields, illustrating the fact that conventional Maxwell-Heaviside electrodynamics are incomplete. Therefore Beltrami electrodynamics are foundational, structuring the vacuum fields of nature and completing the Maxwell-Heaviside view. Transverse plane waves are shown to be solenoidal, complex lamellar and Beltrami, and to obey the Beltrami equation, of which the Evans-Vigier field of O(3) electrodynamics is a rigorously non-zero solution. Therefore the existence of transverse plane waves rigorously implies the existence of the finite Evans-Vigier field.

INTRODUCTION

As argued by Reed, {1} the Beltrami vector field originated in hydrodynamics and is force-free. It is one of the three basic field types: solenoidal; complex lamellar and Beltrami. These vector field types originated in hydrodynamics, and describe properties of the velocity field, flux or streamline, $\boldsymbol{v}$ and the vorticity $\nabla \times \boldsymbol{v}$. The Beltrami field is also a Magnus force-free fluid flow and is expressed as:

$$\boldsymbol{v} \times (\nabla \times \boldsymbol{v}) = \boldsymbol{0}. \tag{1}$$

The solenoidal vector field is

$$\nabla \cdot \boldsymbol{v} = 0 \tag{2}$$

and the complex lamellar vector field is

$$\boldsymbol{v} \cdot (\nabla \times \boldsymbol{v}) = \boldsymbol{0}. \tag{3}$$

The Beltrami condition can also be represented {1} as:

$$\nabla \times \boldsymbol{v} = k\boldsymbol{v} \tag{4}$$

where:

$$k = \frac{1}{v^2} \boldsymbol{v} \cdot \nabla \times \boldsymbol{v} \tag{5}$$

for real valued $\boldsymbol{v}$.

Beltrami fields have been advanced {1} as theoretical models for astrophysical phenomena such as solar flares and spiral galaxies, plasma vortex filaments arising from plasma focus experiments, and superconductivity. Beltrami electromagnetic fields probably have major potential significance to theoretical and empirical science. In plasma vortex filaments for example, energy anomalies arise which cannot be described with the Maxwell-Heaviside equations.

In Section 2, it is shown that the three magnetic components of Yang-Mills electrodynamics with an internal O(3) gauge symmetry (O(3) electrodynamics) are Beltrami fields as well as being complex lamellar and solenoidal fields. In the Beltrami electrodynamics that correspond to O(3) electrodynamics, there exists a magnetic component ($\boldsymbol{B}^{(3)}$), the Evans-Vigier field, {2-10} which is a particular solution of the general one given by Chandrasekhar and Kendall {11} of the general Beltrami equation:

$$\nabla \times \boldsymbol{B} = k\boldsymbol{B}. \tag{6}$$

This shows conclusively that Maxwell-Heaviside electrodynamics is incomplete, because in those electrodynamics, $\boldsymbol{B}^{(3)}$ is zero, while the relevant solution just described is rigorously non-zero. The photon is therefore an object which must be described in Beltrami electrodynamics.

In Section 3, general solutions are given of the Beltrami equation, which is an equation of O(3) electrodynamics. Therefore these solutions are also general solutions of O(3) electrodynamics in the vacuum.

O(3) AND BELTRAMI ELECTRODYNAMICS: SPECIAL CASE OF PLANE WAVES IN THE VACUUM.

One of the central theorems of O(3) electrodynamics is the B cyclic theorem {2-10}:

$$\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} = iB^{(0)}\boldsymbol{B}^{(3)*} \tag{7}$$

which relates in vacuo the three basic magnetic field components $\boldsymbol{B}^{(1)}$, $\boldsymbol{B}^{(2)}$ and $\boldsymbol{B}^{(3)}$. The fields are defined as:

$$\boldsymbol{B}^{(1)} = \frac{B^{(0)}}{\sqrt{2}}(i\boldsymbol{i} + \boldsymbol{j})e^{i\phi} \tag{8}$$

$$\boldsymbol{B}^{(2)} = \frac{B^{(0)}}{\sqrt{2}}(-i\boldsymbol{i} + \boldsymbol{j})e^{-i\phi} \tag{9}$$

$$\boldsymbol{B}^{(3)} = B^{(0)}\boldsymbol{k} \tag{10}$$

$$\phi = \omega t - \kappa Z \tag{11}$$

so $\boldsymbol{B}^{(1)}$ is the complex conjugate of $\boldsymbol{B}^{(2)}$ and both are plane waves; and $\boldsymbol{B}^{(3)}$ is longitudinal and phase free in the vacuum. Here ϕ is the electromagnetic phase and $\boldsymbol{i}$ and $\boldsymbol{j}$ are unit vectors in X and Y. The B Cyclic Theorem satisfies the rules of gauge theory {2-10} because it is a Yang-Mills theory with an O(3) internal gauge symmetry. The three fields $\boldsymbol{B}^{(1)}$, $\boldsymbol{B}^{(2)}$ and $\boldsymbol{B}^{(3)}$ are all solenoidal, complex lamellar and Beltrami. This is a remarkable property of Beltrami electrodynamics recognized as O(3) electrodynamics for the special case where $\boldsymbol{B}^{(1)} = \boldsymbol{B}^{(2)*}$ are plane waves. Therefore O(3) electrodynamics is described by the general Beltrami equation:

$$\nabla \times \boldsymbol{B} = k\boldsymbol{B} \tag{12}$$

applied to electrodynamics instead of hydrodynamics. It is well known that this equation has solutions such as those first described by Chandrasekhar and Kendall {11}, and it is shown in the next section that the Evans-Vigier field, $\boldsymbol{B}^{(3)}$, is one of these solutions. In the Maxwell-Heaviside electrodynamics, this solution does not exist, so Beltrami electrodynamics is at a more advanced level of development than Maxwell-Heaviside electrodynamics. Specifically, we have the results:

$$\nabla\cdot\boldsymbol{B}^{(1)}=0; \qquad \boldsymbol{B}^{(1)}\cdot\nabla\times\boldsymbol{B}^{(1)}=0; \qquad \boldsymbol{B}^{(1)}\times(\nabla\times\boldsymbol{B}^{(1)})=\boldsymbol{0} \tag{13}$$

$$\nabla\cdot\boldsymbol{E}^{(1)}=0; \qquad \boldsymbol{E}^{(1)}\cdot\nabla\times\boldsymbol{E}^{(1)}=0; \qquad \boldsymbol{E}^{(1)}\times(\nabla\times\boldsymbol{E}^{(1)})=\boldsymbol{0} \tag{14}$$

$$\nabla\cdot\boldsymbol{A}^{(1)}=0; \qquad \boldsymbol{A}^{(1)}\cdot\nabla\times\boldsymbol{A}^{(1)}=0; \qquad \boldsymbol{A}^{(1)}\times(\nabla\times\boldsymbol{A}^{(1)})=\boldsymbol{0} \tag{15}$$

(And also for indices (2) and (3)).

for the magnetic flux density (***B***); the electric field strength (***E***) and the vector potential (***A***).

Empirical evidence for $\boldsymbol{B}^{(3)}$ is available in the existence of the third Stokes parameter, S_3, which describes circular polarization {12} through the conjugate product $\boldsymbol{B}^{(1)}\times\boldsymbol{B}^{(2)}$ and in magneto-optical effects due to $\boldsymbol{B}^{(1)}\times\boldsymbol{B}^{(2)}$ {13}. In U(1) gauge theory (Maxwell-Heaviside electrodynamics), this conjugate product is zero, showing conclusively that U(1) gauge field theory is an incomplete theory of electrodynamics.

GENERAL SOLUTIONS OF THE BELTRAMI EQUATION

Multiplying the Beltrami equation

$$\nabla\times\boldsymbol{B}^{(1)}=k\boldsymbol{B}^{(1)} \tag{16}$$

on both sides by $\boldsymbol{B}^{(2)}$, it is seen that

$$\boldsymbol{B}^{(2)}\cdot\nabla\times\boldsymbol{B}^{(1)}=k\boldsymbol{B}^{(1)}\cdot\boldsymbol{B}^{(2)}, \tag{17}$$

so the constant k is not necessarily zero when dealing with complex fields. This result can be illustrated by the use of the Maxwell-Heaviside equations in free space in S.I. units:

$$\nabla\times\boldsymbol{E}=\frac{-\partial\boldsymbol{B}}{\partial t}; \qquad \nabla\times\boldsymbol{B}=\frac{1}{c^2}\frac{\partial\boldsymbol{E}}{\partial t}. \tag{18}$$

It follows that:

$$\begin{aligned}\nabla\times\boldsymbol{B}&=k\boldsymbol{B}\\ \nabla\times\boldsymbol{E}&=k\boldsymbol{E}\\ \nabla\times\boldsymbol{A}&=k\boldsymbol{A}\end{aligned} \tag{19}$$

where $\boldsymbol{B}=\nabla\times\boldsymbol{A}$ as usual, and where $k=\pm\kappa$. Here k is a pseudoscalar which changes sign between left and right circularly polarized radiation. The Beltrami equation for $\boldsymbol{B}^{(3)}$ is:

$$\nabla \times \boldsymbol{B}^{(3)} = k\boldsymbol{B}^{(3)} \tag{20}$$

where $k = 0$.

It is shown as follows, that all components of transverse plane waves are described by Beltrami equations in vacuo. For left handed transverse plane waves:

$$\boldsymbol{E}_L^{(1)} = \frac{E^{(0)}}{\sqrt{2}}(\boldsymbol{i} - i\boldsymbol{j})e^{-i(\omega t - \kappa Z)} = \boldsymbol{E}_L^{(2)*} \tag{21}$$

$$\boldsymbol{B}_L^{(1)} = \frac{B^{(0)}}{\sqrt{2}}(i\boldsymbol{i} + \boldsymbol{j})e^{-i(\omega t - \kappa Z)} = \boldsymbol{B}_L^{(2)*} \tag{22}$$

$$\boldsymbol{A}_L^{(1)} = \frac{A^{(0)}}{\sqrt{2}}(i\boldsymbol{i} + \boldsymbol{j})e^{-i(\omega t - \kappa Z)} = \boldsymbol{A}_L^{(2)*}. \tag{23}$$

For right handed transverse plane waves:

$$\boldsymbol{E}_R^{(1)} = \frac{E^{(0)}}{\sqrt{2}}(\boldsymbol{i} + i\boldsymbol{j})e^{-i(\omega t - \kappa Z)} = \boldsymbol{E}_R^{(2)*} \tag{24}$$

$$\boldsymbol{B}_R^{(1)} = \frac{B^{(0)}}{\sqrt{2}}(-i\boldsymbol{i} + \boldsymbol{j})e^{-i(\omega t - \kappa Z)} = \boldsymbol{B}_R^{(2)*} \tag{25}$$

$$\boldsymbol{A}_R^{(1)} = \frac{A^{(0)}}{\sqrt{2}}(-i\boldsymbol{i} + \boldsymbol{j})e^{-i(\omega t - \kappa Z)} = \boldsymbol{A}_R^{(2)*}, \tag{26}$$

and for the longitudinal $\boldsymbol{B}^{(3)}$ field:

$$\boldsymbol{B}_L^{(3)} = -\boldsymbol{B}_R^{(3)} = B^{(0)}\boldsymbol{k} \tag{27}$$

Therefore

$$\nabla \times \boldsymbol{B}_L^{(1)} = -\kappa\,\boldsymbol{B}_L^{(1)}; \qquad \nabla \times \boldsymbol{B}_R^{(1)} = \kappa\,\boldsymbol{B}_R^{(1)} \tag{28}$$

$$\nabla \times \boldsymbol{E}_L^{(1)} = -\kappa\,\boldsymbol{E}_L^{(1)}; \qquad \nabla \times \boldsymbol{E}_R^{(1)} = \kappa\,\boldsymbol{E}_R^{(1)} \tag{29}$$

$$\nabla \times \boldsymbol{A}_L^{(1)} = -\kappa\,\boldsymbol{A}_L^{(1)}; \qquad \nabla \times \boldsymbol{A}_R^{(1)} = \kappa\,\boldsymbol{A}_R^{(1)} \tag{30}$$

and similarly for index (2). For the longitudinal index (3):

$$\nabla \times \boldsymbol{B}_R^{(3)} = \nabla \times \boldsymbol{B}_L^{(3)} = \boldsymbol{0} \tag{31}$$

and all components are described by Beltrami equations in vacuo.

Since $\boldsymbol{E}$ and $\boldsymbol{B}$ are foundational fields, these equations are valid under all conditions. They are all force-free equations as defined in hydrodynamics by Beltrami.

Eqns. (19) are more general than eqns. (18) because the former are valid for $\boldsymbol{B}^{(1)}$, $\boldsymbol{B}^{(2)}$ and $\boldsymbol{B}^{(3)}$. The latter only for $\boldsymbol{B}^{(1)}$ and $\boldsymbol{B}^{(2)}$ because the $\boldsymbol{B}^{(3)}$ equation has a different structure from the Maxwell equations (2). The $\boldsymbol{B}^{(3)}$ equation has the same structure as the Beltrami equations for $\boldsymbol{B}^{(1)}$ and $\boldsymbol{B}^{(2)}$.

Because of the pseudoscalar k in eqns. (19), there are two distinct Beltrami equations for every Maxwell equation, e.g.

$$\nabla \times \boldsymbol{E}_L^{(1)} = -\frac{\partial \boldsymbol{B}_L^{(1)}}{\partial t} \quad \text{and} \quad \nabla \times \boldsymbol{E}_R^{(1)} = -\frac{\partial \boldsymbol{B}_R^{(1)}}{\partial t} \tag{32}$$

$$\Updownarrow$$

$$\nabla \times \boldsymbol{E}_L^{(1)} = -\kappa\, \boldsymbol{E}_L^{(1)} \quad \text{and} \quad \nabla \times \boldsymbol{E}_R^{(1)} = \kappa\, \boldsymbol{E}_R^{(1)}. \tag{33}$$

The Faraday law does not distinguish between left and right circular polarization. The corresponding Beltrami equations are distinct equations for the two distinct states of radiation.

More accurately, eqns. (19) are Beltrami-Helmholtz equations and $\boldsymbol{B}^{(3)}$ is a solution of the Beltrami equation with $k = 0$, i.e. $\boldsymbol{B}^{(3)}$ is solenoidal and irrotational, complex lamellar and Beltrami. It is illustrated in Fig (8) of Reed {1} as the line on the axis going up the page, and is part of the general solution of the solenoidal Beltrami equation given by Chandrasekhar and Kendall {11}. There is equipartition between toroidal and poloidal components as shown by Chandrasekhar

This proves that $\boldsymbol{B}^{(3)}$ is rigorously non-zero in Beltrami electrodynamics in vacuo. This statement is equivalent to saying that electrodynamics is an O(3) symmetry Yang-Mills theory in vacuo. The $\boldsymbol{B}$ Cyclic Theorem (7) shows that if $\boldsymbol{B}^{(3)}$ vanishes then so does $\boldsymbol{B}^{(1)}$ and $\boldsymbol{B}^{(2)}$.

The general solution in cylindrical coordinates of eqn. (12) is {11, 14}:

$$\boldsymbol{B} = \sum_{m,n} B_{mn} \boldsymbol{b}^{mn}(r,\theta,z) \tag{34}$$

where m is a non-negative integer and where $\boldsymbol{b}^{mn}$ depends on θ and z through $\phi = m\theta + nz$. The expressions for the modes depend on linear combinations of Bessel and Neumann functions, J_m and N_m, similar to the Helmholtz equation {14}. When the domain of solution involves the axis $r = 0$, and we restrict solutions to an axisymmetric wave equation, then:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \psi}{\partial r} \right) + k^2 \psi = 0. \tag{35}$$

The solution is:

$$\psi = C\, J_0(kr) \tag{36}$$

where C is any constant, which specializes to:

$$\boldsymbol{B} = B_0(0, J_1(kr), J_0(kr)) \tag{37}$$

for the mode $m = n = 0$; $a = (0, 0, 1)$. Therefore the unit vector $a = (0, 0, 1)$ designates the Z axis {1}.

The solution for the axial $\boldsymbol{B}^{(3)}$ field is:

$$\boldsymbol{B}^{(3)} = B_0(0, J_1(0), J_0(0)) \tag{38}$$

and depends on the Bessel functions $J_1(0)$ and $J_0(0)$. Therefore:

$$\begin{aligned} \boldsymbol{B}^{(3)} &= \boldsymbol{B}(k=0, m=0, n=0) \\ &= B_0(0,0,1) = B^{(0)}\boldsymbol{k} \end{aligned} \tag{39}$$

and the $\boldsymbol{B}^{(3)}$ field is along the Z axis and is rigorously non-zero.

DISCUSSION

It has been shown that the $\boldsymbol{B}^{(3)}$ equation of O(3) electrodynamics is a Beltrami equation with $k = 0$. The solution:

$$\boldsymbol{B}^{(3)} = B^{(0)}\boldsymbol{k} \tag{40}$$

is a solenoidal, axisymmetric, toroidal solution which is also complex lamellar and Beltrami. The $\boldsymbol{B}^{(3)}$ field is phaseless and irrotational and is the fundamental spin of the electromagnetic field. This concept is missing in Maxwell-Heaviside electrodynamics.

Beltrami electrodynamics are synonymous with O(3) electrodynamics (i.e. a Yang-Mills gauge field theory with an O(3) internal gauge field symmetry). Reed {1} discusses the fact that the Beltrami equation is an equation of higher symmetry electrodynamics, and for the special case of plane waves, the properties of section 2 apply. Therefore the vacuum fields of electrodynamics are Beltrami fields, rather than Maxwell-Heaviside fields, and Beltrami fields are fields of O(3) electrodynamics.

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Appendix: On the nature of the constant *k*

To illustrate that k can be different from zero, consider the complex magnetic plane wave:

$$\boldsymbol{B}^{(1)} = \frac{\boldsymbol{B}^{(0)}}{\sqrt{2}}\left(i\boldsymbol{i} + \boldsymbol{j}\right)e^{i(\omega t - \kappa Z)} \tag{A1}$$

where the complex conjugate is $\boldsymbol{B}^{(2)}$. Therefore, it follows that:

$$\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} = iB^{(0)}\boldsymbol{B}^{(3)*} \tag{A2}$$

where

$$\boldsymbol{B}^{(3)} = B^{(0)}\boldsymbol{k}. \tag{A3}$$

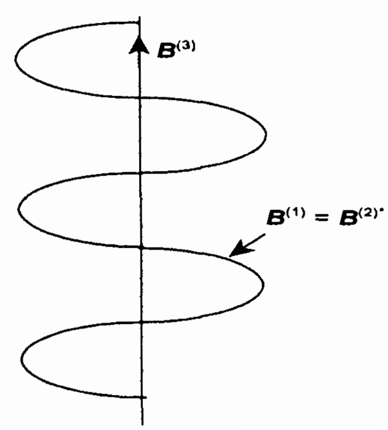
Eqn. (A2) is part of a Lorentz covariant gauge theory {9} related to Beltrami electrodynamics. It is easily seen from eqns. (A1) and (A3) that:

$$\begin{aligned} \nabla \times \boldsymbol{B}^{(1)} &= \kappa \boldsymbol{B}^{(1)} \\ \nabla \times \boldsymbol{B}^{(2)} &= \kappa \boldsymbol{B}^{(2)} \\ \nabla \times \boldsymbol{B}^{(3)} &= 0\boldsymbol{B}^{(3)} \end{aligned} \tag{A4}$$

Therefore, all the components are solutions of the same Beltrami equation:

$$\nabla \times \boldsymbol{B} = k\boldsymbol{B} \tag{A5}$$

The importance of this result is that $\boldsymbol{B}^{(3)}$ is rigorously non-zero in Beltrami electrodynamics, as in O(3) electrodynamics {9}. However, $\boldsymbol{B}^{(3)}$ is **zero** in the received view: Maxwell-Heaviside electrodynamics. The general solution to eqn. (A5) is discussed in the text and shows that Beltrami electrodynamics are more general and more foundational than Maxwell-Heaviside electrodynamics. The general solution is sketched below:



It is very important to note that all three components are solutions of eqn. (A5), and that $\boldsymbol{B}^{(3)}$ is **rigorously non-zero.** Beltrami electrodynamics are therefore more general than Maxwell-Heaviside electrodynamics, a result of foundational significance.