O(3) ELECTRODYNAMICS

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I. TOPOLOGICAL BASIS FOR HIGHER-SYMMETRY ELECTRODYNAMICS

Topology is the study of geometrical configurations invariant under transformation by continuous mappings. It provides what is probably the most fundamental known framework for the description of physical models using the mathematical techniques of group theory [1] and gauge theory [2]. A study of the topology of a

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given experiment can be used to decide whether that experiment is possible or not, and the decision is made in the language of group theory. Topological considerations can be applied to the vacuum itself, so that the vacuum becomes structured, or has a given configuration. On the basis of the fact that topology is a fundamental description, then it is also a fundamental description of the vacuum itself, and decides the structure of physical objects such as electrodynamic field equations [3,4] in the vacuum. The group-theoretic description of the received equations of classical electrodynamics, the Maxwell–Heaviside equations [5], is U(1), homomorphic with O(2) ($U(1) \approx O(2)$). The latter is the group of rotations in two dimensions, and the former is the group of all numbers of the form $e^{i\phi} = \cos\phi + i\sin\phi$, whose group space is a circle. The two groups are homomorphic or similar in form. Each element of O(2) is given uniquely [6] by an angle α, the angle of rotation in a plane. The group space of both O(2) and U(1) is therefore a circle. The received view [5,6] asserts that the classical electromagnetic field is a gauge field invariant under local U(1) gauge transformations. In other words, Maxwell–Heaviside theory is a U(1) symmetry Yang–Mills gauge field theory. Unified field theory proceeds on this assertion, specifically, that the electromagnetic sector has U(1) symmetry. The topological basis for this conclusion in the received view is given by such phenomena as the Aharonov–Bohm effect [6], where the classical vacuum is deduced to have a nontrivial topology [6]. This is combined with the view that electrodynamics is a U(1) gauge theory to give the received explanation of the Aharonov–Bohm effect [3,4,6]. In gauge theory in general, however, the vacuum has a rich topological structure, and this structure is not confined to U(1). Other groups may be used, and each has physical, or measurable, gauge-invariant, consequences. Therefore, the most fundamental basis for the development of field equations, such as those of classical electrodynamics, is the topology of the vacuum itself. In order to understand this further, some topological concepts must be introduced and defined.

Basic to the understanding of topology are simply and non-simply connected spaces. The relevant topological space is the vacuum itself. A simply connected space is one in which all closed curves may be shrunk to a point; and in a non-simply connected space, this is not true in general. In a non-simply connected space, a function may be many-valued, for example $\cos(\phi \pm 2\pi n)$. In this view therefore, the Aharonov–Bohm effect can exist physically if and only if the vacuum *itself* is not simply connected. The group theoretic description of the Aharonov–Bohm effect follows from these considerations. The $U(1) \approx O(2)$ group is not simply connected because its group space (denoted S^1) is a circle. The group space S^1 itself is not simply connected [6]. In the received view, this argument is used to show that the Aharonov–Bohm effect is supported by a vacuum topology described by the group U(1).

In the 1990s, however, there have been several attempts to extend the received view of classical electrodynamics, for example, the work of Barrett

[3,4], Lehnert et al. [7–10], Evans et al. [11–20] and Harmuth et al. [21,22]. These attempts stem from anomalies and self-inconsistencies in classical electrodynamics viewed as a U(1) gauge field theory. Some of these are reviewed in Section III of this chapter. The basis for these developments resides, as it must, in vacuum topology and its subsidiary languages of group and gauge theory. In other words, it may be possible to describe classical electrodynamics with groups other than U(1) in a non-simply connected vacuum, the relevant topological space. Once a particular group is chosen, general gauge field theory [3,4,6] may be used to write down the physical field equations of electrodynamics and the field tensor [3,4,11–20]. The results of the hypothesis are compared with empirical data as usual, and cross compared with the U(1) description. This method is developed and reviewed in this chapter. The basis of our development, therefore, is the topology of the vacuum, which ultimately decides which set of field equations is the more accurate in its description of data. The basis for gauge theory is fiber bundle theory, which is briefly reviewed in Section II.

We will be concerned in this article with the non-simply connected vacuum described by the group O(3), the rotation group. The latter is defined [6] as follows. Consider a spatial rotation in three dimensions of the form

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = (R) \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \quad \text{or} \quad r' = Rr \tag{1}$$

where R is a rotation matrix. Rotations have the property

$$X'^2 + Y'^2 + Z'^2 = X^2 + Y^2 + Z^2 \tag{2}$$

which can be written

$$r'^T r' = r^T r \tag{3}$$

where T denotes "transpose." Therefore

$$\begin{aligned} r^T R^T R r &= r^T r \\ R^T R &= 1 \end{aligned} \tag{4}$$

where R is an orthogonal 3×3 matrix. These matrices form a group. If R_1 and R_2 are orthogonal, then so is $R_1 R_2$:

$$(R_1 R_2)^T R_1 R_2 = R_2^T R_1^T R_1 R_2 = 1 \tag{5}$$

This group is denoted O(3) in three dimensions, and O(n) in n dimensions. The rotation group O(3) is a Lie group (i.e., is a continuous group), and is

non-Abelian (i.e., its rotation matrices do not commute) [6]. A simple example of an O(3) group is the one formed by the unit vectors of a Cartesian frame in three-dimensional space:

$$\begin{aligned} \boldsymbol{i} \times \boldsymbol{j} &= \boldsymbol{k} \\ \boldsymbol{j} \times \boldsymbol{k} &= \boldsymbol{i} \\ \boldsymbol{k} \times \boldsymbol{i} &= \boldsymbol{j} \end{aligned} \tag{6}$$

Therefore, we can adopt as our fundamental hypothesis that the topological space under consideration (i.e., the vacuum) is described by O(3) rather than U(1) and work out the consequences [11–20]. Some of the latter are reviewed in this chapter. An O(3) group can also be formed by the complex unit vectors defined by

$$\begin{aligned} \boldsymbol{e}^{(1)} &= \frac{(\boldsymbol{i} - i\boldsymbol{j})}{\sqrt{2}} \\ \boldsymbol{e}^{(2)} &= \frac{(\boldsymbol{i} + i\boldsymbol{j})}{\sqrt{2}} \\ \boldsymbol{e}^{(3)} &= \boldsymbol{k} \end{aligned} \tag{7}$$

so that

$$\begin{aligned} \boldsymbol{e}^{(1)} \times \boldsymbol{e}^{(2)} &= i\boldsymbol{e}^{(3)*} \\ \boldsymbol{e}^{(2)} \times \boldsymbol{e}^{(3)} &= i\boldsymbol{e}^{(1)*} \\ \boldsymbol{e}^{(3)} \times \boldsymbol{e}^{(1)} &= i\boldsymbol{e}^{(2)*} \end{aligned} \tag{8}$$

forms an O(3) group suitable for the description of circularly polarized radiation, and therefore of radiation in general [11–20]. Here, an asterisk (*) denotes complex conjugate. There are several other ways of defining the O(3) group, one of which is that it is the little group of the Poincaré group of special relativity [6]. A little group with structure O(3) is the group of a particle with mass. So if O(3) is adopted as the group describing classical electrodynamics, the photon, on quantization, may have a tiny mass (empirically estimated [23] as less than 10^{-68} kg). The little group for the massless photon in the received view is unphysical, it is the Euclidean E(2) [6,11–20]. This means that a particle without mass is an unphysical object. The photon without mass is obtained by quantizing a classical U(1) theory, suggesting that the received view is also unphysical. We do not have to search far to find some unphysical properties of the U(1) Yang–Mills gauge field theory of classical electromagnetism. For example, the electromagnetic phase is random, the 4-potential A^μ is unphysical as the result of

Heaviside's development of Maxwell's original concept of a physical vector potential, which was based, in turn, on Faraday's electrotonic state. Barrett [2] has reviewed extensive empirical evidence for a physical classical A^μ, in contradiction to U(1) theory, hereinafter described as "U(1) electrodynamics." The vacuum for the Aharonov–Bohm effect is non-simply connected, and therefore supports a *physical* A^μ [3,6]. The potential A^μ has no physically discernible effect if and only if the space is simply connected. Since U(1) is non-simply connected, there is a self-contradiction in the received view, [3,6] and since A^μ, by definition, is unphysical in U(1) electrodynamics, we must search for a new type of classical electrodynamics. In this chapter, we base this search on the group O(3), and hereinafter describe it as "O(3) electrodynamics." The basic topological space is that of the vacuum, and is described by the O(3) group and gauge theory based on this group. One consequence is that the potential is physical as required, another is that the unphysical random phase of U(1) electrodynamics is replaced by a gauge-invariant physical phase *factor* of O(3) electrodynamics. These changes are shown to have foundational consequences in interferometry and aspects of physical optics, for example. Furthermore, several of the well-developed techniques of non-Abelian gauge field theory [3,4,6] may be brought to bear on classical electrodynamics, because the group O(3) is a non-Abelian group, as argued already. This enriches and develops the subjects of classical and quantum electrodynamics and unified field theory.

The group space of O(3) is doubly connected (i.e., non-simply connected) and can therefore support an Aharonov–Bohm effect (Section V), which is described by a physical inhomogeneous term produced by a rotation in the internal gauge space of O(3) [24]. The existence of the Aharonov–Bohm effect is therefore clear evidence for an extended electrodynamics such as O(3) electrodynamics, as argued already. A great deal more evidence is reviewed in this article in favor of O(3) over U(1). For example, it is shown that the Sagnac effect [25] can be described accurately with O(3), while U(1) fails completely to describe it.

The O(3) group is homomorphic with the SU(2) group, that of 2×2 unitary matrices with unit determinant [6]. It is well known that there is a two to one mapping of the elements of SU(2) onto those of O(3). However, the group space of SU(2) is simply connected in the vacuum, and so it cannot support an Aharonov–Bohm effect or physical potentials. It has to be modified [26] to $SU(2)/Z2 \approx SO(3)$.

Therefore, this is a statement of our fundamental hypothesis, specifically, that the topology of the vacuum defines the field equations through group and gauge field theory. Prior to the inference and empirical verification of the Aharonov–Bohm effect, there was no such concept in classical electrodynamics, the ether having been denied by Lorentz, Poincaré, Einstein, and others. Our development of O(3) electrodynamics in this chapter, therefore, has a well-defined basis in fundamental topology and empirical data. In the course of the development of

this chapter, several misconceptions and inconsistencies of U(1) electrodynamics are brought to light, and these are remedied straightforwardly by changing the gauge group from U(1) to O(3). The implications are briefly reviewed for quantum electrodynamics and unified field theory, starting with electroweak theory. One major result of the latter is the existence of a novel massive boson, the existence of which is consistent with novel empirical data as discussed in Section XII. The gradual and consistent accumulation of evidence leads in this chapter to the conclusion that an O(3) gauge group is to be preferred over a U(1) gauge group in classical electrodynamics.

Some by-products of the development emerge, such as the fact that the acceptance of a structured vacuum described by an O(3) gauge group leads directly to the existence of novel charges and currents in the vacuum. These are conserved, or Noether, currents and charges and are clearly topological in origin. They spring from the fact that the vacuum is a topological space. Four such entities emerge:

1. A topological vacuum electric charge, also proposed empirically by Lehnert et al. [7–10]
2. A topological vacuum electric current, also proposed empirically by Lehnert et al. [7–10]
3. A topological vacuum magnetic charge, proposed also by Barrett [3,4] and Harmuth [21,22]
4. A vacuum topological magnetic current, proposed also by Barrett [3,4] and Harmuth [21,22].

Each of these four objects can provide energy, which can be loosely termed "vacuum energy:" energy coming from the topology of the vacuum.

In well-defined limits, the field equations of O(3) electrodynamics can collapse to a set of two complex conjugate equations that resemble those of U(1) electrodynamics (Maxwell–Heaviside equations), and a third equation for a novel fundamental spin component of O(3) electrodynamics, the $\boldsymbol{B}^{(3)}$ component [11–20] in the basis ((1),(2),(3)). This component also springs from the topology of the vacuum, described by an O(3) gauge group and is therefore a magnetic flux density that exists in the vacuum because of this choice of gauge group. Clearly, the $\boldsymbol{B}^{(3)}$ component is fundamental to O(3) electrodynamics and is not a static magnetic field of U(1) electrodynamics. The $\boldsymbol{B}^{(3)}$ component is an observable of the third Stokes parameter, topological phases, interferometry, and magneto-optics and is a radiated field that propagates with the radiation. In the laboratory, it propagates for all practical purposes at the speed of light, c, as does the third Stokes parameter to which it is proportional [11–20]. It is a fundamental property of the O(3) electromagnetic field that emanates from the topology of the vacuum. It forms an O(3) group with the plane wave $\boldsymbol{B}^{(1)} = \boldsymbol{B}^{(2)*}$

of magnetic flux density in the vacuum in O(3) electrodynamics, giving the B cyclic theorem [11–20]

$$\begin{aligned} \boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} &= iB^{(0)}\boldsymbol{B}^{(3)*} \\ \boldsymbol{B}^{(2)} \times \boldsymbol{B}^{(3)} &= iB^{(0)}\boldsymbol{B}^{(1)*} \\ \boldsymbol{B}^{(3)} \times \boldsymbol{B}^{(1)} &= iB^{(0)}\boldsymbol{B}^{(2)*} \\ B^{(0)} &= \left|\boldsymbol{B}^{(3)}\right| \end{aligned} \tag{9}$$

which is Lorentz-invariant, as it is, within a common factor on both sides, simply a relation between rotation generators of the O(3) group.

An important by-product of the development in this chapter (Section X) is the possible existence of scalar interferometry, which is interferometry between structured scalar potentials, first introduced by Whittaker [27,28] and that can be defined in terms of $\boldsymbol{B}^{(3)}$. This is a type of interferometry that depends on physically meaningful potentials that can exist self-consistently, as we have argued, only in a non-singly connected O(3) vacuum, because potentials in the nonsingly connected U(1) vacuum are assumed to be unphysical.

In summary of this introduction therefore, we develop a novel theory of electrodynamics based on vacuum topology that gives self-consistent descriptions of empirical data where an electrodynamics based on a U(1) vacuum fails. It turns out that O(3) electrodynamics does not incorporate a monopole, as a material point particle, because it is a theory based on the topology of the vacuum. The next section provides foundational justification for gauge field theory using fiber bundle theory.

II. BASIS IN FIBER BUNDLE THEORY

The gauge concept [3] was introduced by Weyl in 1918. In consequence of gauge theory, the absolute magnitude or norm of a physical vector depends on its location in spacetime. This notion is the basis of all contemporary gauge theory, which is expressed in the language [6] of group theory and has been highly developed mathematically [29–32]. For our purposes, it is sufficient to give a brief account of the elements of gauge theory as used in optics and electrodynamics, including O(3) electrodynamics. A gauge theory is a theory of special relativity in O(3) and U(1) electrodynamics, and in electroweak theory, and borrows concepts [6] from general relativity. For example, the homogeneous field equation of both U(1) and O(3) electrodynamics are Jacobi identities akin to the Bianchi identity in general relativity. Several reviews of contemporary gauge theory are given in Ref. 4, and the theory is firmly rooted in rigorous mathematical concepts such as fiber bundle theory. The latter leads to the field equations of O(3) electrodynamics through concepts [29–32] such as principal

bundle, associated vector bundle, connections on principal bundles, covariant derivatives of sections of a vector bundle, exterior covariant derivative, and the curvature of a connection. In optics and electrodynamics however, these mathematical concepts reduce to those of gauge potentials. It is sufficient to know, therefore, that the theory of O(3) electrodynamics is rigorously founded in fiber bundle theory and in the theory of extended Lie algebra [4,15]. The interested reader is referred elsewhere for mathematical details [29–32] because, in natural philosophy, a theory stands or falls by comparison with empirical data, not by mathematical rigor alone. The latter is necessary but not sufficient for a theory in optics and electrodynamics.

A simple example in classical electrodynamics of what is now known as "gauge invariance" was introduced by Heaviside [3,4], who reduced the original electrodynamical equations of Maxwell to their present form. Therefore, these equations are more properly known as the Maxwell–Heaviside equations and, in the terminology of contemporary gauge field theory, are identifiable as U(1) Yang–Mills equations [15]. The subject of this chapter is O(3) Yang–Mills gauge theory applied to electrodynamics and electroweak theory.

The Maxwell–Heaviside field equations are, in SI units

$$\begin{aligned} \nabla \cdot \boldsymbol{E} = 0; \qquad & \nabla \cdot \boldsymbol{B} = 0 \\ \nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{0} \qquad & \\ \nabla \times \boldsymbol{B} - \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} = \boldsymbol{0} \qquad & \end{aligned} \tag{10}$$

where $\boldsymbol{D}$ is the electric displacement, ρ is the electric charge density, $\boldsymbol{B}$ is magnetic flux density, $\boldsymbol{E}$ is the electric field strength, $\boldsymbol{H}$ is the magnetic field strength, and $\boldsymbol{J}$ is the current density. The received view is to assert that in the vacuum:

$$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E}; \qquad \boldsymbol{B} = \mu_0 \boldsymbol{H} \tag{11}$$

where ε_0 and μ_0 are permittivity and permeability in vacuo. Equations (12) then reduce to

$$\begin{aligned} \nabla \cdot \boldsymbol{D} = \rho; \qquad & \nabla \cdot \boldsymbol{B} = 0 \\ \nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0 \qquad & \\ \nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \qquad & \end{aligned} \tag{12}$$

The notion of gauge invariance is illustrated on this level by denoting

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \tag{13}$$

$$\boldsymbol{E} = -\nabla \times \boldsymbol{S} \tag{14}$$

where $\boldsymbol{A}$ is the vector potential of Maxwell and $\boldsymbol{S}$ the vector potential of Stratton. Using the identity

$$\nabla \times \nabla \chi \equiv \boldsymbol{0} \tag{15}$$

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} = \nabla \times (\boldsymbol{A} + \nabla \chi) \tag{16}$$

it is seen that any gradient $\nabla\chi$ can be added to $\boldsymbol{A}$ or $\boldsymbol{S}$, leaving $\boldsymbol{B}$ and $\boldsymbol{E}$ unchanged. Therefore, in the received view, $\boldsymbol{B}$ and $\boldsymbol{E}$ are gauge-invariant, measurable, and physical, whereas $\boldsymbol{A}$ and $\boldsymbol{S}$ are defined only up to an arbitrary gradient function and are therefore mathematical in nature, are not measurable, and have no physical effect. However, this can be true as argued in Section I only if the vacuum is simply connected, whereas the group spaces of U(1) and O(3) are not simply connected. We find empirically [3,4] several experimental verifications of the fact that $\boldsymbol{A}$ and $\boldsymbol{S}$ are in fact physical quantities, and that $\boldsymbol{A}$ and $\boldsymbol{S}$ cannot be changed arbitrarily by adding a gradient of a scalar. However elaborate the mathematical justification for U(1) electrodynamics becomes, this paradox remains.

During the course of this review chapter, we shall unearth several flaws in U(1) electrodynamics, some of which are discussed in Section III. One consequence of the gauge and metric invariance of the free space Maxwell–Heaviside equations is that they are also invariant under the general Lorentz transformation, consisting of boosts, rotations, and spacetime translations [6]. They are invariant also under the fundamental symmetry operations of motion reversal (T) and parity inversion (P). These properties mean that they are unable to describe interferometry and simple optical properties such as normal reflection without self-contradiction. The Maxwell–Heaviside theory and its gauge invariance is rigidly adhered to in the received view, but nevertheless, these basic flaws are there and are discussed systematically in this chapter. In the course of development of O(3) electrodynamics, a more general form of gauge theory is needed, and this more general form is based on vacuum topology and group theory. Therefore, in our view, O(3) electrodynamical equations apply in the vacuum as well as in field matter interaction [11–20]. In general, they must be solved without approximation using numerical techniques, but with well-defined assumptions, analytical solutions emerge. These include the B cyclic theorem [11–20].

The systematic development of gauge theory relies on a rotation of a n dimensional function ψ of the spacetime coordinate x^μ in special relativity. The

rotation is expressed as

$$\psi' = \exp\left(iM^a\Lambda^a(x^\mu)\right)\psi \equiv S(x^\mu)\psi \tag{17}$$

where M^a are group generators, and where Λ^a is an angle that is a function of x^μ through special relativity [6]. In general, M^a are $n \times n$ matrices or tensors. In O(3) electrodynamics, the indices a can be (1), (2), and (3) of the complex basis (7), or Cartesian indices as in the basis (6). From Eq. (17), it is found that

$$\partial_\mu\psi' = S\left(\partial_\mu\psi\right) + \left(\partial_\mu S\right)\psi \tag{18}$$

that is, that $\partial_\mu\psi$ does not transform covariantly. It is well known that this problem is addressed through the introduction of the covariant derivative:

$$D_\mu\psi \equiv (\partial_\mu - igM^aA^a_\mu)\psi \tag{19}$$

where g is in general a proportionality constant giving the right units, and where A^a_μ is the vector potential, sometimes referred to as the "connection." In U(1) electrodynamics, A^a_μ reduces to the familiar 4-potential A^a_μ of the Maxwell–Heaviside theory, a 4-vector. This means that in U(1) electrodynamics, the internal gauge space is a scalar space in which $M = -1$ and in which the covariant derivative reduces to

$$D_\mu(U(1)) = \partial_\mu + igA_\mu \tag{20}$$

which, in momentum space, is the familiar minimal prescription. In O(3) electrodynamics however, A^a_μ is a 12-vector, and can be expressed as

$$\boldsymbol{A}_\mu = A^{(1)}_\mu\boldsymbol{e}^{(1)} + A^{(2)}_\mu\boldsymbol{e}^{(2)} + A^{(3)}_\mu\boldsymbol{e}^{(3)} \tag{21}$$

in the basis ((1),(2),(3)). Similarly, the familiar field tensor $F_{\mu\nu}$ of U(1) electrodynamics becomes

$$\boldsymbol{G}_{\mu\nu} = G^{(1)}_{\mu\nu}\boldsymbol{e}^{(1)} + G^{(2)}_{\mu\nu}\boldsymbol{e}^{(2)} + G^{(3)}_{\mu\nu}\boldsymbol{e}^{(3)} \tag{22}$$

in O(3) electrodynamics. Since ((1),(2),(3)) is a physical space, each of the tensors $G^{(i)}_{\mu\nu}; i = 1, 2, 3$ is well defined in Minkowski spacetime [11–20].

General gauge field theory emerges when the covariant derivative is applied to ψ [6]:

$$D'_\mu\psi = SD_\mu\psi \tag{23}$$

It is useful to go through this derivation in detail because it produces the inhomogeneous term responsible for the Aharonov–Bohm effect in O(3) electrodynamics. The effect of the rotation may be written as

$$(\partial_\mu - igA'_\mu)\psi' = S(\partial_\mu - igA_\mu)\psi \tag{24}$$

which means that

$$\begin{aligned}
\partial_\mu\psi' &= S(\partial_\mu\psi) + (\partial_\mu S)\psi \\
(\partial_\mu S)\psi - igA'_\mu S\psi &= -igSA_\mu\psi \\
igA'_\mu S &= igSA_\mu + \partial_\mu S \\
A'_\mu SS^{-1} &= SA_\mu S^{-1} - \frac{i}{g}(\partial_\mu S)S^{-1} \\
A'_\mu &= SA_\mu S^{-1} - \frac{i}{g}(\partial_\mu S)S^{-1}
\end{aligned} \tag{25}$$

The end result is that the inhomogeneous term $-(i/g)(\partial_\mu S)S^{-1}$ appears in the vacuum. This term originates in the topology of the vacuum, and it is different for U(1) electrodynamics and O(3) electrodynamics. In U(1) electrodynamics, the gauge transformation (25) reduces to

$$A'_\mu \rightarrow A_\mu + \frac{1}{g}\partial_\mu\Lambda \tag{26}$$

which is the covariant form of Eq. (15). In O(3) electrodynamics however, the inhomogeneous term and the vector potential are both physical quantities, as originally envisaged by Maxwell and Faraday. The 12-vector A_μ is the equivalent of Faraday's electrotonic state and of Maxwell's physical vector potential [3,4]. It follows that the effect (25) on the vector potential in O(3) electrodynamics is produced by a physical rotation, and later in this review, it is shown that this physical rotation is the rotation of the platform in the Sagnac effect [20]. More generally, a rotation in the internal gauge space of O(3) electrodynamics produces a phase difference that is also physical and measurable [3,4]. O(3) electrodynamics is therefore able to describe the Sagnac effect precisely, whereas U(1) electrodynamics has no explanation for the Sagnac effect because of its gauge invariance. Quantities such as the 12-vector potential of O(3) electrodynamics are gauge-covariant, not gauge-invariant, because the inhomogeneous term in O(3) electrodynamics is a physical term, not a random mathematical construct as in U(1) electrodynamics.

In general gauge field theory [6], the field tensor is proportional to the commutator of covariant derivatives. This is the result of a round trip or closed

oop with covariant derivatives in Minkowski spacetime, and in condensed notation, the result can be written as

$$G_{\mu\nu} = \frac{i}{g}[D_\mu, D_\nu] \tag{27}$$

$$G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \tag{27a}$$

n U(1) electrodynamics, we recover the familiar 4-curl of Maxwell–Heaviside heory because the commutator $[A_\mu, A_\nu]$ is zero. In O(3) electrodynamics, Eq. (22) applies and each component $G^{(1)}_{\mu\nu}, G^{(2)}_{\mu\nu}$, and $G^{(3)}_{\mu\nu}$ is defined as

$$G^{(i)}_{\mu\nu} = \partial_\mu A^{(i)}_\nu - \partial_\nu A^{(i)}_\mu - ig\varepsilon_{(i)(j)(k)} A^{(j)}_\mu A^{(k)}_\nu, \qquad (i),(j),(k) = (1),(2),(3) \tag{28}$$

n the complex circular basis ((1),(2),(3)), [11–20]. Whereas $F_{\mu\nu}$ of U(1) electrodynamics is gauge- and Lorentz-invariant, $G_{\mu\nu}$ of O(3) electrodynamics ransforms covariantly under rotation in the internal space ((1),(2),(3)), a epresentation of the physical space of three dimensions:

$$G'_{\mu\nu} = SG_{\mu\nu}S^{-1} \tag{29}$$

The homogeneous field equation of O(3) electrodynamics is inferred from he Jacobi identity of covariant derivatives

$$\sum_{\text{cyclic}} [D_\rho, [D_\mu, D_\nu]] \equiv 0 \tag{30}$$

and can be written as the identity [11–20]

$$D_\mu \tilde{\boldsymbol{G}}^{\mu\nu} \equiv \boldsymbol{0} \tag{31}$$

The inhomogeneous field equation is not an identity, but an equation of the Yang–Mills type [6]

$$D_\mu \boldsymbol{H}^{\mu\nu} = \boldsymbol{J}^\nu \tag{32}$$

where $\boldsymbol{H}^{\mu\nu}$ is a generalization of $\boldsymbol{G}^{\mu\nu}$ to include polarization and magnetization, and where $\boldsymbol{J}^\nu$ is the charge current 12-vector, defined as

$$J^{\nu(i)} \equiv \left(\rho^{(i)}, \frac{\boldsymbol{J}^{(i)}}{c}\right) \qquad i = 1,2,3 \tag{33}$$

where c is the speed of light in vacuo for all practical purposes in the laboratory. Equations (31) and (32) are developed fully in Section (IV) and are compared with the Lehnert, Barrett, and Harmuth equations cited in Section I. These equations extend the symmetry of the electromagnetic sector of unified field

theory with many consequences, some of which are discussed in Section (XI) for electroweak theory, and in Part 3 of this three-volume series for grand unified theory.

The development just given illustrates the fact that the topology of the vacuum determines the nature of the gauge transformation, field tensor, and field equations, as inferred in Section (I). The covariant derivative plays a central role in each case; for example, the homogeneous field equation of O(3) electrodynamics is a Jacobi identity made up of covariant derivatives in an internal O(3) symmetry gauge group. The equivalent of the Jacobi identity in general relativity is the Bianchi identity.

Finally, in this section, we develop the concept of electromagnetic phase from U(1) to O(3). This is a nontrivial development [4] that has foundational consequences for interferometry and physical optics for example. In U(1) electrodynamics, the electromagnetic phase is defined up to an arbitrary factor [4] because of gauge invariance. The U(1) phase is therefore

$$\gamma \equiv \omega t - \boldsymbol{\kappa}\cdot\boldsymbol{r} + \alpha \tag{34}$$

where ω is the angular frequency at instant t; $\boldsymbol{\kappa}$ is the wave-vector at coordinate $\boldsymbol{r}$, and α is random. In other words, the U(1) electromagnetic phase factor $\exp(i(\omega t - \boldsymbol{\kappa}\cdot\boldsymbol{r}))$ can be multiplied by the factor $e^{i\alpha}$ because gauge transformation in U(1) is a random rotation in the (scalar) internal gauge space. The random rotation is represented by the operator $e^{i\alpha}$ where α is random. This operation leads to Eq. (26), where the gradient function is random as usual in U(1) electrodynamics. Therefore the U(1) electromagnetic phase is unphysical. This is true despite the fact that the theory of U(1) electrodynamics is the received view, adhered to rigidly. Therefore [4], the field tensor in U(1) electrodynamics, is underdetermined because the phase is arbitrary; and the potential 4-vector of U(1) electrodynamics is overdetermined because it is also arbitrary—an infinite number of A^μ corresponds, in the received view, to one physical condition. Dirac attempted to remedy these flaws by introducing a phase *factor*

$$\Phi(C) = \exp\left(i\frac{e}{\hbar}\oint_C A_\mu dx^\mu\right) \tag{35}$$

where e is electric charge, and $\hbar$ is the Dirac constant. The Dirac phase factor completely defines [4] the system on the U(1) level. The phase factor in O(3) electrodynamics is obtained by generalizing this concept, as first accomplished by Wu and Yang [33]. The phase factor in O(3) electrodynamics can be written as

$$\Phi^*(C) = P\exp\left(ig\oint_C A_\mu dx^\mu\right) \tag{36}$$

$$\Phi^*(C) = P\exp\left(i\oint_C \kappa_\mu dx^\mu\right) = P\exp\left(ig\int\int B^{(3)} dAr\right) \tag{36a}$$

where a magnetic flux of topological origin appears on the right hand side, an area integral over the $\boldsymbol{B}^{(3)}$ (Evans–Vigier) field [11–20]. Here, $\Phi^*(C)$ specifies parallel transport over any loop C in rotation, g is the same factor that appears in the definition of the covariant derivative, [Eq. (20)], and P specifies path dependence in the integral [4]. On the left-hand side appears the line integral corresponding to the dynamical phase factor, which is equal through a non-Abelian Stokes theorem to the topological phase defined by the surface integral over $\boldsymbol{B}^{(3)}$. This result is a clear illustration of the topological origin of $\boldsymbol{B}^{(3)}$, and the phase factor is not a random quantity as in U(1) electrodynamics, but a gauge-covariant quantity. It is the holonomy of the connection A_μ in O(3) electrodynamics and plays a central role in interferometry, including the Aharonov–Bohm effect. Consideration of interferometry leads to the conclusion that O(3) electrodynamics provides a self-consistent description of several situations where U(1) electrodynamics either fails (e.g., the Sagnac effect) or is self-inconsistent (e.g., Michelson interferometry).

III. REFUTATION OF U(1) ELECTRODYNAMICS

From the foregoing, U(1) electrodynamics was never a complete theory, although it is rigidly adhered to in the received view. It has been argued already that the Maxwell–Heaviside theory is a U(1) Yang–Mills gauge theory that discards the basic commutator $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$. However, this commutator appears in the fundamental definition of circular polarity in the Maxwell–Heaviside theory through the third Stokes parameter

$$S_3 = |-i\omega^2 A^{(1)} \times A^{(2)}| = \omega^2 A^{(0)2} \tag{37}$$

so there is an internal inconsistency. In O(3) electrodynamics, on the other hand, the fundamental definition of the $\boldsymbol{B}^{(3)}$ field ensures that circular polarity is consistently defined

$$\boldsymbol{B}^{(3)*} = -ig\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} \tag{38}$$

so that circular polarity in O(3) electrodynamics is due to the $\boldsymbol{B}^{(3)}$ field, which is therefore a foundational physical observable. This argument is a simple and straightforward refutation of U(1) electrodynamics, specifically, of Maxwell–Heaviside theory considered as a U(1) symmetry gauge field theory. The third Stokes parameter is a fundamental signature of circular polarization and was first recognized as such by Stokes in 1852 before the development of Maxwell's original equations in the 1860s [3]. Circular polarization was discovered empirically by Arago in 1811.

There is in effect no circular polarization in U(1) electrodynamics if we choose to define circular polarization in terms of the third Stokes parameter.

This result is inconsistent with the fact that the differential equation developed by Heaviside from Maxwell's original equations describe circular polarization. The root of the inconsistency is that U(1) gauge field theory is made to correspond with Maxwell–Heaviside theory by discarding the commutator $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$. The neglect of the latter results in a reduction to absurdity, because if S_3 vanishes, so does the zero order Stokes parameter:

$$S_0 = \pm S_3 \tag{39}$$

and S_0 describes the intensity of radiation. This result is another self inconsistency of U(1) electrodynamics.

In O(3) electrodynamics, on the other hand, Eq. (38), defining the $\boldsymbol{B}^{(3)}$ field, is consistent with the O(3) field Eq. (31) and (32) because Eq. (38) is part of the definition of the field tensor in O(3) electrodynamics [11–20].

A second simple refutation of U(1) electrodynamics is perfect normal reflection. The explanation of this foundational effect in Maxwell–Heaviside electrodynamics relies on the phase in U(1) electrodynamics, which, as argued already, is a random quantity. If we choose α in Eq. (34) to be zero for simplicity and without loss of generality, then the received view of perfect normal reflection (Fig. 1) is as follows:

$$\exp\left(i(\boldsymbol{\kappa}\cdot\boldsymbol{r} - \omega t)\right) \xrightarrow{R} \exp\left(i(-\boldsymbol{\kappa}\cdot\boldsymbol{r} - \omega t)\right) \tag{40}$$

However, normal reflection, in, for example, the Z axis, is equivalent to the parity inversion operation P. The effect of this operation on the U(1) phase factor is as follows:

$$\exp\left(i(\boldsymbol{\kappa}\cdot\boldsymbol{r} - \omega t)\right) \xrightarrow{P} \exp\left(i(\boldsymbol{\kappa}\cdot\boldsymbol{r} - \omega t)\right) \tag{41}$$

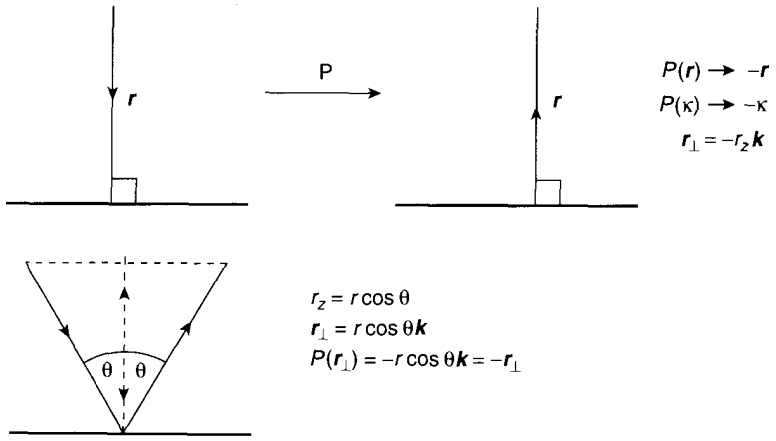


Figure 1. Equivalence of reflection and parity inversion.

Thus the received view of normal reflection (1) in U(1) electrodynamics violates parity. This violation is not allowed in classical physics. For off-normal reflection (Fig. 1), projections on to the normal result in the same paradox using the empirical fact that the angle of reflection is equal to the angle of incidence. In the received view, Eq. (40) is held to rigidly, but is nevertheless in violation of parity. This is true if and only if Snell's law is true. In conclusion, $P(\omega t - \boldsymbol{\kappa}\cdot\boldsymbol{r} = (\omega t - \boldsymbol{\kappa}\cdot\boldsymbol{r})$, which is Snell's law in Maxwell–Heaviside theory.

It is highly significant that this paradox disappears in O(3) electrodynamics through the use of the physical phase factor:

$$\Phi = \exp\left(i\oint \boldsymbol{\kappa}\cdot d\boldsymbol{Z}\right) = \exp\left(ig\int \boldsymbol{B}^{(3)}\cdot d\boldsymbol{S}\right) \tag{42}$$

On the left-hand side appears a line integral, and on the right-hand side, there is an area integral over $\boldsymbol{B}^{(3)}$. If a beam of light originates at an origin O and is normally reflected from a perfectly reflecting mirror at point Z, the line integral is as follows:

$$\oint \boldsymbol{\kappa}\cdot d\boldsymbol{Z} = \int_0^Z \kappa dZ - \int_Z^0 \kappa dZ = 2\kappa Z \tag{43}$$

Note that this gives, fortuitously, the same change, $2\boldsymbol{\kappa}Z$, as in the U(1) description of normal reflection, which therefore fortuitously describes the empirical result.

The area integral on the right-hand side of Eq. (42) is a topological phase [4], because the origin of $\boldsymbol{B}^{(3)}$ is topological as argued already, that is, $\boldsymbol{B}^{(3)}$ springs from the vacuum configuration. Using the relation [11–20]

$$g = \frac{\kappa^2}{B^{(0)}} \tag{44}$$

the right-hand-side exponent becomes $\boldsymbol{\kappa}^2 S$, where S is an area

$$S = 2\frac{\kappa}{Z} \tag{45}$$

If the distance OZ is n wavelengths, λ, then the area becomes

$$S = \frac{n\lambda^2}{\pi} \tag{46}$$

The outcome of these two very simple examples is that all electrodynamics (classical and quantum) must be upgraded to a gauge theory of higher symmetry, such as O(3). Equation (42) is self-consistent, because under P, both sides are negative. The left-hand side is negative because the line integral changes sign under P, and the right-hand side is negative because the integral is negative under P (product of an axial vector $\boldsymbol{B}$ and a polar vector $\boldsymbol{S}$).

Michelson interferometry is dependent on normal reflection from two mirrors at right angles, and so the same foundational argument as just given can be used to show that U(1) electrodynamics does not describe Michelson interferometry self-consistently. Without loss of generality, we can write Eq. (38) as

$$\pi R\kappa A^{(0)}\boldsymbol{k}\cdot R\boldsymbol{k} = \boldsymbol{B}^{(3)}\cdot A\boldsymbol{r}\boldsymbol{k} \tag{47}$$

which can be integrated straightforwardly to give the non-Abelian Stokes theorem [11–20]

$$2\pi\kappa A^{(0)}\oint \boldsymbol{R}\cdot d\boldsymbol{R} = \int\int \boldsymbol{B}^{(3)}\cdot d\boldsymbol{A}\boldsymbol{r} \tag{48}$$

where R is given by

$$R = \frac{1}{\kappa} = \frac{\lambda}{2\pi} \tag{49}$$

and where λ is the wavelength. Multiplying both sides by $g = \boldsymbol{\kappa}/A^{(0)}$ defines the required non-Abelian phase factor in terms of a non-Abelian Stokes theorem

$$\Phi = \exp(2\pi i\oint \boldsymbol{\kappa}\cdot d\boldsymbol{R}) = \exp\left(i\frac{\kappa}{A^{(0)}}\int\int \boldsymbol{B}^{(3)}\cdot d\boldsymbol{A}\boldsymbol{r}\right) \tag{50}$$

which is closely related to Eq. (42). The line integrals must be evaluated over a closed curve [11–20] and have the foundational property

$$\oint_{AO} \boldsymbol{\kappa}\cdot d\boldsymbol{R} = -\oint_{0A} \boldsymbol{\kappa}\cdot d\boldsymbol{R} \tag{51}$$

which is the root cause [34] of Michelson interferometry, and interferometry in general. In U(1) electrodynamics, the change in phase of a light beam originating at the beamsplitter [35] and arriving back at the beamsplitter after normal reflection from either mirror is zero because of the property (41). This is contrary to the empirical observation [35] of the Michelson interferogram, the basis of Fourier transform infrared spectroscopy. In the usual U(1) theory, therefore, the path-dependent part of the electromagnetic phase is the familiar $\boldsymbol{\kappa}\cdot\boldsymbol{r}$, and the complete electromagnetic phase is $\omega t - \boldsymbol{\kappa}\cdot\boldsymbol{r} + \alpha$, where α is random and can be set to zero for simplicity of argument. The phase $\omega t - \boldsymbol{\kappa}\cdot\boldsymbol{r}$ is invariant under both

P and T because it is a dimensionless number, and we shall show that the complete failure of U(1) electrodynamics to describe the Sagnac effect is due to the T invariance of U(1) phase. In Michelson interferometry, as described by O(3) electrodynamics, there is a change in the measurable phase factor after reflection because of the property of line integrals. The phase factors arriving back at the beamsplitter from either mirror are different, and an interferogram appears as observed [35] empirically by changing the length of one arm of the interferometer. The Fourier transformation of this interferogram gives a spectrum.

The inverse Faraday effect depends on the third Stokes parameter empirically in the received view [36], and is the archetypical magneto-optical effect in conventional Maxwell–Heaviside theory. This type of phenomenology directly contradicts U(1) gauge theory in the same way as argued already for the third Stokes parameter. In O(3) electrodynamics, the paradox is circumvented by using the field equations (31) and (32). A self-consistent description [11–20] of the inverse Faraday effect is achieved by expanding Eq. (32):

$$\partial_\mu \boldsymbol{H}^{\mu\nu(1)*} = \boldsymbol{J}^{\nu(1)*} + ig\boldsymbol{A}_\mu^{(2)} \times \boldsymbol{H}^{\mu\nu(3)} \tag{52}$$

$$\partial_\mu \boldsymbol{H}^{\mu\nu(2)*} = \boldsymbol{J}^{\nu(2)*} + ig\boldsymbol{A}_\mu^{(3)} \times \boldsymbol{H}^{\mu\nu(1)} \tag{53}$$

$$\partial_\mu \boldsymbol{H}^{\mu\nu(3)*} = \boldsymbol{J}^{\nu(3)*} + ig\boldsymbol{A}_\mu^{(1)} \times \boldsymbol{H}^{\mu\nu(2)} \tag{54}$$

Using the constitutive relation

$$\boldsymbol{H}^{(3)*} = \frac{1}{\mu}\boldsymbol{B}^{(3)*} \tag{55}$$

gives the magnetic field strength induced in the inverse Faraday effect from first principles of O(3) gauge field theory as

$$\boldsymbol{H}^{(3)*} = -i\frac{g'}{\mu_0}\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} \tag{56}$$

where

$$g' = \frac{\mu_0}{\mu}g \tag{57}$$

Here, μ is the magnetic permeability of the material in which the inverse Faraday effect is observed. We can write Eq. (52) as

$$\partial_\mu \boldsymbol{H}^{\mu\nu(1)*} = \boldsymbol{J}^{\nu(1)*} + \Delta\boldsymbol{J}^{\nu(1)*} \tag{58}$$

so that the transverse current detected [37] in the inverse Faraday effect is given by

$$\Delta\boldsymbol{J}^{\nu(1)*} = ig\varepsilon\boldsymbol{A}_\mu^{(2)} \times \boldsymbol{G}^{\mu\nu(3)} \tag{59}$$

and causes a signal in an induction coil due to the vacuum $\boldsymbol{B}^{(3)}$ field appearing in the O(3) field tensor $\boldsymbol{G}^{\mu\nu(3)}$.

The explanation of the inverse Faraday effect in U(1) electrodynamics relies on the clearly self-inconsistent introduction of $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ phenomenologically: "self-inconsistent" because U(1) gauge field theory sets $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ to zero identically. As argued already, the conjugate product $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ is proportional to the third Stokes parameter in the vacuum and so is a fundamental property of circularly polarized light. As such, it must be considered as a fundamental object in gauge field theory applied to electrodynamics. In U(1) gauge field theory, this is not possible, but it is possible self-consistently in O(3) gauge field theory.

In Maxwell–Heaviside electrodynamics, the field energy, Poynting vector, and Maxwell stress tensor are incorporated in the stress energy momentum tensor [38]. In order to obtain a non-*null* energy and field momentum (Poynting vector), the method of averaging is used. The conventionally defined Poynting vector, for example, becomes proportional to $\boldsymbol{E} \times \boldsymbol{B}^* = \boldsymbol{E}^{(1)} \times \boldsymbol{B}^{(2)}$. This method is inconsistent with electrodynamics considered as a U(1) gauge field theory, but consistent with O(3) electrodynamics.

Recall that in general gauge field theory, for any gauge group, the field tensor is defined through the commutator of covariant derivatives. In condensed notation [6]

$$G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \tag{60}$$

where the commutator is nonzero in general. The connection or generalized potential A_μ is defined in general through the gauge group symmetry. The field tensor $G_{\mu\nu}$ is covariant for all gauge groups, and is always compatible with special relativity for all gauge group symmetries. In this general theory therefore, the homogeneous and inhomogeneous Maxwell equations in the vacuum are the U(1) gauge field equations

$$D^\nu \tilde{G}_{\mu\nu} \equiv 0 \tag{61}$$

$$D^\nu G_{\mu\nu} = 0 \tag{62}$$

where D^ν denotes the covariant derivative pertinent to U(1) and where $\tilde{G}^{\mu\nu}$ is the dual of $G_{\mu\nu}$ as usual. In the U(1) gauge theory, the commutator in Eq. (60) vanishes because the U(1) group has only one structure constant and the internal

symmetry of the gauge theory is a scalar symmetry. The covariant derivative in U(1) is

$$D^\nu = \partial^\nu + igA^\nu \tag{63}$$

Therefore Eqs. (61) and (62) reduce to

$$(\partial^\nu + igA^\nu)\tilde{F}_{\mu\nu} \equiv 0 \tag{64}$$
$$(\partial^\nu + igA^\nu)F_{\mu\nu} \equiv 0 \tag{65}$$

which become the free-space homogeneous and inhomogeneous Maxwell–Heaviside equations if and only if

$$A^\nu \tilde{F}_{\mu\nu} = 0 \tag{66}$$
$$A^\nu F_{\mu\nu} \equiv 0 \tag{67}$$

or in vector notation

$$\begin{aligned} \boldsymbol{A}\cdot\boldsymbol{B} &= 0 \qquad \boldsymbol{A}\times\boldsymbol{E} = \boldsymbol{0} \\ \boldsymbol{A}\cdot\boldsymbol{E} &= 0 \qquad \boldsymbol{A}\times\boldsymbol{B} = \boldsymbol{0} \end{aligned} \tag{68}$$

For plane waves, and using the usual U(1) relation

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \tag{69}$$

the vector potential is proportional to $\boldsymbol{B}$ and so

$$\boldsymbol{B}\times\boldsymbol{E} = \boldsymbol{0} \tag{70}$$

If we attempt to define the free-space field energy and momentum in terms of the products $\boldsymbol{B}\cdot\boldsymbol{B}$ and $\boldsymbol{B}\times\boldsymbol{E}$, the results are zero in U(1) gauge field theory. In order to obtain the conventional field energy and Poynting vector of the free electromagnetic field, products such as $\boldsymbol{B}^{(1)}\times\boldsymbol{B}^{(2)}$ and $\boldsymbol{B}^{(1)}\times\boldsymbol{E}^{(2)}$ have to be used. This procedure, although common place, and referred to in the literature as "time averaging" [38], introduces phenomenology extraneous to U(1), because it introduces the complex internal gauge space ((1),(2),(3)). These inconsistencies in U(1) gauge field theory applied to electrodynamics are therefore summarized as follows: (1) if the U(1) covariant derivative is used, the field energy, momentum, and third Stokes parameter vanish; and (2) if the phenomenological "time averaging" procedure is used, the resultant Poynting vector is proportional to $\boldsymbol{B}^{(1)}\times\boldsymbol{E}^{(2)}$, and is perpendicular to the plane of definition of U(1), whose group space is a circle. This result is another internal inconsistency, because the group space of a gauge theory is a circle, there can be no physical quantity in free space perpendicular to that plane. It is necessary but not sufficient, in this view, that the Lagrangian in U(1) field theory be invariant [6] under U(1) gauge transformation.

In O(3) electrodynamics, the stress energy momentum tensor is defined [11–20] as

$$T_\mu^\nu = \varepsilon_0\left(\boldsymbol{G}^{\mu\rho}\cdot\boldsymbol{G}_{\rho\mu} - \frac{1}{4}\boldsymbol{G}^{\rho\nu}\cdot\boldsymbol{G}_{\rho\nu}\right) \tag{71}$$

giving the field energy self-consistently as

$$U = \varepsilon_0(E^{1(1)}E_1^{(2)} + E^{2(1)}E_2^{(2)} + E^{1(2)}E_1^{(1)} + E^{2(2)}E_2^{(1)} + E^{3(3)}E_3^{(3)*}) \tag{72}$$

The Poynting vector is self-consistently defined as

$$T_1^0 = \varepsilon_0(\boldsymbol{G}^{02}\cdot\boldsymbol{G}_{21} + \boldsymbol{G}^{03}\cdot\boldsymbol{G}_{31}) \tag{73}$$
$$T_2^0 = \varepsilon_0(\boldsymbol{G}^{01}\cdot\boldsymbol{G}_{12} + \boldsymbol{G}^{03}\cdot\boldsymbol{G}_{32}) \tag{74}$$
$$T_3^0 = \varepsilon_0(\boldsymbol{G}^{01}\cdot\boldsymbol{G}_{13} + \boldsymbol{G}^{02}\cdot\boldsymbol{G}_{23}) \tag{75}$$

and is finite. The $\boldsymbol{B}^{(3)}$ component is defined through Eq. (38), giving, self-consistently, the result (39).

The root cause of these further problems with electrodynamics considered as a U(1) gauge field theory is that parallel transport [6] must be used when an internal gauge space is present. The internal gauge space of U(1) is a scalar, and parallel transport results in a covariant derivative whose momentum representation is the minimal prescription [6]. This covariant derivative, however, leads self-inconsistently to a null energy density and Poynting vector as just argued. Therefore, in U(1), the Maxwell–Heaviside equations are obtained if and only if the field energy and Poynting vector are identically zero. A null Poynting vector means null energy and a null third Stokes parameter. The root cause of this is the neglect of $\boldsymbol{A}^{(1)}\times\boldsymbol{A}^{(2)}$, and we have come full circle. The only way out is to adopt a gauge field theory of higher symmetry than U(1).

A related problem is that the linear momentum of radiation in U(1) is defined by

$$\langle p\rangle = \varepsilon_0 c\int \boldsymbol{E}\times\boldsymbol{B}\,dV \tag{76}$$

which is again zero. The linear momentum of a photon, however, is nonzero in quantum theory and is $\hbar\kappa$, leading to the Compton effect and Compton

scattering. It is well known that there is no classical equivalent of the Compton effect [39], so the correspondence principle is lost in the received view based on U(1) gauge field theory. In O(3) electrodynamics, however [11–20], there exists, in general, the longitudinally directed potential $A^{(3)}$ as part of the definition of the field tensor. The classical quantum equivalence in the Compton effect is then given simply as

$$\boldsymbol{p}^{(3)} = e\boldsymbol{A}^{(3)} = \hbar\boldsymbol{\kappa} \tag{77}$$

where e is regarded as the coupling constant in the definition of the constant $g = e/\hbar$, which appears in free space in *both* U(1) and O(3) electrodynamics. This is another characteristic of gauge field theory applied to electrodynamics, that charge e can act as a coupling constant in the covariant derivative. This is true for all internal gauge symmetries, so e need not be defined solely by the charge on the electron. These concepts are discussed further in Ref. 6. Therefore O(3) electrodynamics saves the quantum classical correspondence principle in Planck–Einstein quantization. Equation (77) has the following manifestly covariant form:

$$\begin{aligned} p^{\mu(3)} &= eA^{\mu(3)} = \hbar\kappa^{\mu} \\ A^{\mu(3)} &= \frac{1}{c}(A^{(0)}, cA^{(3)}) \end{aligned} \tag{78}$$

These concepts of O(3) electrodynamics also completely resolve the problem that, in Maxwell–Heaviside electrodynamics, the energy momentum of radiation is defined through an integral over the conventional tensor $T^{\mu\nu}$, and for this reason is not manifestly covariant. To make it so requires the use of special hypersurfaces as attempted, for example, by Fermi and Rohrlich [40]. The O(3) energy momentum (78), in contrast, is generally covariant in O(3) electrodynamics [11–20].

The Maxwell–Heaviside theory seen as a U(1) symmetry gauge field theory has no explanation for the photoelectric effect, which is the emission of electrons from metals on ultraviolet irradiation [39]. Above a threshold frequency, the emission is instantaneous and independent of radiation intensity. Below the threshold, there is no emission, however intense the radiation. In U(1), electrodynamics energy is proportional to intensity and there is, consequently, no possible explanation for the photoelectric effect, which is conventionally regarded as an archetypical quantum effect. In classical O(3) electrodynamics, the effect is simply

$$En = ecA^{(3)} = \text{constant} \times \text{frequency} \tag{79}$$

and in Planck–Einstein quantization, the constant of proportionality is $\hbar$, which turns out to be a universal constant of physics. The concomitant momentum

relation, Eq. (77), is shown empirically by the Compton effect as argued already. Equation (77) means that above a given threshold frequency, there is enough energy in the photon to cause electron emission in the photoelectric effect. All the energy and momentum of the photon are transferred to the electron in a collision above a certain threshold frequency because at this point, the potential energy responsible for keeping the electron in place is exceeded. If we attempt to apply this logic to $\langle \boldsymbol{p} \rangle$ in Eq. (76), there is no threshold frequency possible on the classical level because $\langle \boldsymbol{p} \rangle$ cannot be proportional to frequency, only to beam intensity. The momentum $\boldsymbol{p}^{(3)} = e\boldsymbol{A}^{(3)}$ of classical O(3) electrodynamics is not proportional to intensity; it is proportional to frequency through the gauge equation (77), which also leads to the B cyclic theorem [11–20], the fundamental Lorentz invariant angular momentum relation of O(3) electrodynamics.

In the O(3) Compton effect, the observable change of wavelength is

$$\Delta\lambda = 2\left(\frac{eA^{(3)}}{mc}\right)\lambda_0 \sin^2\frac{\theta}{2} \tag{80}$$

where λ is the wavelength of the incident beam, m is the electron mass, and θ is the scattering angle. If Eq. (77) is applied to this result, we recover the usual quantum description of the Compton effect.

The concept of $\boldsymbol{A}^{(3)}$ can also be used to suggest a way out of the Dirac paradox [41] of U(1) electrodynamics, in which Dirac maintains that so long as we are dealing with transverse waves, we cannot bring in the Coulomb interaction between charged particles. In O(3) electrodynamics, there is a force given by

$$F^{\mu(3)} = e\frac{\partial A^{\mu(3)}}{\partial\tau} \tag{81}$$

whenever the beam interacts with an electron. This interaction results in a longitudinal force with a change of wavelength as just described for the Compton effect. This is not a Coulomb force since $\boldsymbol{E}^{(3)}$ is zero in vacuo [11–20].

Similarly, $\boldsymbol{A}^{(3)}$ can be used to suggest a way out of the de Broglie paradox [42], which points out that momentum and energy transform differently under Lorentz transformation from frequency. This paradox led de Broglie to postulate the existence of empty waves, which, however, have never been observed empirically. It can therefore be suggested that the Lorentz frequency transform must always be applied to

$$e\boldsymbol{A}^{(3)} = \frac{\hbar\omega}{c}\boldsymbol{e}^{(3)} \tag{82}$$

because this momentum is proportional to frequency empirically. If this momentum is interpreted as that of a particle traveling at the speed of light, the momentum becomes indeterminate (massless particle) or infinite (massive particle) unless it is always interpreted as being a constant ($\hbar$) multiplied by ω/c, which always exists empirically as the speed of light. The energy must evidently be interpreted in the same way, namely, as a constant multiplied by frequency. The Lorentz transform applied to frequency produces the aberration of light as usual [39] in special relativity. In this interpretation, there is no de Broglie paradox and no need to postulate the existence of empty waves [42].

The Sagnac effect cannot be described by U(1) electrodynamics [4,43] because of the invariance of the U(1) phase factor under motion reversal symmetry *(T)*:

$$\Phi = \exp\left(i(\omega t - \boldsymbol{\kappa}\cdot\boldsymbol{r})\right) \xrightarrow{T} \exp\left(i(\omega t - \boldsymbol{\kappa}\cdot\boldsymbol{r})\right) \tag{83}$$

The T operator generates the counterclockwise (A) loop from the clockwise (C) loop in the Sagnac effect, with the result that there is no difference in phase factor for journeys around the A and C loops, and no interferogram. This is contrary to observation when the Sagnac platform is at rest [43]. When the platform of the Sagnac interferometer [3] is rotated, there is the well-known Sagnac phase shift, which was first detected in 1913. This defies description by U(1) electrodynamics because the Maxwell–Heaviside field equations in the vacuum are invariant to rotation, which is part of the most general type of Lorentz transform [6]. The Maxwell–Heaviside equations in vacuo are also gauge- and metric-invariant, and are not capable of describing the Sagnac effect at all. The O(3) electrodynamics, in contrast, are completely successful in describing the interferogram with platform at rest and with a rotating platform. The details of this important advantage of O(3) electrodynamics are discussed in Section (VI), where a kinematic explanation of the Sagnac effect is also given using O(3) gauge theory. More details of magneto-optical effects are given in Section (VII).

The Aharonov–Bohm effect is self-inconsistent in U(1) electrodynamics because [44] the effect depends on the interaction of a vector potential $\boldsymbol{A}$ with an electron, but the magnetic field defined by $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ is zero at the point of interaction [44]. This argument can always be used in U(1) electrodynamics to counter the view that the classical potential $\boldsymbol{A}$ is physical, and adherents of the received view can always assert in U(1) electrodynamics that the potential must be unphysical by gauge freedom. If, however, the Aharonov–Bohm effect is seen as an effect of O(3) electrodynamics, or of SU(2) electrodynamics [44], it is easily demonstrated that the effect is due to the physical inhomogeneous term appearing in Eq. (25). This argument is developed further in Section VI.

Barrett has argued convincingly that there are several effects in classical electrodynamics [3,4] where the potential must be physical, and Ref. 3 lists

empirically observed effects where this is the case. The arguments in this section point to the fact that U(1) electrodynamics, defined as U(1) gauge field theory applied to electrodynamics, is self-inconsistent in the vacuum, as well as in field–matter interaction. In the next section, the field equations of electrodynamics seen as an O(3) gauge field theory applied to electrodynamics are given in full, revealing the presence in free space of conserved topological charges and currents that do not appear in U(1) electrodynamics and that in general are not zero.

IV. FIELD EQUATIONS OF O(3) ELECTRODYNAMICS IN THE COMPLEX CIRCULAR BASIS

In their most condensed form, the field equations are Eqs. (31) and (32), respectively, and, in general, must be solved without approximation on a computer with constitutive relations, as usual in classical electrodynamics. The familiar field tensors $\tilde{G}^{\mu\nu}$ and $H^{\mu\nu}$of the homogeneous and inhomogeneous Heaviside–Maxwell equations [U(1) Yang–Mills gauge field theory] become vectors in the O(3) symmetry internal gauge space of Eqs. (31) and (32), which are equations of O(3) symmetry Yang–Mills gauge field theory. Therefore an object such as $\tilde{\boldsymbol{G}}^{\mu\nu}$ is a vector in the internal gauge space and a tensor in Minkowski spacetime, and an object such as $\boldsymbol{J}^{\mu}$ is a 3-vector in the internal O(3) space and a 4-vector in Minkowski spacetime. The ordinary derivatives of the Maxwell–Heaviside equations are replaced in Eqs. (31) and (32) by covariant derivatives in an internal gauge space, with three rotation generators [11–20]. Eqs. (31) and (32) are gauge-covariant, and not gauge-invariant, under all conditions, including the vacuum. As argued already, the homogeneous Eq. (31) is a Jacobi identity of the O(3) group, and the tilde denotes dual tensor as usual. The homogeneous field equation, Eq. (31), originates in the cyclic identity between O(3) covariant derivatives, Eq. (30), and can be developed by writing out the covariant derivative in terms of the coupling constant g, which has the classical units $\boldsymbol{\kappa}/A^{(0)}$ [11–20]. The coupling constant, as usual in gauge theory [6], couples the dynamical field to its source, so in Eqs. (31) and (32), the dynamical field is never free of its source, and there is no source-free region. This is also true in U(1) electrodynamics on a rigorous level because g also appears in the U(1) covariant derivative as argued already. A field propagating without a source is a violation of causality. On Planck quantization, the coupling constant g has the units $e/\hbar$ in both O(3) and U(1) gauge theory, and for one photon in free space

$$eA^{(0)} = \hbar\boldsymbol{\kappa} \tag{84}$$

signaling that the photon is always coupled to its source. The quantity e has the dual role [6] of coupling constant and charge on the electron. The presence of g in the theory does not mean that the gauge bosons are charged after quantization,

anymore than it means that the U(1) gauge bosons are charged after quantization. The role of g is to measure the "strength" with which the dynamical electromagnetic field couples to its source. This aspect of g [6] is a consequence of the gauge principle, and g originates in parallel transport—it is a coefficient needed to ensure that units are balanced [6].

The homogeneous field equation (31) can be expanded in terns of the O(3) covariant derivative [6,11–20]:

$$(\partial_\mu + g\boldsymbol{A}_\mu\times)\tilde{\boldsymbol{G}}^{\mu\nu} \equiv \boldsymbol{0} \tag{85}$$

A particular solution is

$$\partial_\mu\tilde{\boldsymbol{G}}^{\mu\nu} = \boldsymbol{0} \tag{86}$$

the first equation of which gives

$$\boldsymbol{A}_\mu \times \tilde{\boldsymbol{G}}^{\mu\nu} = 0 \tag{87}$$

$$\partial_\mu \tilde{\boldsymbol{G}}^{\mu\nu(i)} = 0; \qquad i = 1,2,3 \tag{88}$$

that is, Heaviside–Maxwell-type equations and an equation for $\boldsymbol{B}^{(3)}$, which in vector form is

$$\frac{\partial \boldsymbol{B}^{(3)}}{\partial t} = \boldsymbol{0} \tag{89}$$

The latter equation can be interpreted to mean that the third Stokes parameter does not vary with time in a circularly polarized beam of light. The particular solution (87) gives the B cyclic theorem (9) self-consistently [11–20].

In the vacuum (in the absence of matter), the inhomogeneous O(3) field equation (32) can be interpreted as

$$\partial_\mu \boldsymbol{G}^{\mu\nu} = 0 \tag{90}$$

$$\boldsymbol{J}^\nu = g\varepsilon_0 \boldsymbol{A}_\mu \times \boldsymbol{G}^{\mu\nu} \tag{91}$$

where $\boldsymbol{J}^\nu$ is a conserved vacuum current. Equation (90) gives the component equations:

$$\partial_\mu G^{\mu\nu(i)} = 0; \qquad i = 1,2,3 \tag{92}$$

The first two are Maxwell–Heaviside-type equations, and the third, in vector form, is

$$\nabla \times \boldsymbol{B}^{(3)} = \boldsymbol{0} \tag{93}$$

which can be interpreted to mean that the third Stokes parameter is irrotational in the vacuum. It can be shown [17] that the current $\boldsymbol{J}^\nu$ self-consistently gives the vacuum energy

$$En^{(3)} = \frac{1}{\mu_0}\int \boldsymbol{B}^{(3)}\cdot\boldsymbol{B}^{(3)}dV \tag{94}$$

due to the $\boldsymbol{B}^{(3)}$ field.

In the presence of matter (electrons and protons), the inhomogeneous field equation (32) can be expanded as given in Eqs. (52)–(54) and interprets the inverse Faraday effect self-consistently as argued already. Constitutive relations such as Eq. (55) must be used as in U(1) electrodynamics.

The fundamental field equations (31) and (32) can be expanded out fully in the (1),(2),(3) basis defined by Eqs. (8) to give four field equations: the O(3) equivalents of the Coulomb, Gauss, Ampère–Maxwell, and Faraday laws. This expansion shows clearly that the adoption of an O(3) configuration for the vacuum produces conserved vacuum charges and currents from the first principles of gauge field theory. The vacuum electric charge and vacuum electric current were introduced empirically and developed by Lehnert [7–10]; and the magnetic equivalents were introduced and developed empirically by Harmuth [21,22] and later developed from gauge theory by Barrett [3,4], whose field equations in SU(2) gauge group symmetry are isomorphic with the field equations in O(3) gauge group symmetry given here.

The Gauss law in O(3) electrodynamics is

$$\nabla\cdot\boldsymbol{B}^{(1)*} \equiv ig(\boldsymbol{A}^{(2)}\cdot\boldsymbol{B}^{(3)} - \boldsymbol{B}^{(2)}\cdot\boldsymbol{A}^{(3)}) \tag{95}$$

$$\nabla\cdot\boldsymbol{B}^{(2)*} \equiv ig(\boldsymbol{A}^{(3)}\cdot\boldsymbol{B}^{(1)} - \boldsymbol{B}^{(3)}\cdot\boldsymbol{A}^{(1)}) \tag{96}$$

$$\nabla\cdot\boldsymbol{B}^{(3)*} \equiv ig(\boldsymbol{A}^{(1)}\cdot\boldsymbol{B}^{(2)} - \boldsymbol{B}^{(1)}\cdot\boldsymbol{A}^{(2)}) \tag{97}$$

and allows for the possibility of a topological magnetic monopole originating in the vacuum configuration defined by the O(3) gauge group. Empirical evidence for such a monopole has been reviewed by Mikhailov [4] and interpreted by Barrett [45]. However, the right-hand side of Eqs. (95) to (97) can also be zero for particular solutions [11–20], in which case no magnetic monopole exists. In general, Eqs. (95)–(97) must be solved numerically and simultaneously with the other three equations [Eqs. (98)–(100)] given next. This is not a trivial task, but would give a variety of solutions not present in U(1) electrodynamics, solutions can be compared with empirical data.

The Faraday law on the O(3) level is

$$\nabla \times \boldsymbol{E}^{(1)*} + \frac{\partial \boldsymbol{B}^{(1)*}}{\partial t} \equiv -ig(cA_0^{(3)}\boldsymbol{B}^{(2)} - cA_0^{(2)}\boldsymbol{B}^{(3)} + \boldsymbol{A}^{(2)} \times \boldsymbol{E}^{(3)} - \boldsymbol{A}^{(3)} \times \boldsymbol{E}^{(2)}) \tag{98}$$

$$\nabla \times \boldsymbol{E}^{(2)*} + \frac{\partial \boldsymbol{B}^{(2)*}}{\partial t} \equiv -ig(cA_0^{(1)}\boldsymbol{B}^{(3)} - cA_0^{(3)}\boldsymbol{B}^{(1)} + \boldsymbol{A}^{(3)} \times \boldsymbol{E}^{(1)} - \boldsymbol{A}^{(1)} \times \boldsymbol{E}^{(3)}) \tag{99}$$

$$\nabla \times \boldsymbol{E}^{(3)*} + \frac{\partial \boldsymbol{B}^{(3)*}}{\partial t} \equiv -ig(cA_0^{(2)}\boldsymbol{B}^{(1)} - cA_0^{(1)}\boldsymbol{B}^{(2)} + \boldsymbol{A}^{(1)} \times \boldsymbol{E}^{(2)} - \boldsymbol{A}^{(2)} \times \boldsymbol{E}^{(1)}) \tag{100}$$

and contains on the right-hand sides terms proportional to a conserved topological vacuum magnetic current, which was introduced empirically by Harmuth [21,22] and developed by Barrett [3,4] using SU(2) gauge field theory. This vacuum magnetic current provides energy, in the same way as the current J^ν leads to the energy in Eq. (94), and this energy emanates from the vacuum configuration. In principle, therefore, it can be used as a source of mechanical energy provided devices are available to convert the vacuum topological magnetic current into mechanical energy. The same is true of the topological magnetic charge in Eqs. (95)–(97). These charges and currents vanish only in very special cases [11–20], and in general are nonzero. They originate from fundamental topological considerations as argued in Section I.

The O(3) Coulomb law in field–matter interaction is

$$\nabla \cdot \boldsymbol{D}^{(1)*} = \rho^{(1)*} + ig(\boldsymbol{A}^{(2)} \cdot \boldsymbol{D}^{(3)} - \boldsymbol{D}^{(2)} \cdot \boldsymbol{A}^{(3)}) \tag{101}$$

$$\nabla \cdot \boldsymbol{D}^{(2)*} = \rho^{(2)*} + ig(\boldsymbol{A}^{(3)} \cdot \boldsymbol{D}^{(1)} - \boldsymbol{D}^{(3)} \cdot \boldsymbol{A}^{(1)}) \tag{102}$$

$$\nabla \cdot \boldsymbol{D}^{(3)*} = \rho^{(3)*} + ig(\boldsymbol{A}^{(1)} \cdot \boldsymbol{D}^{(2)} - \boldsymbol{D}^{(1)} \cdot \boldsymbol{A}^{(2)}) \tag{103}$$

In the vacuum, the quantities $\rho^{(i)}, i = 1,2,3$, disappear, but the topological Noether charges proportional to the remaining right-hand-side terms do not disappear, leaving one of the Lehnert equations [7–10]. Lehnert introduced the vacuum charge empirically. Lehnert and Roy [10] have given clear empirical evidence for the existence of vacuum charge and current. The latter appears in the O(3) Ampère–Maxwell law, which in field–matter interaction is

$$\nabla \times \boldsymbol{H}^{(1)*} - \boldsymbol{J}^{(1)*} - \frac{\partial \boldsymbol{D}^{(1)*}}{\partial t} = -ig(cA_0^{(2)}\boldsymbol{D}^{(3)} - cA_0^{(3)}\boldsymbol{D}^{(2)} + \boldsymbol{A}^{(2)} \times \boldsymbol{H}^{(3)} - \boldsymbol{A}^{(3)} \times \boldsymbol{H}^{(2)}) \tag{104}$$

$$\nabla \times \boldsymbol{H}^{(2)*} - \boldsymbol{J}^{(2)*} - \frac{\partial \boldsymbol{D}^{(2)*}}{\partial t} = -ig(cA_0^{(3)}\boldsymbol{D}^{(1)} - cA_0^{(1)}\boldsymbol{D}^{(3)} + \boldsymbol{A}^{(3)} \times \boldsymbol{H}^{(1)} - \boldsymbol{A}^{(1)} \times \boldsymbol{H}^{(3)}) \tag{105}$$

$$\nabla \times \boldsymbol{H}^{(3)*} - \boldsymbol{J}^{(3)*} - \frac{\partial \boldsymbol{D}^{(3)*}}{\partial t} = -ig(cA_0^{(1)}\boldsymbol{D}^{(2)} - cA_0^{(2)}\boldsymbol{D}^{(1)} + \boldsymbol{A}^{(1)} \times \boldsymbol{H}^{(2)} - \boldsymbol{A}^{(2)} \times \boldsymbol{H}^{(1)}) \tag{106}$$

In the vacuum, the terms $\boldsymbol{J}^{(i)}, i = 1,2,3$ disappear, but the topological Noether electric vacuum currents on the right-hand sides of these equations do not. These are the equivalents of the vacuum current introduced empirically by Lehnert [7–10]. These vacuum charges and currents originate in the vacuum configuration and provide energy as argued already. This can loosely be called "vacuum energy." In principle, it can be converted to useful form, and this type of energy does not originate in point electric charge; it originates in the topology of the vacuum itself.

The Lehnert field equations in the vacuum also exist in U(1) form, and were originally postulated [7–10] in U(1) gauge field theory. It can be demonstrated as follows, that they originate from the U(1) gauge field equations when matter is not present:

$$(\partial^\nu - igA^{\nu*})F_{\mu\nu} = 0 \tag{107}$$

This equation can also be written as

$$\partial^\nu F_{\mu\nu} = igA^{\nu*}F_{\mu\nu} \qquad g = \kappa/A^{(0)} \tag{108}$$

giving the first Lehnert equation in the form

$$\nabla \cdot \boldsymbol{D} = -ig\boldsymbol{A}^* \cdot \boldsymbol{D} \equiv \rho \tag{109}$$

Similarly, Eq. (107) shows that the second Lehnert equation is

$$\nabla \times \boldsymbol{H} - \frac{\partial \boldsymbol{D}}{\partial t} = \boldsymbol{J} = ig(cA^{0*}\boldsymbol{D} + \boldsymbol{A}^* \times \boldsymbol{H}) \tag{110}$$

and vacuum charge and current emanate directly from U(1) gauge field theory as well as from O(3) gauge field theory as just argued. The constant $e/\hbar$ must be regarded as a coupling constant in both cases [6], because it arises from the gauge principle. Similarly, the vacuum magnetic monopole and charge can be obtained from the U(1) gauge equation:

$$(\partial^\nu - igA^{\nu*})\tilde{F}_{\mu\nu} \equiv 0 \tag{111}$$

and in vector form are

$$\nabla \cdot \boldsymbol{B} = ig\boldsymbol{A}^* \cdot \boldsymbol{B} \tag{112}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times \boldsymbol{E} = ig\,(cA^{0*}\boldsymbol{B} + \boldsymbol{A}^* \times \boldsymbol{E}) \tag{113}$$

In both U(1) and O(3), the existence of vacuum charges and currents depends on the existence of the coupling constant g, which is due fundamentally to the notion of covariant derivative, and can be traced, therefore, to the original gauge principle of Weyl, as discussed in Section II. The coupling constant g must be introduced in vacuo if we accept special relativity and the gauge principle. The existence of vacuum charges and currents follows. The arguments in Section III lead us to reject the U(1) gauge theory of electrodynamics in favor of another theory such as O(3) electrodynamics. These vacuum charges and currents are conserved in the sense that they are Noether currents, and therefore do not violate the Noether theorem [6], specifically, conservation of charge/current, energy, and momentum.

It is seen that as the gauge group is changed from U(1) to a higher symmetry, more solutions are allowed for the field equations, and therefore for the vacuum charges and currents. Mikhailov has detected a magnetic monopole in six independent experiments [4], interpreted as a topological magnetic monopole by Barrett [4,5], and a magnetic monopole means the presence of magnetic current. This has also been detected empirically [46]. Both the magnetic charge and the current are topological in origin. In the case of U(1) gauge field theory applied to electrodynamics, the vacuum configuration is described by a U(1) group symmetry, and in O(3) electrodynamics by an O(3) gauge group symmetry.

All gauge theory depends on the rotation of an n-component vector whose 4-derivative does not transform covariantly as shown in Eq. (18). The reason is that $\psi(x)$ and $\psi(x+dx)$ are measured in different coordinate systems; the field ψ has different values at different points, but $\psi(x)$ and $\psi(x)+d\psi$ are measured with respect to different coordinate axes. The quantity $d\psi$ carries information about the nature of the field ψ itself, but also about the rotation of the axes in the internal gauge space on moving from $x+dx$. This leads to the concept of parallel transport in the internal gauge space and the resulting vector [6] is denoted $\psi(x)+d\psi$. The notion of parallel transport is at the root of all gauge theory and implies the introduction of g, defined by

$$\delta\psi = igM^a A^A_\mu \, dx^\mu \psi \tag{114}$$

where dx^μ is the distance over which the vector is carried, M^a are group rotation generators, and A^a_μ are generalized vector potentials for the given internal gauge symmetry [e.g., U(1) or O(3)]. The covariant derivative is therefore

$$D_\mu \equiv \partial_\mu - igM^a A^a_\mu \tag{115}$$

and is defined in this way under all conditions, in the presence and absence of matter (electrons and protons). It follows that the electromagnetic field tensor

under all conditions for all gauge groups is

$$G_{\mu\nu} \equiv \frac{i}{g}[D_\mu, D_\nu] \tag{116}$$

and if g is zero, the field tensor becomes infinite for any gauge group, including U(1). Here, [,] denotes commutator as usual. The constant g interpreted in this way is neither a property of the source (an electron) nor of the field, but a constant that couples source and field. Note that gauge theory is a necessary condition for the existence of vacuum charges and currents, but not sufficient. The actual existence of these entities must be determined empirically, as in the experiments by Mikhailov [4] and in the work of Lehnert and Roy [10]. The gauge equations on both the U(1) and O(3) levels allow for the fact that vacuum charges and currents may be zero [11–20]. The $\boldsymbol{B}^{(3)}$ field of O(3) electrodynamics, however, is always nonzero in the vacuum, as it is the direct result of a vacuum configuration described by O(3) symmetry. If vacuum charges and currents do exist, however, they provide the possibility of extracting energy from the vacuum as developed in Section XI.

V. FIELD EQUATIONS OF O(3) ELECTRODYNAMICS IN THE CARTESIAN BASIS: REDUCTION TO THE LAWS OF ELECTROSTATICS

In this section, it is shown that the field equations of O(3) electrodynamics written in the Cartesian basis have a substantially different meaning from those written in the complex circular basis of Section IV. The latter basis essentially introduces motion and dynamics, while Eqs. (31) and (32), written in the Cartesian basis, produce the laws of electrostatics self-consistently. This is confirmation of the mathematical and physical correctness of Eqs. (31) and (32).

In the Cartesian basis, the O(3) field tensor is

$$\boldsymbol{G}_{\mu\nu} = G^X_{\mu\nu}\boldsymbol{i} + G^Y_{\mu\nu}\boldsymbol{j} + G^Z_{\mu\nu}\boldsymbol{k} \tag{117}$$

and the O(3) potential is

$$\boldsymbol{A}_\mu = A^X_\mu \boldsymbol{i} + A^Y_\mu \boldsymbol{j} + A^Z_\mu \boldsymbol{k} \tag{118}$$

where the upper indices (X, Y, Z) denote an O(3) internal space defined by the Cartesian unit vectors in Eq. (6). The components of the field tensor are

$$G^{\mu\nu}_X = \partial^\mu A^\nu_X - \partial^\nu A^\mu_X - ig[A^\mu_Y, A^\nu_Z] \tag{119}$$

$$G^{\mu\nu}_Y = \partial^\mu A^\nu_Y - \partial^\nu A^\mu_Y - ig[A^\mu_Z, A^\nu_X] \tag{120}$$

$$G^{\mu\nu}_Z = \partial^\mu A^\nu_Z - \partial^\nu A^\mu_Z - ig[A^\mu_X, A^\nu_Y] \tag{121}$$

where the potentials are real quantities. Therefore the commutators vanish:

$$[A_Y^\mu, A_Z^\nu] = [A_Z^\mu, A_X^\nu] = [A_X^\mu, A_Y^\nu] = 0 \tag{122}$$

The covariant derivative of O(3) electrodynamics in the Cartesian basis is

$$D_\mu = \partial_\mu - igM^a A_\mu^a = \partial_\mu - ig(A_\mu^X + A_\mu^Y + A_\mu^Z) \tag{123}$$

and a rotation in the internal gauge space is denoted by

$$\begin{aligned}\psi' &= e^{i(M^a \Lambda^a(x^\mu)}\psi \\ &= e^{i\Lambda_X(x^\mu)} e^{i\Lambda_Y(x^\mu)} e^{i\Lambda_Z(x^\mu)}\psi\end{aligned} \tag{124}$$

For a rotation about the Z axis

$$\psi' = e^{i\Lambda_Z(x^\mu)}\psi \equiv S\psi \tag{125}$$

producing the gauge transformation:

$$A_Z \rightarrow A_Z + \frac{1}{g}\partial_Z \Lambda \tag{126}$$

This is, self-consistently, the same result as for O(3) electrodynamics in the complex circular basis [11–20] because of the relation $\boldsymbol{k} = \boldsymbol{e}^3$.

The use of Cartesian indices for the internal O(3) gauge space produces the laws of electrostatics as follows. For clarity, the derivation is given in detail. First, the components of the magnetic field disappear:

$$B_X = G_X^{32} = \partial^3 A_X^2 - \partial^2 A_X^3 - ig[A_Y^3, A_Z^2] = 0 \tag{127}$$

$$B_Y = G_Y^{13} = \partial^1 A_Y^3 - \partial^3 A_Y^1 - ig[A_Z^1, A_X^3] = 0 \tag{128}$$

$$B_Z = G_Z^{21} = \partial^2 A_Z^1 - \partial^1 A_Z^2 - ig[A_X^2, A_Y^1] = 0 \tag{129}$$

This means that a magnetic field is always a quantity that depends on motion, or a current. If there is no magnetic field, there is no electric current, that is, no motion of charge. The use of Cartesian indices for the internal O(3) gauge space therefore corresponds to an electrostatic situation where there is no movement of charge. The use of complex circular indices corresponds to electrodynamics.

The nonzero static electric field components are given by equations such as:

$$G_X^{01} = \partial^0 A_X^1 - \partial^1 A_X^0 - ig[A_Y^0, A_Z^1] \tag{130}$$

$$G_X^{10} = \partial^1 A_X^0 - \partial^0 A_X^1 - ig[A_Y^1, A_Z^0] \tag{131}$$

which correspond to

$$-E_X = \frac{1}{c}\frac{\partial}{\partial t}A_X^1 + \frac{\partial}{\partial X}A_X^0 \tag{132}$$

$$E_x = -\frac{1}{c}\frac{\partial}{\partial t}A_X^1 - \frac{\partial}{\partial X}A_x^0 \tag{133}$$

The static electric field is therefore given self-consistently by

$$\boldsymbol{E} = -\nabla A^0 - \frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{A} \tag{134}$$

The vector potential $\boldsymbol{A}$ is zero, however, because the magnetic field is zero, and we arrive at the familiar law of electrostatics:

$$\boldsymbol{E} = -\nabla A^0 \tag{135}$$

Using the vector identity (16), it is found that $\boldsymbol{E}$ is irrotational:

$$\nabla \times \boldsymbol{E} = \boldsymbol{0} \tag{136}$$

In the Cartesian basis, the homogeneous field equation of O(3) electrodynamics can be written out as three component equations:

$$\partial_\mu \tilde{G}_X^{\mu\nu} = ig(A_\mu^Y \tilde{G}_Z^{\mu\nu} - A_\mu^Z \tilde{G}_Y^{\mu\nu}) \tag{137}$$

$$\partial_\mu \tilde{G}_Y^{\mu\nu} = ig(A_\mu^Z \tilde{G}_X^{\mu\nu} - A_\mu^X \tilde{G}_Z^{\mu\nu}) \tag{138}$$

$$\partial_\mu \tilde{G}_Z^{\mu\nu} = ig(A_\mu^X \tilde{G}_Y^{\mu\nu} - A_\mu^Y \tilde{G}_X^{\mu\nu}) \tag{139}$$

For $\nu = 0$

$$\partial_\mu \tilde{G}_X^0 = A_\mu^Y \tilde{G}_Z^{\mu 0} - A_\mu^Z \tilde{G}_Y^{\mu 0} \tag{140}$$

and using

$$\tilde{G}^{10} = B_X \tag{141}$$

this gives the result

$$\partial_X B_X = 0 \tag{142}$$

The complete result for $\nu = 0$ is therefore

$$\nabla \cdot \boldsymbol{B} = 0 \tag{143}$$

which is self-consistent with Eqs. (127)–(129), indicating the absence of a magnetic field because of the absence of moving charges.

For $\nu = 1$, we obtain

$$\partial_0 \tilde{G}_X^{01} + \partial_2 \tilde{G}_X^{21} + \partial_3 \tilde{G}_X^{31} = ig(A_0^Y \tilde{G}_Z^{01} + A_2^Y \tilde{G}_Z^{21} + A_3^Y \tilde{G}_Z^{31} - A_0^Z \tilde{G}_Y^{01} - A_2^Z \tilde{G}_Y^{21} - A_3^Z \tilde{G}_Y^{31}) \tag{144}$$

that is, $\partial_0 B_X^X = 0$. Repeating this procedure gives

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{0} \tag{145}$$

which is self-consistent with $\boldsymbol{B} = \boldsymbol{0}$.

The inhomogeneous field equation (32) in the Cartesian basis must be written in the static limit where

$$J^\nu = (\rho, \boldsymbol{0}) \tag{146}$$

The component equations look like

$$\partial_\mu H_X^{\mu\nu} + ig(A_\mu^Y H_Z^{\mu\nu} - A_\mu^Z H_Y^{\mu\nu}) = J_X^\nu \tag{147}$$

For $\nu = 0$, we obtain

$$\partial_1 H_X^{10} + \partial_2 H_X^{20} + \partial_3 H_X^{30} + ig(A_1^Y H_Z^{10} - A_1^Z H_Y^{10} + A_2^Y H_Z^{20} - A_2^Z H_Y^{20} + A_3^Y H_Z^{30} - A_3^Z H_Y^{30}) = J_X^0 = \rho \tag{148}$$

and this results in the equation

$$\nabla \cdot \boldsymbol{D} = \rho \tag{149}$$

which is the Coulomb law of electrostatics. The Coulomb law is well known to be self-consistent with Eqs. (135) and (136). For $\nu = 1$ and other indices, we obtain the self-consistent result

$$\frac{\partial \boldsymbol{D}}{\partial t} = \boldsymbol{0} \tag{150}$$

which is true for an electrostatic displacement $\boldsymbol{D}$.

In summary, the laws of O(3) electrodynamics in the Cartesian basis reduce to the laws of electrostatics:

$$\begin{aligned} \boldsymbol{E} &= -\nabla A^0 \\ \nabla \times \boldsymbol{E} &= \boldsymbol{0} \\ \nabla \cdot \boldsymbol{D} &= \rho \end{aligned} \tag{151}$$

and this is an indication of the correctness and self-consistency of Eqs. (31) and (32). The need for the complex circular basis now becomes clear: this basis introduces dynamics into the O(3) laws. The Cartesian representation of the gauge space describes a static situation where there is charge but no current (movement of charge). A magnetic field always requires the movement of charge. It has therefore been shown that the laws of electrostatics are laws of a gauge field theory of O(3) internal symmetry. This is another refutation of the received view, that the laws of electrostatics are laws of a gauge field theory of U(1) internal symmetry.

The Gauss and Ampère laws of magnetism are obtained mathematically, and somewhat artificially, from the fact that using a Cartesian basis gives Eq. (143) (the Gauss law); and from the fact that there is no current and no $\boldsymbol{B}$, so we have

$$\boldsymbol{J} = \nabla \times \boldsymbol{B} = \boldsymbol{0} \tag{152}$$

and the Ampère law follows. However, there is a more satisfactory way of obtaining the Gauss and Ampère laws by using the complex circular basis. The latter is needed because magnetism is not a static phenomenon, as evidenced by the both the Ampère and Faraday laws. Magnetism is always a dynamic phenomenon, so we always need complex circular indices. Therefore the Gauss and Ampère laws are obtained from the particular solutions (87) and (91) leading to Eqs. (88) and (92). The phenomenon of radiation is then removed by removing the Maxwell displacement current in Eqs. (88) and (92). This removes the radiated $\boldsymbol{B}^{(3)}$ field and leaves the Gauss, Ampère, Coulomb, and Faraday laws of the received view at the expense of generality. This procedure is a method of obtaining the old laws from O(3) electrodynamics, which is, however, more general and self-consistent. In forcing a reduction of O(3) electrodynamics to the received view, we lose the vacuum charges and currents and a great deal of information.

Information is also lost if we replace the ((1),(2),(3)) basis by the (X, Y, Z) basis for the internal gauge space. The reason is that the former basis is essentially dynamical and the latter is essentially static. This is again a self-consistent result, because electrodynamics, by definition, requires the movement of charge. The misnamed subject of "magnetostatics" also requires the movement of charge, and so is not static.

VI. EXPLANATION OF INTERFEROMETRY AND RELATED PHYSICAL OPTICAL EFFECTS USING O(3) ELECTRODYNAMICS

The explanation of interferometric effects in U(1) electrodynamics is in general self-inconsistent, and sometimes, as in the Sagnac effect, nonexistent. In this

section, the theory of interferometry and related physical optical effects is developed with O(3) electrodynamics, which is found to give an accurate and self-consistent explanation, for example, of the Sagnac effect in terms of the fundamental component $\boldsymbol{B}^{(3)}$. The latter is therefore a physical observable in all interferometry.

In order to understand interferometry at a fundamental level in gauge field theory, the starting point must be the non-Abelian Stokes theorem [4]. The theorem is generated by a round trip or closed loop in Minkowski spacetime using covariant derivatives, and in its most general form is given [17] by

$$\exp\left(\oint D_\mu dx^\mu\right) = \exp\left(-\frac{1}{2}\int [D_\mu, D_\nu] d\sigma^{\mu\nu}\right) \tag{153}$$

where the integral over the closed loop on the left-hand side is related to an integral over the hypersurface $\sigma^{\mu\nu}$ of the commutator of covariant derivatives. The electromagnetic phase factor in O(3) electrodynamics is developed as an exponential from Eq. (153) and is given most generally by

$$\exp\left(g\oint D_\mu dx^\mu\right) = \exp\left(-\frac{1}{2}g\int [D_\mu, D_\nu] d\sigma^{\mu\nu}\right) \tag{154}$$

The observable phase is the real part of this exponential, specifically, the cosine. Recall that in ordinary U(1) electrodynamics, the phase factor is given by the exponent

$$\phi = \exp\left(i(\omega t - \boldsymbol{\kappa}\cdot\boldsymbol{r} + \alpha)\right) \tag{155}$$

where α is random.

To reduce Eq. (153) to the ordinary Stokes theorem, the U(1) covariant derivative is used

$$D_\mu = \partial_\mu + igA_\mu \tag{156}$$

to give the result

$$\oint A_\mu dx^\mu = -\frac{1}{2}\int F_{\mu\nu} d\sigma^{\mu\nu} \tag{157}$$

The space part of this expression is the ordinary, or Abelian, Stokes theorem

$$\oint \boldsymbol{A}\cdot d\boldsymbol{r} = \int \boldsymbol{B}\cdot d\boldsymbol{A}\boldsymbol{r} = \int \nabla\times\boldsymbol{A}\cdot d\boldsymbol{A}\boldsymbol{r} \tag{158}$$

with the following fundamental property:

$$\oint_{OA} \boldsymbol{A}\cdot d\boldsymbol{r} = -\oint_{AO} \boldsymbol{A}\cdot d\boldsymbol{r} \tag{159}$$

In U(1) electrodynamics in free space, there are only transverse components of the vector potential, so the integral (158) vanishes. It follows that the area integral in Eq. (157) also vanishes, and so the U(1) phase factor cannot be used to describe interferometry. For example, it cannot be used to describe the Sagnac effect. The latter result is consistent with the fact that the Maxwell–Heaviside and d'Alembert equations are invariant under T, which generates the clockwise (C) Sagnac loop from the counterclockwise (A) loop [17]. It follows that the phase difference observed with platform at rest in the Sagnac effect [47] cannot be described by U(1) electrodynamics. This result is also consistent with the fact that the traditional phase of U(1) electrodynamics is invariant under T as discussed already in Section (III). The same result applies for the Michelson–Gale experiment [48], which is a Sagnac effect.

From Eqs. (157) and (158) the integral

$$\text{Int} = -\frac{1}{2}\int F_{\mu\nu} d\sigma^{\mu\nu} = 0 \tag{160}$$

vanishes in interferometry as described by U(1) electrodynamics. Therefore, in order to explain interferometry and related optical effects by gauge theory, a non-Abelian Stokes theorem and a non-Abelian phase factor are required. This means that O(3) electrodynamics is capable of describing interferometry but U(1) electrodynamics is not. An area integral is needed that does not vanish, as in Eq. (160), and equated through the theorem (157) to a line integral. It is straightforward to show that the only possible solution for the O(3) phase factor is

$$P\exp\left(ig\oint \boldsymbol{A}^{(3)}\cdot dr\right) = P'\exp\left(ig\int \boldsymbol{B}^{(3)}\cdot d\boldsymbol{A}\boldsymbol{r}\right) \tag{161}$$

and since $g = \boldsymbol{\kappa}/A^{(0)}$ classically the phase factor reduces to

$$P\exp\left(i\oint \boldsymbol{\kappa}^{(3)}\cdot d\boldsymbol{r}\right) = P'\exp\left(ig\int B^{(3)}\cdot d\boldsymbol{A}\boldsymbol{r}\right) \tag{162}$$

for all interferometry and physical optics. Equation (162) is nonzero if and only if the Evans–Vigier field $\boldsymbol{B}^{(3)}$ is nonzero, and the latter is therefore responsible for all interferometry and related physical optical effects.

The P on the left-hand side of Eq. (162) denotes path ordering and the P' denotes area ordering [4]. Equation (162) is the result of a round trip or closed loop in Minkowski spacetime with O(3) covariant derivatives. Equation (161) is a direct result of our basic assumption that the configuration of the vacuum can be described by gauge theory with an internal O(3) symmetry (Section I). Henceforth, we shall omit the P and P' from the left- and right-hand sides, respectively, and give a few illustrative examples of the use of Eq. (162) in interferometry and physical optics.

The Sagnac effect with a platform at rest [47] is explained as the phase factor:

$$\exp\left(i\oint_{A-C}\boldsymbol{\kappa}^{(3)}\cdot d\boldsymbol{r}\right) = \exp\left(2i\oint\boldsymbol{\kappa}^{(3)}\cdot d\boldsymbol{r}\right) \tag{163}$$

which is nonzero and gives an observable interferogram, a cosine function:

$$\gamma = \cos\left(2\oint\boldsymbol{\kappa}^{(3)}\cdot d\boldsymbol{r} \pm 2\pi n\right) \tag{164}$$

Using the relation:

$$B^{(0)} = \left|\boldsymbol{B}^{(3)}\right| = gA^{(0)2} \tag{165}$$

the right-hand side of Eq. (162) may be written as

$$\Phi = \exp(i\kappa^2 Ar) \tag{166}$$

and so Eq. (164) becomes

$$\gamma = \cos\left(2\frac{\omega^2}{c^2}Ar \pm 2\pi n\right) \tag{167}$$

This is an expression for the observed phase difference with the platform at rest in the Sagnac experiment [47]; it is a rotation in the internal gauge space. In U(1) electrodynamics, there is no phase difference when the platform is at rest, as discussed already.

When the platform is rotated in the Sagnac effect, there is an additional rotation in the internal gauge space described by

$$\psi' = \exp(iJ_Z\alpha(x^\mu))\psi \tag{168}$$

where $\alpha(x^\mu)$ is an angle in the plane of the Sagnac platform [48]. The effect on the gauge potential $A_\mu^{(3)}$ is as follows:

$$A_\mu^{(3)} \rightarrow A_\mu^{(3)} + \frac{1}{g}\partial_\mu\alpha \tag{169}$$

The angular frequency of rotation of the platform is

$$\Omega = \frac{\partial\alpha}{\partial t} \tag{170}$$

and so Eq. (169) implies that the additional rotation of the platform has the effect

$$\omega \rightarrow \omega \pm \Omega \tag{171}$$

on frequency, depending on the sense of rotation of the platform, which therefore produces the phase factor difference

$$\Delta\gamma = \exp\left(i\left(\frac{Ar}{c^2}\left((\omega+\Omega)^2 - (\omega-\Omega)^2\right)\right)\right) \tag{172}$$

and an interferogram

$$\mathrm{Re}\,(\Delta\gamma) = \cos\left(4\frac{\omega\Omega Ar}{c^2} \pm 2\pi n\right) \tag{173}$$

as observed [49] to very high accuracy. This formula was first given by Sagnac [50] using kinematic methods. There is no explanation for it in U(1) electrodynamics [4].

The calculation can be repeated using matter waves, because the Sagnac effect exists in electrons [51] as well as in photons. The starting point is the same, namely, the assumption that the vacuum configuration is described by an O(3) gauge group symmetry. The same structured vacuum applies to both electrodynamics and dynamics, wherein the energy momentum tensor is also a vector in the internal gauge space:

$$\begin{aligned}\boldsymbol{p}^\mu &= p^{\mu(1)}\boldsymbol{e}^{(1)} + p^{\mu(2)}\boldsymbol{e}^{(2)} + p^{\mu(3)}\boldsymbol{e}^{(3)} \\ &= \hbar(\boldsymbol{\kappa}^{\mu(1)}\boldsymbol{e}^{(1)} + \boldsymbol{\kappa}^{\mu(2)}\boldsymbol{e}^{(2)} + \boldsymbol{\kappa}^{\mu(3)}\boldsymbol{e}^{(3)})\end{aligned} \tag{174}$$

where

$$\omega^2 = c^2\boldsymbol{\kappa}^2 + \frac{m_0^2c^4}{\hbar^2} \tag{175}$$

Here, ω is the angular frequency of a matter wave, such as that of an electron, $\boldsymbol{\kappa}$ is its wave number magnitude, and m_0 is the rest mass of the particle corresponding to the matter wave. The rest mass could be the photon's rest mass, estimated to be less than 10^{-68} kg.

Both p_μ and $\boldsymbol{\kappa}_\mu$ are governed by a gauge transformation

$$p_\mu \rightarrow S p_\mu S^{-1} - i(\partial_\mu S)S^{-1} \tag{176}$$

and similarly for $\boldsymbol{\kappa}_\mu$. The rotation of the Sagnac platform is governed by Eq. (168), from which we obtain

$$\boldsymbol{\kappa}^{0(3)} \rightarrow \boldsymbol{\kappa}^{0(3)} \pm \partial^0 \alpha \tag{177}$$

which is the same as Eq. (171). This is a topological result given by the structure of the vacuum and is valid for all matter waves, including the electromagnetic wave as argued already. The holonomy difference with platform at rest for A and C loops [round trips in Minkowski spacetime with O(3) covariant derivatives] for matter waves is

$$\Delta\gamma = \exp(2i\boldsymbol{\kappa}^2 Ar) \tag{178}$$

where, from Eq. (175)

$$\boldsymbol{\kappa}^2 = \frac{\omega^2}{c^2} - \frac{m_0^2 c^4}{\hbar^2} \tag{179}$$

The extra holonomy difference due to the rotating platform is the same as for electromagnetic waves:

$$\Delta\Delta\gamma = \exp\left(\frac{4i\omega\Omega Ar}{c^2}\right) \tag{180}$$

This result is true for all matter waves and also in the Michelson–Gale experiment, where it has been measured to a precision of one part in 10^{23} [49]. Hasselbach et al. [51] have demonstrated it in electron waves. We have therefore shown that the electrodynamic and kinematic explanation of the Sagnac effect gives the same result in a structured vacuum described by O(3) gauge group symmetry.

The preceding is a result of special relativity precise to one part in 10^{23} [49]. Its explanation in standard special relativity is as follows. Let the tangential velocity of the disk be v_1 and the velocity of the particle be v_2 in the laboratory frame [52]. When the particle and disk are moving in the same direction, the velocity of the particle is $v_2 - v_1 = v_3$ relative to an observer on the periphery of the disk. Vice-versa, the relative velocity is $v_2 + v_1 = v_4$. The special theory of relativity states that time for the two particles will be dilated to different

extents, so the time dilation difference relative to the observer on the periphery of the disk is

$$\Delta t = \left(1 - \frac{v_3^2}{c^2}\right)^{1/2} - \left(1 - \frac{v_4^2}{c^2}\right)^{-1/2} \tag{181}$$

using the binomial theorem. When the disk is stationary [53]:

$$t = \frac{2\pi r}{v_2} \tag{182}$$

where r is its radius. So the observable time difference of the Sagnac effect is

$$\Delta\Delta t = \frac{4\pi r v_1}{c^2} = \frac{4\Omega A r}{c^2} \tag{183}$$

as deduced already as a rotation in the O(3) gauge space of a structured vacuum.

The Maxwell–Heaviside theory of electrodynamics has no explanation for the Sagnac effect [4] because its phase is invariant under T, as argued already, and because the equations are invariant to rotation in the vacuum. The d'Alembert wave equation of U(1) electrodynamics is also T-invariant. One of the most telling pieces of evidence against the validity of the U(1) electrodynamics was given experimentally by Pegram [54] who discovered a little known [4] cross-relation between magnetic and electric fields in the vacuum that is denied by Lorentz transformation.

It can be shown straightforwardly, as follows, that there is no holonomy difference if the phase factor (154) is applied to the problem of the Sagnac effect with U(1) covariant derivatives. In other words, the Dirac phase factor [4] of U(1) electrodynamics does not describe the Sagnac effect. For C and A loops, consider the boundary

$$X^2 + Y^2 = 1 \tag{184}$$

of the assumed circular paths of the two light beams of the Sagnac effect. The line integral vanishes around the boundary

$$\begin{aligned} \oint dr &= \int_0^{2\pi} dX + \int_0^{2\pi} dY \\ &= -\int_0^{2\pi} \sin\phi\, d\phi + \int_0^{2\pi} \cos\phi\, d\phi \\ &= 0 = -\int dr \end{aligned} \tag{185}$$

and so

$$\oint \boldsymbol{\kappa} \cdot d\boldsymbol{r} = -\oint \boldsymbol{\kappa} \cdot d\boldsymbol{r} = 0 \tag{186}$$

in U(1) electrodynamics and the relevant holonomy in this symmetry of electrodynamics is the same

$$\exp\left(i \oint_C \boldsymbol{\kappa} \cdot d\boldsymbol{r}\right) = \exp\left(-i \oint_C \boldsymbol{\kappa} \cdot d\boldsymbol{r}\right) = 1 \tag{187}$$

for both beams. There is no interferogram with the platform at rest, contrary to observation.

Furthermore, the only electromagnetic vector present in free space in the Maxwell–Heaviside theory is the plane wave [11–20]:

$$\boldsymbol{A}^{(1)} = \boldsymbol{A}^{(2)*} = \frac{A^{(0)}}{\sqrt{2}}(\boldsymbol{i} + i\boldsymbol{j})e^{i(\omega t - \boldsymbol{\kappa} \cdot \boldsymbol{r})} \tag{188}$$

which is always perpendicular to $\boldsymbol{r}$, so we obtain Eq. (187) self-consistently. Owing to the gauge invariance of the Maxwell–Heaviside theory, there is no extra effect of a moving platform, again contrary to observation. The principle of gauge invariance, and U(1) electrodynamics in general, fail to describe the Sagnac effect.

On the O(3) level, it can be shown that if we write out the commutator of covariant derivatives in Eq. (153) the phase factor becomes [6]

$$\gamma = \exp\left(\int [D_\mu, D_\nu] d\sigma^{\mu\nu}\right) \tag{189}$$

$$\gamma = \exp\left(-i\frac{g}{2}\int (\partial_\mu A_\nu - \partial_\nu A_\mu) d\sigma^{\mu\nu} - g^2 \int [A_\mu, A_\nu] d\sigma^{\mu\nu}\right) \tag{190}$$

ut as just argued, integrals such as

$$I(U(1)) = \int (\partial_\mu A_\nu - \partial_\nu A_\mu) d\sigma^{\mu\nu} \tag{191}$$

vanish for both A and C loops, leaving the only source of nonzero holonomy, Eq. (162), leading to the observable interferogram in Eq. (167). This derivation can be self-checked using a closed loop with O(3) covariant derivatives in Minkowski spacetime [6] whereupon the holonomy in one direction is

$$\gamma_A = \exp\left(-i\frac{g}{2}\int G_{\mu\nu} dS^{\mu\nu}\right) \tag{192}$$

and in the other direction is

$$\gamma_C = \exp\left(i\frac{g}{2}\int G_{\mu\nu} dS^{\mu\nu}\right) \tag{193}$$

where $S^{\mu\nu}$ is the area enclosed by the loop. The holonomy represents a rotation in the internal O(3) gauge space and is a general result for all gauge group symmetries. If the internal basis of the space of O(3) is (a, b, c), the holonomy can be expressed as

$$\gamma = \exp\left(\mp i\frac{g}{2}\int (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - ig\varepsilon_{abc} A^b_\mu A^c_\nu) dS^{\mu\nu}\right) \tag{194}$$

If the internal symmetry is U(1), the holonomy in either direction is

$$\begin{aligned}\gamma(U(1)) &= \exp\left(\mp ig \int (\partial_\mu A_\nu - \partial_\nu A_\mu) dS^{\mu\nu}\right) \\ &= \exp\left(\mp ig \oint A_\mu dx^\mu\right) = 1\end{aligned} \tag{195}$$

and the ordinary Stokes theorem can be used to show that there is no holonomy difference.

If the internal group symmetry is O(3) in the basis ((1),(2),(3)), we obtain:

$$\begin{aligned}&\exp\left(\mp i\frac{g}{2}\int (\partial_\mu A^{(1)}_\nu - \partial_\nu A^{(1)}_\mu) dS^{\mu\nu}\right) = 1 \\ &\exp\left(\mp i\frac{g}{2}\int (\partial_\mu A^{(2)}_\nu - \partial_\nu A^{(2)}_\mu) dS^{\mu\nu}\right) = 1 \\ &\exp\left(\mp i\frac{g}{2}\int (\partial_\mu A^{(3)}_\nu - \partial_\nu A^{(3)}_\mu) dS^{\mu\nu}\right) = 1\end{aligned} \tag{196}$$

and the only source of holonomy difference is the commutator term, which is written in general as [17]

$$\gamma = \exp\left(\mp\frac{g^2}{2}\int \varepsilon_{abc} A^a_\mu A^c_\nu dS^{\mu\nu}\right) \tag{197}$$

Considering the special case

$$\gamma = \exp\left(\mp\frac{g^2}{2}\int (A^{(1)}_X A^{(2)}_Y - A^{(2)}_X A^{(1)}_Y) dS^{XY}\right) \tag{198}$$

and using Eqs. (165) and (188), it is found that the holonomy is

$$\gamma = \exp(\mp i\kappa^2 Ar) \tag{199}$$

The difference in holonomy is Eq. (178), and the interferogram can be written as

$$\gamma = \cos(2\kappa^2 Ar \pm 2\pi n) \tag{200}$$

with the platform at rest.

The Sagnac effect caused by the rotating platform is therefore due to a rotation in the internal gauge space ((1),(2),(3)), which results in the frequency shift in Eq. (171). The frequency shift is experimentally the same to an observer on and off the platform and is independent of the shape of the area Ar. The holonomy difference (172) derived theoretically depends only on the magnitudes ω and Ω, and these scalars are frame-invariant, as observed experimentally. There is no shape specified for the area Ar in the theory, and only its scalar magnitude enters into Eq. (172), again in agreement with experiment.

In the one photon limit, O(3) electrodynamics [11–20] produces the result:

$$eA^{(0)} = \hbar\kappa \tag{201}$$

Substituting this into

$$\gamma = \exp\left(\mp i\frac{e}{\hbar}B^{(3)}Ar\right) \tag{202}$$

for a beam made up of one photon, the flux $B^{(3)}Ar$ becomes $\hbar/e$ and so, in the one photon limit

$$\gamma = \exp(\pm i) \tag{203}$$

The observable phase difference is therefore nonzero for one photon in O(3) electrodynamics. The effect with platform in motion is the same as Eq. (172) for one photon.

Equations leading to Eq. (162) apply in general in O(3) electrodynamics and to interferometry and physical optics in general. They imply the existence of the quantity

$$g_m \equiv \frac{1}{V}\int B^{(3)}dAr \tag{204}$$

in which the units of a topological magnetic monopole are directly dependent on the vacuum configuration. We therefore have the relation

$$\Phi = gg_m V \tag{205}$$

and the observation of phase Φ implies the existence of both $\boldsymbol{B}(3)$ and g_m. The latter must not be confused with the Dirac point magnetic monopole or with the quantities on the right-hand sides of Eqs. (95) to (97).

In the Maxwell–Heaviside theory of electrodynamics, the electromagnetic phase is a product of two 4-vectors together with a random quantity α:

$$\phi = \kappa_\mu x^\mu + \alpha = \omega t - \boldsymbol{\kappa}\cdot\boldsymbol{r} + \alpha \tag{206}$$

Let $\alpha = 0$ without loss of generality, because it is a random number. Then the remaining part of the phase in Eq. (206) is invariant under parity inversion, which is the same as perfect normal reflection as argued in Section III. Therefore the phase arriving back at the beam splitter [55] in one arm of the Michelson interferometer is unchanged for all r, the length of the arm. The same is true for the other arm, and so there is no interferogram, because the phases arriving back from either arm are always the same as the phase in the beam that initially entered the beam splitter. This result is clearly contrary to observation, and U(1) electrodynamics is unable to explain Michelson interferometry, the basis of Fourier transform infrared spectral techniques and instruments.

In O(3) electrodynamics, the interferogram is described by the holonomy

$$\exp\left(i\oint_{1-2}\boldsymbol{\kappa}^{(3)}\cdot d\boldsymbol{r}\right) = \exp\left(2i\int B^{(3)}dAr\right) \tag{207}$$

where 1–2 represents a path traversal from beam splitter to mirror and back to beamsplitter. Using the property

$$\oint_1 \boldsymbol{\kappa}^{(3)}\cdot d\boldsymbol{r} = -\oint_2 \boldsymbol{\kappa}^{(3)}d\boldsymbol{r} \tag{208}$$

this is nonzero, and the interferogram is the cosine function [17]

$$\mathrm{Re}(\gamma) = \cos(2\boldsymbol{\kappa}^{(3)}\cdot\boldsymbol{r} \pm 2\pi n) \tag{209}$$

which is nonzero and depends on r. By varying r, an interferogram is generated as observed empirically [55]. Its Fourier transform is a spectral function, and in general the beam is polychromatic.

The principle of interferometry in O(3) electrodynamics follows from the fact that it is caused by a rotation in the internal gauge space

$$\exp\left(i\oint_{1-2}\kappa_\mu dx^\mu\right) = \exp(iJ_Z\Lambda(x^\mu))\exp\left(i\oint_{\cdot}\kappa_\mu dx^\mu\right) \tag{210}$$

or more succinctly

$$\gamma' = e^{iJ_z\Lambda(x^\mu)}\gamma \tag{211}$$

In Michelson interferometry, for example, the left-hand-side of Eq. (210) becomes

$$\gamma = \exp(2i\kappa_\mu x^\mu) \tag{212}$$

whose real part is Eq. (209), the interferogram. This result follows from the fact that the rotation (211) in the O(3) internal gauge space results in

$$\begin{aligned} A_\mu^{(3)} &\rightarrow A_\mu^{(3)} + \frac{1}{g}\partial_\mu\Lambda \\ \omega &\rightarrow \omega + \frac{\partial\Lambda}{\partial t} \\ \kappa &\rightarrow \kappa + \frac{1}{c}\frac{\partial\Lambda}{\partial t} \end{aligned} \tag{213}$$

and if $\omega = \partial\Lambda/\partial t$, Eq. (212) follows. We have already applied Eq. (210) to the Sagnac effect.

In U(1) electrodynamics, the equivalent of Eq. (210) is the rotation in the U(1) internal gauge space:

$$e^{i(\omega t - \boldsymbol{\kappa}\cdot\boldsymbol{r} + \Lambda)} = e^{i\Lambda}e^{i(\omega t - \boldsymbol{\kappa}\cdot\boldsymbol{r})} \tag{214}$$

in other words

$$\psi' = e^{i\Lambda}\psi \tag{214a}$$

where Λ is random. The electromagnetic phase in U(1) electrodynamics is defined only up to a random number Λ, whereas the phase in O(3) electrodynamics is fully defined and gives rise to physical effects in interferometry. The details of the effect depend on the geometry of the interferometer.

Another example of a physical effect of this type is the Aharonov–Bohm effect, which is supported by a multiply connected vacuum configuration such as that described by the O(3) gauge group [6]. The Aharonov–Bohm effect is a gauge transform of the true vacuum, where there are no potentials. In our notation, therefore the Aharonov–Bohm effect is due to terms such as $(1/g)\partial_\mu\Lambda$, depending on the geometry chosen for the experiment. It is essential for the Aharonov–Bohm effect to exist such that $(1/g)\partial_\mu\Lambda$ be physical, and not random. It follows therefore that the vacuum configuration defined by the U(1) group does not support the Aharonov–Bohm effect [26]. The vacuum configuration defined by the SU(2) group cannot support the effect because SU(2) is singly connected [6], leaving O(3) as the only possibility. This is another strong indication of the need for O(3) electrodynamics. Barrett [26] has also reasoned that the U(1) vacuum configuration cannot support the Aharonov–Bohm effect. First, there is a fundamental topological flaw in Heaviside's reduction of the potential to a mathematical convenience because this can apply only in singly connected spaces, whereas U(1) itself is not singly connected, and Maxwell–Heaviside theory is asserted to be a U(1) Yang–Mills gauge field theory. This is another self-inconsistency of the received view. In fact, any polarized classical wave such as a circularly polarized wave has two vectorial components that form the O(3) symmetry basis ((1),(2),(3)) [3]. Another inconsistency of the received view of the Aharonov–Bohm effect is that it depends on the interaction of an assumed physical vector potential A with an electron. However [26], the magnetic field $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ is always zero at the point of interaction, and the effect is described self-inconsistently [6] as an integral over the flux due to $\boldsymbol{B}$. At the point of interaction this flux is always zero. The effect actually depends on the inhomogeneous term generated by the gauge transform of the vacuum [6] into regions where both the magnetic field and the potential are zero. So the effect is an interferometric effect determined by gauge transformed terms such as

$$A_i' = -\frac{i}{g}(\partial_i S_i)S_i^{-1} = \frac{1}{g}\partial_i\Lambda(i); \qquad i = 1,2,3 \tag{215}$$

in O(3) electrodynamics, where these terms are physical. The Aharonov–Bohm effect is therefore a rotation in the internal gauge space of a vacuum configuration described by the O(3) group, and not the U(1) group, where terms such as (215) are random.

VII. EXPLANATION OF MAGNETO-OPTICS AND OTHER EFFECTS USING O(3) ELECTRODYNAMICS

The subject of O(3) electrodynamics was initiated through the inference of the $\boldsymbol{B}^{(3)}$ field [11] from the inverse Faraday effect (IFE), which is the magnetization of matter using circularly polarized radiation [11–20]. The phenomenon of radiatively induced fermion resonance (RFR) was first inferred [15] as the resonance equivalent of the IFE. In this section, these two interrelated effects are reviewed and developed using O(3) electrodynamics. The IFE has been observed several times empirically [15], and the term responsible for RFR was first observed empirically as a magnetization by van der Ziel et al. [37] as being proportional to the conjugate product $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ multiplied by the Pauli matrix

σ in europium-ion-doped glasses. Good agreement was obtained [37] between theory and experiment, implying that the resonance equivalent of this term is present in nature. In other words, resonance can be induced between the states of the Pauli matrix by circularly polarized radiation. This resonance phenomenon is potentially of widespread utility as argued in this section because (1) it has a much higher resolution than ESR or NMR, (2) it has its own spectral fingerprint or chemical shift pattern, and (3) RFR can be observed without the use of superconducting magnets. In O(3) electrodynamics, it is essentially due to the product of the Pauli matrix with the $\boldsymbol{B}^{(3)}$ field and also exists [20] in O(3) quantum electrodynamics.

The IFE was inferred phenomenologically by Pershan [56] in terms of the conjugate product of circularly polarized electric fields, $\boldsymbol{E} \times \boldsymbol{E}^* = \boldsymbol{E}^{(1)} \times \boldsymbol{E}^{(2)}$. In O(3) electrodynamics, it is described from the first principles of gauge field theory by the inhomogeneous field equation (32), which can be expanded as

$$\partial_\mu \boldsymbol{H}^{\mu\nu(1)*} = \boldsymbol{J}^{\nu(1)*} + ig\boldsymbol{A}_\mu^{(2)} \times \boldsymbol{H}^{\mu\nu(3)} \tag{216}$$

$$\partial_\mu \boldsymbol{H}^{\mu\nu(2)*} = \boldsymbol{J}^{\nu(2)*} + ig\boldsymbol{A}_\mu^{(3)} \times \boldsymbol{H}^{\mu\nu(1)} \tag{217}$$

$$\partial_\mu \boldsymbol{H}^{\mu\nu(3)*} = \boldsymbol{J}^{\nu(3)*} + ig\boldsymbol{A}_\mu^{(1)} \times \boldsymbol{H}^{\mu\nu(2)} \tag{218}$$

that is, as three cyclically symmetric equations in the $O(3)$ symmetry basis ((1),(2),(3)) without empiricism. In order to make further progress, a constitutive relation must be used, as follows, but there is no need to assume the existence of $\boldsymbol{E} \times \boldsymbol{E}^*$ empirically. This is proportional to $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ which is part of the fundamental definition of the O(3) field tensor [11–20]. The constitutive relation used is [20]

$$\boldsymbol{H}^{\mu\nu(3)*} = \varepsilon \boldsymbol{G}^{\mu\nu(3)*} \tag{219}$$

so that

$$\boldsymbol{H}^{(3)*} = -i\frac{g}{\mu}\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} \tag{220}$$

where ϵ and μ are the electric permittivity and magnetic permeability of the sample being magnetized by a circularly polarized electromagnetic field whose signature, the third Stokes parameter, is proportional to $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ and therefore to $\boldsymbol{B}^{(3)}$ (Section III). If the vacuum configuration is assumed to be described by an O(3) group, it follows that the inverse Faraday effect is due to $\boldsymbol{B}^{(3)}$, and is empirical evidence for $\boldsymbol{B}^{(3)}$, leading to the development of O(3) electrodynamics.

The magnetization in the IFE is now defined as

$$\partial_\mu \boldsymbol{H}^{\mu\nu(1)*} = \boldsymbol{J}^{\nu(1)*} + \Delta\boldsymbol{J}^{\nu(1)*} \tag{221}$$

where

$$\Delta\boldsymbol{J}^{\nu(1)*} = ig\varepsilon\boldsymbol{A}_\mu^{(2)} \times \boldsymbol{G}^{\mu\nu(3)} \tag{222}$$

It can be worked out precisely [15] in an electron gas for a visible frequency laser such as that used by van der Ziel et al. [37]. The magnetic flux density set up in the electron gas is

$$B^{(3)}_{\text{sample}} = \frac{N}{V}\left(\frac{\mu_0 e^3 c^2 B^{(0)}}{2m^2\omega^3}\right) B^{(3)}_{\text{free space}} \tag{223}$$

where there are N electrons in a volume V, and where m is the mass of the electron. It is inversely proportional to the cube of the angular frequency of the circularly polarized laser. The free-space value of $\boldsymbol{B}^{(3)}$ is

$$B^{(3)}_{\text{free space}} = \left(\frac{\mu_0 I}{c}\right)^{1/2} \tag{224}$$

in terms of the intensity I (W/m^2) of the laser and so

$$B^{(3)}_{\text{sample}} = \frac{N}{V}\left(\frac{\mu_0^2 e^3 c}{2m^2}\right)\frac{I}{\omega^3} e^{(3)} \tag{225}$$

For example, for a pulsed Nd-YaG laser [57] where $I = 5.5 \times 10^{12}$ W/m^2, and $\omega = 1.77 \times 10^{16}$ rad/s, we obtain

$$|\boldsymbol{B}^{(3)}_{\text{sample}}| = 1.06 \times 10^{-35}\frac{N}{V}$$
$$\approx 10^{-9}\,\text{T} = 10^{-5}\,\text{G} \tag{226}$$

which for $N/V = 10^{26}\,\text{m}^{-3}$ (Avogadro's number) is the same order of magnitude as that observed experimentally by van der Ziel et al. [37] in the first inverse Faraday effect experiment. More generally, g/μ is a frequency dependent hyper polarizability [58], giving the possibility of the as yet undeveloped IFE spectroscopy with its characteristic [58] spectral fingerprint. IFE spectroscopy is magnetization near optical resonance caused by the $\boldsymbol{B}^{(3)}$ field in O(3) electrodynamics and is potentially as useful as infrared or Raman spectroscopy.

We can write Eq. (216) as

$$\partial_\mu \boldsymbol{H}^{\mu\nu(1)*} = \boldsymbol{J}^{\nu(1)*} + \Delta\boldsymbol{J}^{\nu(1)*} \tag{227}$$

where the transverse current can be developed as

$$\Delta\boldsymbol{J}^{\nu(1)*} = \varepsilon g^2 \boldsymbol{A}_\mu^{(2)} \times (\boldsymbol{A}^{\mu(1)} \times \boldsymbol{A}^{\nu(2)}) \tag{228}$$

causes a signal in an induction coil due to the vacuum $\boldsymbol{B}^{(3)}$ field, a component of $\boldsymbol{G}^{\mu\nu(3)}$. This transverse current causes the inverse Faraday effect as observed experimentally in an induction coil [37].

The explanation of the IFE in the Maxwell–Heaviside theory relies on phenomenology that is self-inconsistent. The reason is that $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ is introduced phenomenologically [56] but the same quantity (Section III) is discarded in U(1) gauge field theory, which is asserted in the received view to be the Maxwell–Heaviside theory. In O(3) electrodynamics, the IFE and third Stokes parameter are both manifestations of the $\boldsymbol{B}^{(3)}$ field proportional to the conjugate product that emerges from first principles [11–20] of gauge field theory, provided the internal gauge space is described in the basis ((1),(2),(3)).

Equation (228) can be developed further using the following result:

$$\boldsymbol{F} \times (\boldsymbol{G} \times \boldsymbol{H}) = \boldsymbol{G}(\boldsymbol{F} \cdot \boldsymbol{H}) - \boldsymbol{H}(\boldsymbol{F} \cdot \boldsymbol{G}) \tag{229}$$

This vector relation shows that

$$\begin{aligned} \boldsymbol{A}_{\mu}^{(2)} \times (\boldsymbol{A}^{\mu(1)} \times \boldsymbol{A}^{\nu(2)}) &= \boldsymbol{A}^{\mu(1)}(\boldsymbol{A}_{\mu}^{(2)} \cdot \boldsymbol{A}^{\nu(2)}) - \boldsymbol{A}^{\nu(2)}(\boldsymbol{A}_{\mu}^{(2)} \cdot \boldsymbol{A}^{\mu(1)}) \\ &= -A^{(0)2}\boldsymbol{A}^{\nu(2)} \end{aligned} \tag{230}$$

Using

$$g = \frac{\kappa}{A^{(0)}} = \frac{\omega}{cA^{(0)}} \tag{231}$$

it is found that

$$\Delta \boldsymbol{J}^{\nu(2)} = -\frac{\varepsilon}{c}\omega^2 \boldsymbol{A}^{\nu(2)} \tag{232}$$

On the one-electron level, the 4-current can be written in terms of the energy momentum:

$$\Delta \boldsymbol{J}^{\nu(2)} = \frac{e}{mcV}\boldsymbol{p}^{\nu(2)} \tag{233}$$

defined through the minimal prescription. From Eqs. (232) and (233), we obtain

$$\varepsilon = \frac{\chi}{c^2 V} = -\frac{e^2}{m\omega^2 V} \tag{234}$$

where χ is the one-electron susceptibility.

This result is self-consistent with the demonstration [15] that the IFE can be described through χ by using the Hamilton–Jacobi equation for one electron in the classical electromagnetic field, but the O(3) derivation is far simpler. The current $\Delta \boldsymbol{J}^{(2)}$ is due to the field-induced transverse electronic linear momentum [20].

Consider now the development of Eq. (218). From Eq. (219)

$$\partial_{\mu} \boldsymbol{H}^{\mu\nu(3)*} = \boldsymbol{0} \tag{235}$$

and so

$$\boldsymbol{J}^{\nu(3)*} = -ig\boldsymbol{A}_{\mu}^{(1)} \times \boldsymbol{H}^{\mu\nu(2)} \tag{236}$$

Equation (235) follows from the theoretical and experimental finding that [11–20]

$$\frac{\partial \boldsymbol{B}^{(3)}}{\partial t} = \nabla \times \boldsymbol{B}^{(3)} = \boldsymbol{0} \tag{237}$$

in the vacuum. In Eq. (236), $\boldsymbol{J}^{\nu(3)*}$ is induced self-consistently in the IFE as follows.

Use the constitutive relation

$$\boldsymbol{H}^{\mu\nu(2)} = \varepsilon \boldsymbol{G}^{\mu\nu(2)} \tag{238}$$

and the definition

$$\boldsymbol{G}^{\mu\nu(2)} = c(\partial^{\mu}\boldsymbol{A}^{\nu(2)} - \partial^{\nu}\boldsymbol{A}^{\mu(2)} - ig\boldsymbol{A}^{\mu(3)} \times \boldsymbol{A}^{\nu(1)}) \tag{239}$$

with

$$\boldsymbol{A}_{\mu}^{(1)} \times (\boldsymbol{A}^{\mu(3)} \times \boldsymbol{A}^{\nu(1)}) = 0 \tag{240}$$

Set $\nu = 3$ in Eq. (236) to obtain

$$\boldsymbol{J}^{3(3)*} = 2ig\varepsilon \boldsymbol{A}^{(1)} \times \boldsymbol{B}^{(2)} \tag{241}$$

which is the current induced by the nonlinear cross-product $\boldsymbol{A}^{(1)} \times \boldsymbol{B}^{(2)}$. Using

$$\boldsymbol{B}^{(2)} = \nabla \times \boldsymbol{A}^{(2)} \tag{242}$$

this current is equal to that of the orbital IFE [36]

$$\begin{aligned} \boldsymbol{J}^{3(3)*} &= ig\varepsilon c\kappa \boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} \\ &= -\frac{e^2}{m\omega V}\boldsymbol{B}^{(3)} \end{aligned} \tag{243}$$

and so $\boldsymbol{J}^{3(3)*}$ is the magnetization current due to $\boldsymbol{B}^{(3)}$ for one electron. There is no longitudinal source current in Eq. (218) because the source current of circularly polarized radiation is necessarily transverse, the charge in the source goes around in a circle whose plane is perpendicular to the (3) axis and the source does not move forward along the (3) axis. There is therefore no current in the (3) axis, that is, no source current in the (3) axis as argued.

The technique of RFR is simply the resonance equivalent of the IFE as argued already, but is potentially of major utility. The techniques of nuclear magnetic resonance (NMR), electron spin resonance (ESR), and magnetic resonance imaging (MRI), are widely used in contemporary analytical science and medicine, and all rely on the principle of fermion resonance induced between states of the Pauli spinor. The resonance pattern is distinct for each sample, and in MRI, an image can be built up. Optical methods have been used to enhance the subject considerably [59–65] using laser frequencies. In conventional ESR and NMR, the resonance is induced by a circularly polarized radio frequency (RF) or microwave frequency coil, and the population of the energy states of the Pauli matrices of electron or proton are separated by a very tiny amount by a powerful and homogeneous magnet, usually a superconducting magnet. The resolving power of these techniques is limited by the magnetic flux density of the magnet. This limitation can be removed by replacing the magnet with a circularly polarized electromagnetic field, resulting in RFR. In theory, the latter technique has a much greater resolving capability than does NMR or ESR and can be developed into an MRI technique based on the same principle, the induction of resonance between the states of the Pauli matrix by a circularly polarized RF field. The multi-million-dollar superconducting magnet of a conventional ESR or NMR spectrometer could be replaced in principle by an ordinary RF field.

This result emerges self-consistently at all levels of physics, from the classical nonrelativistic to the quantum electrodynamic. On the nonrelativistic classical level, the technique of RFR is due to the interaction of $\boldsymbol{B}^{(3)}$ with the Pauli matrix. One way of demonstrating this result, which has been observed empirically [37], is to extend the minimal prescription to complex $\boldsymbol{A}$, starting [66] with the Newtonian kinetic energy of the classical electron

$$H_{\mathrm{KE}} = \frac{1}{2m}\boldsymbol{p}\cdot\boldsymbol{p} \tag{244}$$

where $\boldsymbol{p}$ is its linear momentum and m is its mass. The electron interacts with the classical electromagnetic field through the O(3) covariant derivative written in momentum space, in other words, with the minimal prescription with complex $\boldsymbol{A}$, with $\boldsymbol{A}^{(1)} = \boldsymbol{A}^{(2)*}$. The interaction kinetic energy is therefore the real part of:

$$H_{\mathrm{KE}} = \frac{1}{2m}(\boldsymbol{p} - e\boldsymbol{A}^{(1)})\cdot(\boldsymbol{p} - e\boldsymbol{A}^{(2)}) \tag{245}$$

where $\boldsymbol{A}^{(1)}$ and $\boldsymbol{A}^{(2)}$ are complex conjugate transverse plane waves for simplicity of argument. The energy in Eq. (245) can be written out as

$$H_{\mathrm{KE}} = \frac{1}{2m}\boldsymbol{p}\cdot\boldsymbol{p} - \frac{e}{2m}\mathrm{Re}\,(\boldsymbol{A}^{(1)}\cdot\boldsymbol{p}) - \frac{e}{2m}\mathrm{Re}\,(\boldsymbol{p}\cdot\boldsymbol{A}^{(2)}) + \frac{e^2}{2m}\boldsymbol{A}^{(1)}\cdot\boldsymbol{A}^{(2)} \tag{246}$$

a well-known result of numerous textbooks [67]. The only difference is one of notation. In the textbooks, $\boldsymbol{A} = \boldsymbol{A}^{(1)}$ and $\boldsymbol{A}^* = \boldsymbol{A}^{(2)}$. In order to derive the RFR term, we use Pauli matrices as a basis for three-dimensional space following Sakurai [68] in his Eq. (3.18). The interaction between the classical electron and the classical electromagnetic field in this basis is described on the classical level by

$$H_{\mathrm{KE}} = -\frac{1}{2m}\boldsymbol{\sigma}\cdot(\boldsymbol{p} - e\boldsymbol{A}^{(1)})\boldsymbol{\sigma}\cdot(\boldsymbol{p} - e\boldsymbol{A}^{(2)}) \tag{247}$$

and consists of four terms: (1) the magnetic dipole term

$$H_1 = -\frac{e}{2m}\boldsymbol{p}\cdot(\boldsymbol{A}^{(1)} + \boldsymbol{A}^{(2)}) = \frac{e}{2m}\boldsymbol{m}_0\cdot\mathrm{Re}\,\boldsymbol{B} \tag{248}$$

where $\boldsymbol{m}_0$ is the magnetic dipole moment of the electron or proton and $\mathrm{Re}\,\boldsymbol{B}$ is the real part of the magnetic component of the electromagnetic field, (2) the spin–flip term

$$H_2 = -i\frac{e}{2m}\boldsymbol{\sigma}\cdot\boldsymbol{p}\times(\boldsymbol{A}^{(2)} - \boldsymbol{A}^{(1)}) \tag{249}$$

which, for an electron or proton moving in the Z axis, can be expressed as

$$H_2 = -e\frac{A^{(0)}}{\sqrt{2}}p_Z\boldsymbol{\sigma}_Z\cdot(\boldsymbol{j}\cos\phi + \boldsymbol{i}\sin\phi) \tag{250}$$

where

$$\phi = \omega t - \kappa Z = \omega\left(t - \frac{Z}{c}\right) \tag{251}$$

[it can be seen that if $\phi = 0$, the Pauli matrix (or "spin") points in the Y axis; when $\phi = \pi/2$, in the X axis; when $\phi = \pi$, in the $-Y$ axis; when $\phi = 3\pi/2$, in the $-X$ axis; and when $\phi = 2\pi$, back in the Y axis].

Thirdly, the polarizability term which appears in the textbooks [67], is given by

$$H_3 = \frac{e^2}{2m}\boldsymbol{A}^{(1)}\cdot\boldsymbol{A}^{(2)} = \frac{e^2}{2m}A^{(0)2} \tag{252}$$

and is the basis [69] of susceptibility theory, and (4) the RFR term, which is missing from the textbooks, is given by the real-valued expression

$$H_4 = i\frac{e^2}{2m}\boldsymbol{\sigma}\cdot\boldsymbol{A}^{(1)}\times\boldsymbol{A}^{(2)} = -\frac{e^2}{2m}A^{(0)2}\boldsymbol{\sigma}\cdot\boldsymbol{k} \tag{253}$$

All four terms have been observed empirically. Terms 1–3 are well known, and term 4 has been observed as a magnetization in europium ion doped glasses by van der Ziel et al. [37] as argued already. The RFR term therefore emerges self-consistently with three other well-known and well-observed terms from what is effectively the O(3) covariant derivative.

This analysis of the classical non-relativistic level can be confirmed by writing the four Stokes parameters [70] in terms of potentials in free space:

$$\begin{aligned} S_0 &= A_X^{(1)}A_X^{(2)} + A_Y^{(1)}A_Y^{(2)} \\ S_1 &= A_X^{(1)}A_X^{(2)} - A_Y^{(1)}A_Y^{(2)} \\ S_2 &= -(A_X^{(1)}A_Y^{(2)} + A_Y^{(1)}A_X^{(2)}) \\ S_3 &= -i(A_X^{(1)}A_Y^{(2)} - A_Y^{(1)}A_X^{(2)}) \end{aligned} \tag{254}$$

For elliptically polarized electromagnetic radiation

$$S_0^2 = S_1^2 + S_2^2 + S_3^2 = S_3^2 \tag{255}$$

and for circularly polarized radiation

$$S_0 = \pm S_3 \tag{256}$$

Therefore, the existence of $\boldsymbol{A}^{(1)}\cdot\boldsymbol{A}^{(2)}$, which is proportional to S_0 and to field intensity, implies the existence of $\pm i\boldsymbol{A}^{(1)}\times\boldsymbol{A}^{(2)}$, which is an observable proportional to S_3. If the light intensity tensor [70] is defined as

$$\rho_{\alpha\beta} = \frac{A_\alpha^{(1)}A_\beta^{(2)}}{A^{(0)2}} \tag{257}$$

then from Eqs. (254) and (256), in circular polarization:

$$\rho_{\alpha\beta} = \frac{1}{2A^{(0)2}}\begin{bmatrix} S_0 & iS_3 \\ -iS_3 & S_0 \end{bmatrix} \tag{258}$$

Now define the Pauli matrices [6,68]

$$\sigma_X \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \sigma_Y \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}; \sigma_Z \equiv \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \tag{259}$$

which are interrelated by the following cyclic relation:

$$\left[\frac{\sigma_X}{2}, \frac{\sigma_Y}{2}\right] = i\frac{\sigma_Z}{2} \tag{260}$$

The intensity tensor becomes

$$\rho_{\alpha\beta} = \frac{1}{2A^{(0)2}}(S_0 - i\boldsymbol{\sigma}_Z\cdot\boldsymbol{A}^{(1)}\times\boldsymbol{A}^{(2)}) \tag{261}$$

showing that the RFR term occurs in the fundamental definition of this tensor for circularly polarized radiation. The RFR term is as fundamental as the intensity itself, through Eq. (256).

For practical purposes, the critically important feature of the RFR term is its dependence for a given beam intensity on the inverse of frequency squared of the beam. This means that the spectral resolution [15] in RFR has the same dependence. This critically important feature is shown straightforwardly from the O(3) relations

$$\boldsymbol{B}^{(1)} = \nabla\times\boldsymbol{A}^{(1)}; \qquad \boldsymbol{B}^{(2)} = \nabla\times\boldsymbol{A}^{(2)} \tag{262}$$

so from Eq. (188), the magnetic transverse plane waves are

$$\boldsymbol{B}^{(1)} = \boldsymbol{B}^{(2)*} = \frac{B^{(0)}}{\sqrt{2}}(\boldsymbol{i}\boldsymbol{i} + \boldsymbol{j})e^{i(\omega t - \kappa Z)} \tag{263}$$

and the electric transverse plane waves are

$$\boldsymbol{E}^{(1)} = \boldsymbol{E}^{(2)*} = \frac{E^{(0)}}{\sqrt{2}}(\boldsymbol{i} - \boldsymbol{i}\boldsymbol{j})e^{i(\omega t - \kappa Z)} \tag{264}$$

an analysis that results in the relation between conjugate products [15]

$$\boldsymbol{A}^{(1)}\times\boldsymbol{A}^{(2)} = \frac{c^2}{\omega^2}\boldsymbol{B}^{(1)}\times\boldsymbol{B}^{(2)} = \frac{1}{\omega^2}\boldsymbol{E}^{(1)}\times\boldsymbol{E}^{(2)} \tag{265}$$

Expressing $\boldsymbol{B}^{(1)}\cdot\boldsymbol{B}^{(2)}$ in terms of beam power density (I in W/m^2) results in

$$\boldsymbol{B}^{(1)}\times\boldsymbol{B}^{(2)} = i\frac{\mu_0}{c}I\boldsymbol{e}^{(3)*} \tag{266}$$

where $\boldsymbol{\mu}_0$ is the vacuum permeability in SI units.

The basis of the RFR technique is that a probe photon at a resonance angular frequency ω_{res} can be absorbed under the resonance condition

$$\hbar\omega_{res} = \frac{e^2c^2B^{(0)2}}{2m\omega^2}(1-(-1)) \tag{267}$$

defined by the transition from the negative to the positive states of the Pauli matrix σ_Z. This is precisely analogous to the basic mechanism of ESR and NMR and is a spectral absorption. The RFR resonance frequency is therefore

$$f_{res} = \frac{\omega_{res}}{2\pi} = \left(\frac{e^2\mu_0 c}{2\pi\hbar m}\right)\frac{I}{\omega^2} \tag{268}$$

and is inversely proportional to the square of the angular frequency ω_{res} of the circularly polarized pump electromagnetic field replacing the superconducting magnet of ESR, NMR, and MRI [69].

For ^{1}H proton resonance, the result (268) is adjusted empirically for the different experimentally observed g factors of the electron (2.002) and proton (5.5857). A more complete theory must rest on the internal structure of the proton or other nuclei. The basic theory of RFR is straightforward, however, and a term emerges with three other well-known terms. In principle, RFR can investigate nuclear properties using microwave or RF generators instead of multi-million superconducting magnets.

For proton resonance therefore, the RFR equation [15] is

$$\omega_{res} = \left(\frac{5.5857e^2\mu_0 c}{2.002\hbar m}\right)\frac{I}{\omega^2} = 1.532\times10^{25}\frac{I}{\omega^2} \tag{269}$$

and some data from this equation are shown in Table I, where it is seen that RFR proton resonances can be far higher than those in conventional NMR. The

TABLE I
RFR Frequencies from Eq. (27) for the Proton for $I = 10$ W/cm^2

Pump Frequency	Resonance Frequency
5000 cm^{-1} (visible)	0.28 Hz
500 cm^{-1} (infrared)	28.0 Hz
1.8 GHz	1.8 GHz (autoresonance)
1.0 GHz (microwave)	6.18 GHz
0.1 GHz (RF)	20.6 cm^{-1} (far infrared)
10.0 MHz (RF)	2,060 cm^{-1} (infrared)
1.0 MHz (RF)	206,000 cm^{-1} (ultraviolet)

concomitant resolution in RFR is also far higher than in NMR, and as will be shown, the RFR technique has its own spectral fingerprint or chemical shift pattern. The spinup–spindown population difference in RFR is also orders of magnitude greater [15] than in NMR, and because of this, the homogeneity of the pump electromagnetic field is not critical theoretically. This is another advantage of the RFR technique. Any remaining objection to the existence of RFR is removed by the empirical fact that the term (253) has been observed experimentally as a magnetization [37]. The only remaining experimental challenge is to induce resonance between the states of σ in term (253).

If RFR is applied to the electron, the same overall advantage is obtained; the equivalent of Eq. (269) is

$$\omega_{res} = 1.007\times10^{28}\frac{I}{\omega^2} \tag{270}$$

These conclusions can be obtained on the nonrelativistic level, and it is possible in theory to practice proton and electron spin resonance without permanent magnets, at much higher resolution, without the need for very high homogeneity, and with a novel chemical shift pattern, or spectral fingerprint, determined by a site-specific molecular property tensor, to be described later in this section.

On the classical relativistic level, the starting point is the Einstein equation

$$p^\mu p_\mu = m^2c^2 \tag{271}$$

where p^μ and p_μ are energy/momentum 4-vectors. In order to demonstrate RFR, Eq. (271) is rewritten in the basis (260) using the gamma matrices [68]

$$\gamma^\mu p_\mu \gamma^\mu p_\mu = m^2c^2 \tag{272}$$

and the classical electromagnetic field is introduced through the O(3) minimal prescription:

$$\gamma^\mu(p_\mu - eA^{(1)}_\mu)\gamma^\mu(p_\mu - eA^{(2)}_\mu) = m^2c^2 \tag{273}$$

In the compact Feynman slash notation [68], Eq. (272) becomes

$$\not{p}\not{p} = m^2c^2 \tag{274}$$

and Eq. (273) becomes

$$(\not{p} - e\not{A}^{(1)})(\not{p} - e\not{A}^{(2)}) = m^2c^2 \tag{275}$$

This is the classical relativistic expression for the interaction of an electron or proton with the classical electromagnetic field. The quantized version of Eq. (275) is the van der Waerden equation [1] as described by Sakurai [68] in his Eq. (3.24). The RFR term in relativistic classical physics is contained within the term $e^2 \not{A}^{(1)} \not{A}^{(2)}$, a result that can be demonstrated by expanding this term as follows

$$\begin{aligned} e^2 \not{A}^{(1)} \not{A}^{(2)} &= e^2 \gamma^\mu A_\mu^{(1)} \gamma^\mu A_\mu^{(2)} \\ &= e^2 (\gamma^0 A_0^{(1)} - \boldsymbol{\gamma} \cdot \boldsymbol{A}^{(1)})(\gamma^0 A_0^{(2)} - \boldsymbol{\gamma} \cdot \boldsymbol{A}^{(2)}) \end{aligned} \tag{276}$$

Using the well-known relation between the gamma and Pauli matrices [68]

$$\begin{aligned} (\boldsymbol{\gamma} \cdot \boldsymbol{p})(\boldsymbol{\gamma} \cdot \boldsymbol{p}) &= \begin{bmatrix} \mathbf{0} & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & \mathbf{0} \end{bmatrix} \cdot \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix} \begin{bmatrix} \mathbf{0} & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & \mathbf{0} \end{bmatrix} \cdot \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix} \\ &= \begin{bmatrix} (\boldsymbol{\sigma} \cdot \boldsymbol{p})(\boldsymbol{\sigma} \cdot \boldsymbol{p}) & \mathbf{0} \\ \mathbf{0} & (\boldsymbol{\sigma} \cdot \boldsymbol{p})(\boldsymbol{\sigma} \cdot \boldsymbol{p}) \end{bmatrix} \end{aligned} \tag{277}$$

it is found that

$$e^2 \not{A}^{(1)} \not{A}^{(2)} = e^2 (A_0^{(1)} A_0^{(2)} - \boldsymbol{A}^{(1)} \cdot \boldsymbol{A}^{(2)} - i \boldsymbol{\sigma} \cdot \boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}) \tag{278}$$

an expression that includes the RFR term

$$T_{\mathrm{RFR}} = -ie^2 \boldsymbol{\sigma} \cdot \boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} \tag{279}$$

On the nonrelativistic quantum level, both the time-independent and time-dependent Schrödinger equations can be used to demonstrate the existence of RFR. As shown by Sakurai [68], the time-independent Schrödinger–Pauli equation can be used to demonstrate ordinary ESR and NMR in the nonrelativistic quantum limit. This method is adopted here to demonstrate RFR in nonrelativistic quantum mechanics with the time-independent Schrödinger–Pauli equation [68]:

$$\hat{H} \psi = En \, \psi \tag{280}$$

where the Hamiltonian operator is

$$\hat{H} = \frac{1}{2m} (\boldsymbol{\sigma} \cdot \boldsymbol{p})(\boldsymbol{\sigma} \cdot \boldsymbol{p}) + V_0 \tag{281}$$

Here, V_0 is the potential energy, which, however, does not affect the RFR term. This method is first checked for its self-consistency using a real-valued potential function $\boldsymbol{A}$ corresponding to a static magnetic field, then the same equation is used to demonstrate the existence of the RFR term.

In a static magnetic field, the minimal prescription shows that the time-independent Schrödinger–Pauli equation of a fermion in a classical field is

$$\hat{H} \rightarrow \frac{1}{2m} (\boldsymbol{\sigma} \cdot (\boldsymbol{p} + e\boldsymbol{A}))(\boldsymbol{\sigma} \cdot (\boldsymbol{p} + e\boldsymbol{A})) + V \tag{282}$$

The usual ESR or NMR term is obtained from

$$\begin{aligned} \hat{H} \psi &= i \frac{e}{2m} (\boldsymbol{\sigma} \cdot \boldsymbol{p} \times \boldsymbol{A} + \boldsymbol{\sigma} \cdot \boldsymbol{A} \times \boldsymbol{p}) \psi + \cdots \\ &= \frac{e\hbar}{2m} \boldsymbol{\sigma} \cdot (\nabla \times (\boldsymbol{A} \psi) + \boldsymbol{A} \times \nabla \psi) + \cdots \\ &= \frac{e\hbar}{2m} \boldsymbol{\sigma} \cdot ((\nabla \times \boldsymbol{A}) \psi + (\nabla \psi) \times \boldsymbol{A} + \boldsymbol{A} \times (\nabla \psi)) + \cdots \\ &= \frac{e\hbar}{2m} \boldsymbol{\sigma} \cdot \boldsymbol{B} \psi + \cdots \end{aligned} \tag{283}$$

and is the famous "half-integral spin" first derived by Dirac in relativistic quantum mechanics. However, it also exists in nonrelativistic quantum mechanics as just shown [68], but is a purely quantum term with no classical equivalent because it depends on the operator relation:

$$\boldsymbol{p} \rightarrow -i\hbar \nabla \tag{284}$$

This is the spin Zeeman effect and in perturbation theory [69] gives the nonzero ground-state energy:

$$En = \frac{e\hbar}{2m} \langle 0 | \boldsymbol{\sigma} \cdot \boldsymbol{B} | 0 \rangle \neq 0 \tag{285}$$

It is the basis for all ESR and NMR.

To obtain the RFR term on this level, the same method is used for complex-valued $\boldsymbol{A}$. This gives an extra classical term, or expectation value, which can be written as

$$En = \frac{ie^2}{2m} \boldsymbol{\sigma} \cdot \boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} \tag{286}$$

Perturbation theory gives the ground-state term

$$En = \frac{ie^2}{2m} \langle 0 | \boldsymbol{\sigma} \cdot \boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} | 0 \rangle \tag{287}$$

which is again classical and real-valued. It has the inverse square frequency dependence described already and exists on the nonrelativistic quantum level according to the correspondence principle. Therefore the RFR term is unlike the ESR or NMR terms in that the RFR term is classical while the other two are quantum.

The time-dependent Schrödinger equations

$$H\Psi = i\hbar\frac{\partial\Psi}{\partial t} \tag{288}$$

$$H = H^{(0)} + H^{1}(t) \tag{289}$$

$$\Psi(t) = \Psi_n e^{-iEnt} \tag{290}$$

can also be applied to the RFR phenomenon. A two-level system can be considered to consist of the fermion in its spinup and spindown states (states of the Pauli matrix). The unperturbed two-level system has energies E_1 and E_2 and eigenfunctions ψ_1 and ψ_2. These are solutions of [69]:

$$H^{(0)}\psi_n = E_n\psi_n \tag{291}$$

In the presence of a time-dependent perturbation $H^{(1)}(t)$, the state of the system is described by a linear combination of basis functions:

$$\Psi(t) = a_1(t)\Psi_1(t) + a_2(t)\Psi_2(t) \tag{292}$$

and the system evolves under the influence of the perturbation, so a_1 and a_2 are also time-dependent. If it starts as state 1, it may evolve to state 2. The probability at any instant that the system is in state 2 is $a_2(t)a_2^*(t)$, and the probability that it remains in state 1 is.

$$a_1(t)a_1^*(t) = 1 - a_2(t)a_2^*(t) \tag{293}$$

Therefore

$$\begin{aligned} H\Psi &= a_1H^{(0)}\Psi_1 + a_1H^{(1)}(t)\Psi_1 + a_2H^{(0)}\Psi_2 + a_2H^{(1)}(t)\Psi_2 \\ &= i\hbar\frac{\partial}{\partial t}(a_1\Psi_1 + a_2\Psi_2) \\ &= i\hbar a_1\frac{\partial\Psi_1}{\partial t} + i\hbar\frac{\partial a_1}{\partial t}\Psi_1 + i\hbar a_2\frac{\partial\Psi_2}{\partial t} + i\hbar\frac{\partial a_2}{\partial t}\Psi_2 \end{aligned} \tag{294}$$

Each basis function satisfies

$$H^{(0)}\Psi_n = i\hbar\frac{\partial\Psi_n}{\partial t} \tag{295}$$

and therefore

$$a_1H^{(1)}(t)\Psi_1 + a_2H^{(1)}(t)\Psi_2 = i\hbar\dot{a}_1\Psi_1 + i\hbar\dot{a}_2\Psi_2 \tag{296}$$

This equation is

$$a_1H^{(1)}(t)\psi_1e^{-iE_1t/\hbar} + a_2H^{(1)}(t)\psi_2e^{-iE_2t/\hbar} = i\hbar\dot{a}_1\psi_1e^{-iE_1t/\hbar} + i\hbar\dot{a}_2\psi_2e^{-iE_2t/\hbar} \tag{297}$$

and can be multiplied through by ψ_i^* and integrated over all space. Since ψ_1 and ψ_2 are orthonormal

$$a_1H_{11}^{(1)}(t)e^{-iE_1t/\hbar} + a_2H_{12}^{(1)}(t)e^{-iE_2t/\hbar} = i\hbar\dot{a}_1e^{-iE_1t/\hbar} \tag{298}$$

Similarly, multiply through by ψ_2^*:

$$a_1H_{21}^{(1)}(t)e^{-iE_1t/\hbar} + a_2H_{22}^{(1)}(t)e^{-iE_2t/\hbar} = i\hbar\dot{a}_2e^{-iE_2t/\hbar} \tag{299}$$

Here

$$H_{ij}^{(1)}(t) \equiv \int\psi_i^*H^{(1)}(t)\psi_j\,d\tau \tag{300}$$

and ψ_1 and ψ_2 are time-dependent parts of the wavefunction of states 1 and 2 of the unperturbed fermion. Thus

$$H_{11}^{(1)}(t) \equiv \int\psi_1^*H^{(1)}(t)\psi_1\,d\tau \equiv \langle 1|H^{(1)}(t)|1\rangle \tag{301}$$

and so on.

At this point, the RFR Hamiltonian is inputted:

$$H^{(1)}(t) = i\frac{e^2}{2m}\boldsymbol{\sigma}\cdot\boldsymbol{A}^{(1)}\times\boldsymbol{A}^{(2)} \tag{302}$$

so the existence of $H_{11}^{(1)}(t)$ and $H_{12}^{(1)}(t)$ and so on depends on the properties of σ between fermion states.

Define

$$\boldsymbol{S} \equiv \frac{1}{2}\hbar\boldsymbol{\sigma} \tag{303}$$

and

$$\alpha \equiv \left|\frac{1}{2}, \frac{1}{2}\right\rangle \equiv \text{state 1}$$
$$\beta \equiv \left|\frac{1}{2}, -\frac{1}{2}\right\rangle \equiv \text{state 2} \tag{304}$$

then

$$S_Z\alpha = \frac{1}{2}\hbar\alpha; \qquad S_Z\beta = -\frac{1}{2}\hbar\beta \tag{305}$$

and

$$\langle\alpha|S_Z|\alpha|\rangle = \frac{1}{2}\hbar = \frac{1}{2}\hbar\int\alpha^*\alpha d\tau$$
$$\langle\alpha|S_Z|\beta\rangle = 0 = -\frac{1}{2}\hbar\int\alpha^*\beta d\tau \tag{306}$$

Now define

$$\boldsymbol{B}^{(3)*} \equiv -i\frac{e}{\hbar}\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} \tag{307}$$

and

$$H^{(1)}(t) = -\frac{e}{m}\boldsymbol{S}\cdot\boldsymbol{B}^{(3)} = -\frac{e}{m}S_Z B_Z^{(3)} \tag{308}$$

So Eqs. (252) and (253) become

$$a_1 H_{11}^{(1)}(t) = i\hbar\dot{a}_1 \tag{309}$$
$$a_2 H_{22}^{(1)}(t) = i\hbar\dot{a}_2 \tag{310}$$

because

$$H_{11}^{(1)}(t) = -\frac{e\hbar}{2m}B_Z^{(3)}; \qquad H_{12}^{(1)}(t) = 0$$
$$H_{22}^{(2)}(t) = \frac{e\hbar}{2m}B_Z^{(3)}; \qquad H_{21}^{(1)}(t) = 0 \tag{311}$$

Equations (309) and (310) are decoupled differential equations of the form

$$\dot{a}_1 = i\frac{eB_Z^{(3)}}{2m}a_1; \qquad \dot{a}_2 = -i\frac{eB_Z^{(3)}}{2m}a_2 \tag{312}$$

where

$$B_Z^{(3)} \equiv \frac{e}{\hbar}A^{(0)2} \tag{313}$$

with the constraint:

$$a_1 a_1^* + a_2 a_2^* = 1 \tag{314}$$

A particular solution of Eqs. (312) and (313) is

$$a_1 = \frac{1}{\sqrt{2}}\exp\left(i\frac{etB_Z^{(3)}}{2m}\right); \quad a_2 = \frac{1}{\sqrt{2}}\exp\left(-i\frac{etB_Z^{(3)}}{2m}\right) \tag{315}$$

The perturbed wave function is therefore:

$$\Psi = \frac{\Psi_1}{\sqrt{2}}\exp\left(i\frac{etB_Z^{(3)}}{2m}\right) + \frac{\Psi_2}{\sqrt{2}}\exp\left(-i\frac{etB_Z^{(3)}}{2m}\right) \tag{316}$$

and

$$p_1 = a_1 a_1^* = 0.5$$
$$p_2 = a_2 a_2^* = 0.5 \tag{317}$$

The probability of finding the system in one state or the other remains constant at 50%, and:

$$\Psi = \frac{\Psi}{\sqrt{2}}\exp(i\omega_{\text{res}}t) + \frac{\Psi}{\sqrt{2}}\exp(-i\omega_{\text{res}}t) \tag{318}$$

where

$$\omega_{\text{res}} = \frac{eB_Z^{(3)}}{2m} \tag{319}$$

is the radiatively induced resonance frequency defined by

$$\hbar\omega_{\text{res}} = H^{(1)}(t) \tag{320}$$

The final result is:

$$\Psi = \frac{\Psi_1}{\sqrt{2}}e^{i\omega_{\text{res}}t} + \frac{\Psi_2}{\sqrt{2}}e^{-i\omega_{\text{res}}t} \tag{321}$$

where

$$H\Psi = i\hbar\frac{\partial\Psi}{\partial t} \tag{322}$$

which is a combination of states with energies $\pm\hbar\omega_{\rm res}$. The RFR term prepares or dresses the fermion in a combination of α and β spin states analogously with ESR or NMR.

On the relativistic quantum level, the Einstein equation becomes the van der Waerden equation [1,68] with the usual operator rules

$$\begin{aligned} p^\mu &\rightarrow i\hbar\partial^\mu \\ p_\mu &\rightarrow i\hbar\partial_\mu \end{aligned} \tag{323}$$

to give

$$(i\gamma^\mu\partial_\mu)(i\gamma^\mu\partial_\mu)\psi_W = \frac{m^2c^2}{\hbar^2}\psi_W \tag{324}$$

where ψ_W is a two component wave function as described by Sakurai [68] in his Eq. (3.24). The classical electromagnetic field is introduced into eq. (324) using O(3) covariant derivatives to give the term $e^2\not{A}^{(1)}\not{A}^{(2)}$ on the quantum relativistic level. The Dirac equation is obtained from the van der Waerden equation [68] using standard methods, and the two equations are equivalent. The RFR term was indeed first derived [15] using the Dirac equation.

On the level of quantum electrodynamics [17], a classical expression such as

$$H = \frac{e^2}{2m}(\boldsymbol{\sigma}\cdot\boldsymbol{A}^{(1)})(\boldsymbol{\sigma}\cdot\boldsymbol{A}^{(2)}) \tag{325}$$

becomes the interaction Hamiltonian

$$H = \frac{e^2}{4m\hbar\varepsilon_0 V}\sum_k\left(\frac{I}{\omega_k}a_k^+a_k + \sum_q\frac{\sigma^{(3)}}{\omega_q}(a_q^+a_{k-q} - a_qa_{k-q}^+)\right) \tag{326}$$

describing the exchange of a photon that results in the change of the spin of the electron. This process is equivalent [17] to the absorption of a photon in the atomic transition $i \rightarrow j$ and the absorption of a photon in the atomic transition $j \rightarrow i$.

The free Hamiltonian term quadratic in $\boldsymbol{B}^{(3)}$ must also be considered and is

$$H_1 = \frac{e}{2\omega_q\varepsilon_0 V}\sum_{k,k',q}(a_{k+q}^+a_ka_{k'-q}^+a_{k'}) \tag{327}$$

This term appears only in O(3) quantum electrodynamics and describes the interaction between four photons [17]: the absorption of photons with modes $k+q$ and $k'-q$ and the emission of photons with modes k and k'. This is a physical process where two photons interact and mutually exchange momenta, and is a process that is observable only in O(3) quantum electrodynamics. The effect has been observed empirically by Tam and Happer [71] in two interacting circularly polarized lasers and was explained using the concept of long range spins by Naik and Pradhan [72]. If the direction of the rotation of the polarization is the same, the two beams attract and vice versa. In O(3) quantum electrodynamics [17], the effect is a form of self-focusing or photon bunching that would result if the spins of the photons were aligned in the same direction, as observed empirically [71]. This result also suggests that O(3) quantum electrodynamics could account for light-squeezing effects and also photon anti-bunching if the photon spins were opposite.

The O(3) quantum electrodynamic equivalent of the RFR effect has been numerically analyzed by Crowell [17] using the Hamiltonian (327). Numerically, it is possible to consider only a finite number of photon modes, and the difference in energy between these modes is set equal to the difference between the two spin states of the fermion. More complex situations were also analyzed [17]. Crowell discovered a variety of effects numerically, including modified Rabi flopping, which has an inverse frequency dependence similar to that observed in the solid state in reciprocal noise [73]. The latter is also explained by Crowell [17] using a non-Abelian model. A variety of other effects of RFR on the quantum electrodynamical level was also reported numerically [17]. The overall result is that the occurrence, classically, of the $\boldsymbol{B}^{(3)}$ field means that there is a quantum electrodynamical Hamiltonian generated by the classical term proportional to $\frac{1}{2}B^{(3)2}$. This induces transitional behavior because it contributes to the dynamics of probability amplitudes [17]. The Hamiltonian is a quartic potential where the value of $\boldsymbol{B}^{(3)}$ determines the value of the potential. The latter has two minima: one where $\boldsymbol{B}^{(3)} = 0$ and the other for a finite value of the $\boldsymbol{B}^{(3)}$ field, corresponding to states that are invariants of the Lagrangian but not of the vacuum.

Another potentially useful feature of RFR is that its site specificity is different from that of NMR or ESR, because RFR relies on a different molecular property tensor [74]. In a precursor to RFR, called optical NMR (ONMR) [59–65], site specificity has been demonstrated at a spatial resolution corresponding to quantum dots, a dramatic demonstration of the enhancement possible with the use of circularly polarized lasers or circularly polarized microwave fields such as in RFR.

The calculation of the chemical shift in RFR is straightforward [74] and relies on a calculation of the second-order perturbation energy (SI units)

$$En = \sum_n\frac{\langle 0|H|n\rangle\langle n|H|0\rangle}{\hbar\omega_{0n}} \tag{328}$$

with the perturbation Hamiltonian

$$H = \frac{1}{2m}(\boldsymbol{p} + e(\boldsymbol{A} + \boldsymbol{A}_N))^2 + V \tag{329}$$

where

$$\boldsymbol{A}_N = \frac{\mu_0}{4\pi r^3}\boldsymbol{m}_N \times \boldsymbol{r} \tag{330}$$

is the vector potential [69] due to the nuclear dipole moment $\boldsymbol{m}_N$. The perturbation term relevant to the RFR chemical shift is the one photon off-resonance population term [74], which is by far the dominant chemical shift term (where c.c. = complex conjugates):

$$En = i\frac{e^3}{m^2\hbar\omega_{0n}}\sum_n \langle 0|\boldsymbol{p}\cdot\boldsymbol{A}|n\rangle\langle n|\boldsymbol{A}_N\cdot\boldsymbol{A}^*|0\rangle + \text{c.c.} \tag{331}$$

The transition electric dipole moment is defined by [74]

$$\langle 0|\mu|n\rangle = \frac{e}{m\omega_{0n}}\langle 0|\boldsymbol{p}|n\rangle \tag{332}$$

and the vector relations:

$$\begin{aligned} &i(\boldsymbol{\mu}\times(\boldsymbol{m}_N\times\boldsymbol{r}))\cdot(\boldsymbol{A}^{(1)}\times\boldsymbol{A}^{(2)}) \\ &\quad = i(\boldsymbol{\mu}\cdot\boldsymbol{A})((\boldsymbol{m}_N\times\boldsymbol{r})\cdot\boldsymbol{A}^{(2)}) - i(\boldsymbol{\mu}\cdot\boldsymbol{A}^{(2)})((\boldsymbol{m}_N\times\boldsymbol{r})\cdot\boldsymbol{A}) \end{aligned} \tag{333}$$

and

$$\boldsymbol{\mu}\times(\boldsymbol{\mu}_N\times\boldsymbol{r}) = (\boldsymbol{\mu}\cdot\boldsymbol{r})\boldsymbol{m}_N - (\boldsymbol{\mu}\cdot\boldsymbol{m}_N)\boldsymbol{r} \tag{334}$$

demonstrate that Eq. (331) may be written as

$$E = \zeta\left(i\frac{e^2}{2m}\boldsymbol{\sigma}\cdot\boldsymbol{A}^{(1)}\times\boldsymbol{A}^{(2)}\right) \tag{335}$$

where

$$\zeta = \frac{g_N e\mu_0}{8\pi m}\sum_n \langle o|\boldsymbol{\mu}|n\rangle\langle n|\frac{\boldsymbol{r}}{r^3}|0\rangle \tag{336}$$

Here, $\boldsymbol{m}_N = g_N(e/4m)\hbar\boldsymbol{\sigma}$ and Eq. (335) defines the RFR chemical shift factor or shielding constant. This depends on the novel molecular property tensor in Eq. (336), which is not the tensor that defines the well-known NMR chemical shift through the Lamb shift formula of NMR [69]. The order of magnitude of ζ is about 10^{-6}, roughly the same as in NMR. The complete RFR spectrum from the protons in atoms and molecules is therefore

$$E_{\text{int}} = i\frac{e^2}{2m}(1+\zeta)\boldsymbol{\sigma}\cdot\boldsymbol{A}^{(1)}\times\boldsymbol{A}^{(2)} \tag{337}$$

and is site-specific because of the site specificity of ζ.

The experimental or empirical demonstration of RFR is a logical consequence of the detection of a term proportional to $\boldsymbol{\sigma}\cdot\boldsymbol{A}^{(1)}\times\boldsymbol{A}^{(2)}$ by van der Ziel et al. [37], and some experimental details are suggested here. It would be necessary to work initially on the interaction of a fermion beam with an electromagnetic beam. All levels of one fermion theory given in this section could then be tested under conditions that most closely approximate the theory. A successful demonstration of RFR would require careful engineering in the matter of beam interaction. The IFE has been demonstrated at 3.0 GHz by Deschamps et al. [75], and this experiment provides clues as to how to go about detecting RFR. It seems that the simplest demonstration is autoresonance, where the circularly polarized pump frequency (ω) is adjusted to be the same as the RFR frequency (ω_{res}):

$$\omega_{\text{res}} = \omega \tag{338}$$

Under this condition, the pump beam is absorbed at resonance because the pump frequency matches the resonance frequency exactly. Equation (270) simplifies to

$$\omega_{\text{res}}^3 = 1.007\times 10^{28} I \tag{339}$$

Therefore, we can tune ω_{res} for a given I, or vice versa, using interacting fermion and electromagnetic beams. Since autoresonance must appear in the gigahertz range if the pump frequency is in this microwave range, the setup in Ref. 75 can be used as a starting point for the RFR design. Essentially, the magnetization 75 must be converted into a resonance. In Ref. 75, a pulsed microwave signal at 3.0 GHz was detected from a klystron delivering megawatts of power over 12 μs with a repetition rate of 10 Hz. The TE_{11} mode was circularly polarized inside a circular waveguide of 7.5 cm diameter. A plasma was created by the very intense microwave pulse. To detect RFR experimentally, the same standard of engineering would have to be reached with an electromagnetic beam interacting with an electron beam, rather than a plasma, which contains positive ions [15]. To detect resonance, the intensity of the microwave radiation would be much lower, and governed by the autoresonance equation (339). As in the design used

by Deschamps et al. [75], the section of the waveguide surrounding the tube would perhaps be made of nylon coated with a micrometer-range layer of copper. The incoming electron beam would have to be guided carefully into the circular waveguide used to circularly polarize the microwave radiation. The engineering design for RFR probably has to be at least as accurate as in the experiment [75] in which magnetization was detected in the IFE at 3.0 GHz in a plasma. Cross-referencing with the detection of the term $\boldsymbol{\sigma}\cdot\boldsymbol{A}^{(1)}\times\boldsymbol{A}^{(2)}$ in Ref. 37, at least part of the signal detected by Deschamps et al. must be due to the RFR term, which is the interaction of $\boldsymbol{B}^{(3)}$ with the Pauli spinor. Contemporary IFE experiments [76] in plasma routinely detect this term and so routinely detect the $\boldsymbol{B}^{(3)}$ field. Equation (339) predicts that the resonance occurs at 3.0 GHz if I is tuned to 0.0665 W/cm^2 for an electron beam. For a circular waveguide of 7.0 cm diameter, this requires only 2.94 W of power.

The preceding estimate is based on one-fermion theory, so the observed resonance frequency in a fermion beam may be different as a result of fermion–fermion interaction. Therefore, it is strongly advisable that I be tunable over a wide range to search for the actual resonance pattern. The same experiment can then be repeated in a proton, atomic or molecular beam and the RFR effect should be I/ω^2-dependent with a pattern of resonance determined by the novel chemical shift factor ζ. Spin–spin interaction between fermions would split the spectrum as in ordinary NMR, but the RFR fingerprint would be unique.

It is to be emphasized finally that the RFR technique is simply the resonance equivalent of a magnetization term proportional to $\boldsymbol{\sigma}\cdot\boldsymbol{A}^{(1)}\times\boldsymbol{A}^{(2)}$ that has now been observed on numerous occasions [76] in the IFE in paramagnetic materials and plasma. The experimental challenge is to convert this magnetization to resonance.

VIII. CORRECTIONS TO QUANTUM ELECTRODYNAMICS IN O(3) ELECTRODYNAMICS

As discussed by Crowell [17], quantized electric and magnetic fields exist in a vacuum that is composed of virtual photons that are the result of the Heisenberg uncertainty fluctuations in the electric and magnetic fields. These fluctuations can be considered as first-order terms, and second-order terms involve fluctuations with electrons and positrons. These virtual pairs [17] are randomly distributed in the vacuum, but an electric field will preferentially align, or polarize, the virtual charge separation. Therefore a photon, with its oscillating electric field, will be associated with these virtual pairs of electrons and positrons that are polarized with the photon electric field. In the formal language of quantum electrodynamics, this is represented by Feynman diagrams [6,17].

The magnetic field is oriented perpendicular to the plane inscribed by a completely polarized electron–positron pair [17]. The virtual electron–positron

is accompanied by a virtual electromagnetic field, and as discussed by Crowell [17], the charges of the virtual pair will separate under the influence of the photon electric field. The magnetic field lines of the virtual electron–positron pair will preferentially align with the magnetic field of the photon. Therefore quantum theory is the action of the vacuum on particles and fields, so there are terms such as $\boldsymbol{E}^{(1,2)}+\delta\boldsymbol{E}^{(1,2)}$ and $\boldsymbol{B}^{(1,2)}+\delta\boldsymbol{B}^{(1,2)}$ where the variational terms are quantum fluctuations. Now, following the argument by Crowell [17], consider the differential form $\boldsymbol{F}=\mathrm{d}\boldsymbol{A}$, which can be written in spacetime as

$$\boldsymbol{F}=F_{\mu\nu}dx^{\mu}\wedge dx^{\nu} \tag{340}$$

The Yang–Mills functional [17] is defined by the integration of the wedge product $\boldsymbol{F}\wedge^{*}\boldsymbol{F}$, where * denotes the Hodge dual-star operator

$$k=\frac{1}{8\pi^2}\int_{(m,g)}F_{\mu\nu}F_{\alpha\beta}\,dx^{\mu}\wedge dx^{\nu}\wedge dx^{\alpha}\wedge dx^{\beta} \tag{341}$$

and where k is the instanton number. The electric and magnetic fields on the manifold of three dimensions are

$$E_i=\varepsilon_{0ji}F^{0j};\qquad B_i=\varepsilon_{kji}F^{kj} \tag{342}$$

and the Yang–Mills functional is

$$k=\frac{1}{16\pi}\int([E_i,B_j]+[\delta E_i,\delta B_j])\,d^4x \tag{343}$$

leading to the equal time commutator [17]

$$[\delta E_i^a(\boldsymbol{r},t),\delta B_j^b(\boldsymbol{r}',t)]=\hbar\delta_{ij}\delta^{ab}\delta(\boldsymbol{r}-\boldsymbol{r}')\delta(t-t') \tag{344}$$

where the O(3) indices are included. Quantum-mechanically, the electric and magnetic fields are conjugate variables, and the uncertainty relationship is dictated by the fluctuations in these fields in the vacuum.

These field fluctuations in the vacuum will interact with the photon's electric and magnetic fields. The fluctuation in the interaction energy due to the magnetic field is given by [17]

$$\delta\boldsymbol{E}=\int\delta(\boldsymbol{j}\cdot\boldsymbol{A})=\int\boldsymbol{H}\cdot\delta\boldsymbol{B}\,d^3r \tag{345}$$

and can be estimated from the quantized flux $2\pi\hbar/e$. This term is responsible for the Lamb shift in the energy levels of atoms such as the hydrogen atom. The

magnetic field fluctuation is defined as the magnetic flux quanta multiplied by the small area enclosed by the electron-positron pair, an area that is determined by the coordinate fluctuations of the electron and positron, and that can be estimated by using the energy fluctuation $\delta E = \delta mc^2$, the uncertainty relation between the energy and the time $\delta E \delta t = \hbar$ and the uncertainty in the position $\delta x = c\delta t$.

The magnetic field fluctuation is approximately 5.6×10^4 T over a range of about 10^{-15} m, and lasts for about 10^{-23} s. Fluctuations on this scale occur at about the classical radius of the electron.

O(3) electrodynamics predicts the existence of the $\boldsymbol{B}^{(3)}$ field, which must also have an effect on the stochastic motion of an electron on a fine scale [17]. There exists in theory [17] the commutator

$$[\delta E_i^{(3)}(\boldsymbol{r},t), \delta B_j^{(3)}(\boldsymbol{r}',t)] = \hbar\delta_{ij}\delta(\boldsymbol{r}-\boldsymbol{r}')\delta(t-t') \tag{346}$$

and the uncertainty fluctuations:

$$\delta\boldsymbol{B}^{(3)} = \frac{e}{\hbar}(\delta\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} + \boldsymbol{A}^{(1)} \times \delta\boldsymbol{A}^{(2)}) \tag{347}$$

The magnetic vector potentials will have the magnitude $|\boldsymbol{B}^{(3)}|/k$, so the magnitude of the $\boldsymbol{B}^{(3)}$ fluctuation is expected to be [17]

$$|\delta\boldsymbol{B}^{(3)}| = \frac{2e}{\hbar k^2}(|\delta\boldsymbol{B}||\boldsymbol{B}|) \tag{348}$$

The fluctuation in the ordinary magnetic field in this expression is

$$\delta\boldsymbol{B} = \frac{\pi}{2}\frac{(\delta m)^2}{e\hbar} \tag{349}$$

which is about 5.6×10^4 T. The magnetic field associated with the photon, without quantum fluctuations, is about 3×10^{-14} T, so the fluctuation in $\boldsymbol{B}^{(3)}$ is approximately 6×10^{-7} T. These result from virtual electron–positron pairs and are expected to be 10 orders of magnitude smaller than the standard magnetic field, giving measurable contributions to quantum electrodynamics in the 10-GeV range [17].

Crowell [17] argues that the vacuum contribution to the virtual $\boldsymbol{B}^{(3)}$ field is a very small effect, about a millionth of the Lamb shift.

The nonrelativistic estimate of the contribution of $\boldsymbol{B}^{(3)}$ to the Lamb shift was first carried out by Crowell [17] as follows. The interaction of the radiation field with the electron is given by

$$H = \frac{e}{c}\int d^3r\boldsymbol{j}(\boldsymbol{r})\cdot\boldsymbol{A}(r) \tag{350}$$

The Ampère law is next used with a covariant definition of the curl operator

$$\nabla \rightarrow D\times = \nabla\times + i\frac{e}{\hbar}\sum_i A^i\times \tag{351}$$

implying

$$\begin{aligned}\boldsymbol{j}(\boldsymbol{r})\cdot\boldsymbol{A}(\boldsymbol{r}) &= \boldsymbol{D}(\boldsymbol{r}) \times \boldsymbol{H}(\boldsymbol{r})\cdot\boldsymbol{A}(\boldsymbol{r}) \\ &= \boldsymbol{H}(\boldsymbol{r})\cdot\boldsymbol{D} \times \boldsymbol{A}(\boldsymbol{r}) + \boldsymbol{D}\cdot\boldsymbol{H}(\boldsymbol{r}) \times \boldsymbol{A}(\boldsymbol{r})\end{aligned} \tag{352}$$

The last term is a boundary operator and is discarded, leaving a $\boldsymbol{B}^{(3)}$ contribution

$$H = -i\frac{e^2}{\hbar c}\int d^3r\boldsymbol{H}\cdot\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} \tag{353}$$

which leads to the Lamb shift due to $\boldsymbol{B}^{(3)}$. The interaction Hamiltonian (353) will induce the spontaneous emission of a photon with wave number $\omega = ck$ and an atomic state transition $|n\rangle \rightarrow |n'\rangle$, which gives the second-order perturbation shift in energy

$$\Delta E_n = \sum_{n'}\sum_{k,\varepsilon}\left(\frac{|\langle n',k,\varepsilon|H_{\text{int}}|n,0\rangle|^2}{E_n - E_{n'} - ck}\right) \tag{354}$$

where ϵ is the polarization state of the emitted photon. First, consider the term $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ with $\boldsymbol{A} = \boldsymbol{A}\,\epsilon$. The matrix elements of the interaction Hamiltonian are

$$\langle n',k,\varepsilon|H_{\text{int}}|n,0\rangle = \frac{e^4}{\mu^2c^2}A^3\langle\boldsymbol{p}\times\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}\times\boldsymbol{\varepsilon}^*\rangle \tag{355}$$

and if the sum over the photon numbers goes to the continuum, the energy shift is

$$\Delta E_n = -\frac{e^4}{\mu^2c^2}A^3\int\frac{d^3k}{k^3}\sum_{n',\varepsilon}\frac{|\langle n',k|\boldsymbol{p}\cdot\boldsymbol{\varepsilon}^*|n,0\rangle|^2}{E_n - E_{n'} - ck'} \tag{356}$$

where the factor $(2\pi\hbar)^{-3}$ is absorbed into $\boldsymbol{A}^{(3)}$. Now Crowell [17] sums over the polarization states and puts the integral in spherical coordinate form:

$$\Delta E_n = -\frac{e^4}{\mu^2c^2}A^3\int_0^\infty\frac{dk}{k}\sum_{n'}\frac{|\langle n'|\boldsymbol{p}\cdot\boldsymbol{\varepsilon}^*|n\rangle|^2}{E_n - E_{n'} - ck} \tag{357}$$

The integration of this result leads to

$$\Delta E_n = -\frac{e^4}{\mu^2c^2}A^3\sum_{n'}\frac{|\langle n'|\boldsymbol{p}\cdot\boldsymbol{\varepsilon}^*|n\rangle|^2}{E_n - E_{n'}}\lim_{k\rightarrow 0}\ln\left(1 + \frac{E_n' - E_n}{\hbar kc}\right) \tag{358}$$

which is divergent. This divergence is dealt with by recognizing that the probability of emitting a photon depends on the electron current as a function of wave number, so that the dipole approximation becomes

$$|\langle n', k\varepsilon|\boldsymbol{p}|n, 0\rangle| = |\langle n'|\boldsymbol{p}|n, 0\rangle|^2 |j(k)|^2 \tag{359}$$

where $j(k)$ is a current for each wave number k divided by the total current, a ratio that reflects the percentage of photons that are emitted with a given k. For a finite number of photons, this will be a Poisson distribution. If the sample size of photons is very large, but if the number of photons emitted is far less, then $|j(k)| \sim k$, and the following result is obtained

$$\Delta E_n = -\frac{e^4}{\mu^2 c^2} A^3 \sum_{n'} \frac{|\langle n'|\boldsymbol{p}\cdot\boldsymbol{\varepsilon}^*|n\rangle|^2}{E_n - E_{n'}} \ln\left(1 - \frac{\hbar kc}{E_n - E_n'}\right)\Bigg|_0^\infty \tag{360}$$

an integral that is logarithmically divergent in the ultraviolet range [17].

In U(1) quantum electrodynamics, the ultraviolet divergence is removed [17] by countering it with a similar term. For the free electron, there is the infinite term

$$\Delta E_{e^-} = \frac{2e^2}{3\pi m^2 c^2} \sum_q \langle q|\boldsymbol{p}|p\rangle^2 \int_0^\infty dk \tag{361}$$

leading to the mass renormalization of the electron from the energy shift:

$$E_{e^-} = E_{e^-}^0 + \Delta E_{e^-} = \frac{1}{p}\langle|\boldsymbol{p}|\rangle + \frac{2e^2}{3\pi m^2 c^2}\langle|\boldsymbol{p}|\rangle^2 \int_0^\infty dk \tag{362}$$

An analogous process in O(3) quantum electrodynamics involves, following Crowell [17], the coupling of the electron with a nonlinear photon coupling corresponding to the energy shift:

$$\begin{aligned}\Delta E_{e^-}^{B^{(3)}} &= \sum_{k,n} |\langle n, k, \varepsilon|p^2|A|^2 A|0, 0\rangle|^2 \\ &= -\frac{8\pi}{3}\frac{\hbar^2 e^4}{m^2 c^2 \mu^4} A^3 \sum_{n'} |\langle n'|p|n\rangle|^2 \int_0^\infty dk\end{aligned} \tag{363}$$

This correction is added to the energy shift due to the $\boldsymbol{B}^{(3)}$ field to give

$$\Delta E_{e^-}^{B^{(3)}} = -\frac{8\pi}{3}\frac{\hbar^2 e^4}{m^2 c^2 \mu^4}\frac{1}{\hbar c} A^3 \sum_{n'} |\langle n'|p|n\rangle|^2 \int_0^\infty dk \frac{(E_{n'} - E_n)}{E_{n'} - E_n - \hbar kc} \tag{364}$$

which is logarithmically divergent, a divergence that is countered by the fact that the amplitudes drop off sharply for processes with frequencies $\hbar\omega > 2mc^2$, where m is the mass of the virtual electron and positron. The integral (364) can be cut off at this value, giving the final result:

$$\Delta E_{e^-}^{B^{(3)}} = -\frac{8\pi}{3}\frac{\hbar^2 e^4}{m^2 c^2 \mu^4} \alpha A^3 \sum_{n'} |\langle n'|p|n\rangle|^2 \ln\left(\frac{2mc^2}{E_{n'} - E_n - \hbar kc}\right) \tag{365}$$

The calculation of the Lamb shift due to $\boldsymbol{B}^{(3)}$ is completed by using the equations

$$H|n\rangle = E_n|n\rangle \tag{366}$$

and

$$\sum_{n'} \frac{|\langle n'|p|n\rangle|^2}{(E_{n'}' - E_n)} = \langle n'|\frac{\boldsymbol{p}}{(H_0 - E_n)}\cdot\boldsymbol{p}|n\rangle \tag{367}$$

The momentum operator acts on $(H_0 - E_n)^{-1}$ as

$$\frac{\boldsymbol{p}}{(H_0 - E_n)} = -\frac{\boldsymbol{p}}{(H_0 - E_n)^2} H_0 \tag{368}$$

and the action of the two momentum operators on the free Hamiltonian is

$$\frac{\boldsymbol{p}}{(H_0 - E_n)}\cdot\boldsymbol{p} = [\boldsymbol{p}\cdot[H_0, \boldsymbol{p}]] \tag{369}$$

In the Lamb shift, the Coulomb potential between proton and electron contributes to the commutator in the hydrogen atom, and the commutator with the free Hamiltonian becomes $(\hbar^2 e^2/2)\nabla^2(1/r)$, which gives a delta function that is evaluated in the matrix element when written out by completeness as an integral over space:

$$\frac{e^2\hbar^2}{2}\langle n|\nabla^2\frac{1}{r}|n\rangle = \frac{e^2\hbar^2}{2}\int d^3r\psi^*(\boldsymbol{r})2\pi\delta(\boldsymbol{r})\psi(\boldsymbol{r}) \tag{370}$$

For an atom in the s state, we have $|\psi|^2 = [1/\pi(na_0)^3]$, where n is the principle atomic number and a_0 is the Bohr radius. The Lamb shift due to $\boldsymbol{B}^{(3)}$ is therefore

$$\text{Lamb}(\boldsymbol{B}^{(3)}) = \frac{1}{3\pi^2}\left(\frac{e^2}{a_0}\right)\alpha^5 \ln\left(\frac{2mc^2}{E_m - E_n}\right) \tag{371}$$

which is 5.33×10^{-5} of the standard Lamb shift. This answer is about five times the quantum fluctuation estimate made already.

On the relativistic level in O(3) quantum electrodynamics [O(3) QED], the Lagrangian density is

$$\mathscr{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} \tag{372}$$

with the gauge covariant field:

$$F^a_{\mu\nu} = \partial_\nu A^a_\mu - \partial_\mu A^a_\nu + ig\varepsilon^{abc}[A^b_\nu, A^c_\mu] \tag{373}$$

Variational calculus with this Lagrangian density leads [17] to the field equation:

$$\partial_\mu F^{a\mu\nu} + ig\varepsilon^{abc} A^b_\mu F^{c\mu\nu} = 0 \tag{374}$$

with electric and magnetic components:

$$E^a_i = F^a_{i0} = -\dot{A}^a_i - \nabla_i A^a_0 + ig\varepsilon^{abc} A^b_0 A^c_i \tag{375}$$

In O(3) QED, the components of the vector potentials are expanded [17] in a Fourier series of

$$\varepsilon^k_{ij} B^a_k = \nabla_i A^a_j - \nabla_j A^a_i + ig\varepsilon^{abc} A^b_i A^c_j \tag{376}$$

modes, with creation and annihilation operators that act on the Fock space of states, with box normalization within a quantization volume V that has periodic boundary conditions, thus giving:

$$A^a_i(\boldsymbol{r}, t) = \sum_k \frac{1}{(2\omega V)^{1/2}} (e_i a^a(k) e^{i\boldsymbol{k}\cdot\boldsymbol{r}} + e_i a^{a^\dagger}(k) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}) \tag{377}$$

The electric and magnetic components within O(3) QED are then

$$E^a_i = \sum_k \frac{1}{(2\omega V)^{1/2}} \left(\frac{|k|}{c} e_i a^a(k) e^{i\boldsymbol{k}\cdot\boldsymbol{r}} + \frac{|k|}{c} e_i a^{a^\dagger}(k) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} \right) \tag{378}$$

$$\varepsilon^k_{ij} B^a_k = \sum_k \frac{1}{(2\omega V)^{1/2}} (k_{[je_i]} a^a(k) e^{i\boldsymbol{k}\cdot\boldsymbol{r}} + k_{[je_i]} e_i a^{a^+}(k) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}) + ig\varepsilon^{abc} \sum_{kk'} e_{[je_i]} (a^b(k) e^{i\boldsymbol{k}\cdot\boldsymbol{r}} + a^{b^+}(k) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}})(a^c(k') e^{i\boldsymbol{k}'\cdot\boldsymbol{r}} + a^{c^+}(k') e^{-i\boldsymbol{k}'\cdot\boldsymbol{r}}) \tag{379}$$

and the Hamiltonian for this non-Abelian field theory [17] contains novel quartic terms.

If $A^{(3)}_i$ is phase-free, as discussed in Section III, and in Ref. 15, there are no longitudinal electric field components. This also occurs if $A^{(3)}_i$ is zero [17]. The $\boldsymbol{B}^{(3)}$ field is then a Fourier sum over modes with operators $a^+_{k-q} a_q$ and is perpendicular to the plane defined by $\boldsymbol{A}^{(1)}$ and $\boldsymbol{A}^{(2)}$. The four-dimensional dual to this term is defined on a time-like surface, following Crowell [17], which can be interpreted as $\boldsymbol{E}^{(3)}$ under dyad vector duality in three dimensions. The $\boldsymbol{E}^{(3)}$ field vanishes because of the nonexistence of the raising and lowering operators $a^{(3)}, a^{(3)+}$. The $\boldsymbol{B}^{(3)}$ is nonzero because of the occurrence of raising and lowering operators in the expansion of $\boldsymbol{A}^{(1)}$ and $\boldsymbol{A}^{(2)}$. These facts imply that $\boldsymbol{B}^{(3)}$ is phaseless and longitudinal, but they do not necessarily represent a breakdown of duality because [15] $c\boldsymbol{B}^{(3)}$ can be dual to an imaginary valued $i\boldsymbol{E}^{(3)}$.

The effect of a local gauge transformation (Sction II) on the classical $\boldsymbol{B}^{(3)}$ field is described as

$$\boldsymbol{B}^{(3)'} = ig\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} g^{-1} \tag{380}$$

where the group element g is an algebraic generator $g = e^{iX}$. So in $\hbar = c = 1$ units the effect on $\boldsymbol{B}^{(3)}$ is generated as

$$dA' = g(dA + A \wedge A)g^{-1} \tag{381}$$

where g is the group element for the O(3) theory. In the case of quantum field theory, a gauge transformation

$$A^a_\mu \rightarrow A^a_\mu + \delta A^a_\mu \tag{382}$$

is associated [17] with a unitary transform of the fermion field:

$$\psi \rightarrow \psi + \delta\psi \tag{383}$$

In quantum field theory, the gauge field is determined by its Lagrangian density, and the fermion field, by the Dirac Lagrangian density:

$$\mathscr{L}_D = -\bar{\psi}(\gamma^\mu \partial_\mu + m)\psi \tag{384}$$

In order to describe the interaction between the gauge and fermion fields, the following equation is used:

$$\partial_\mu F^{a\mu\nu} + ig\varepsilon^{abc} A^b_\mu F^{c\mu\nu} = j^\nu \tag{385}$$

Here

$$j^{\nu} = \frac{\partial \mathscr{L}}{\partial A_{\nu}} \tag{386}$$

and the addition of an interaction Lagrangian density $\mathscr{L}_i = j^{\nu} A_{\nu}$ is implied. The current term is determined by the Dirac field and is

$$j^{\nu} = \bar{\psi}\gamma^{\nu}\psi \tag{387}$$

Mass renormalization requires [15] that an additional term $\bar{\psi}\gamma^{\nu}\psi\delta m$ be added where δm is the difference between the physical and bare masses [77].

The total Lagrangian is then

$$\mathscr{L} = \mathscr{L}_G + \mathscr{L}_D + \mathscr{L}_i \tag{388}$$

and describes the interaction between the fermions and the gauge field. The Dirac field is the electron field and the gauge field is the non-Abelian electromagnetic field. The theory describes the interaction between quantized electrons and quantized photons on the O(3) level. Because it is a gauge theory, it conveys momentum from one electron to another by the virtual creation and destruction of a vector boson (the photon). There is no creation of any averaged momentum from the virtual quantum fluctuation [17].

In order to upgrade these well-known methods [6,17] of U(1) quantum field theory to involve the classical $\boldsymbol{B}^{(3)}$ field, the following prescription is used:

$$A_{\mu} \rightarrow t^a A_{\mu}^a \tag{389}$$

Here t^a is a group structure constant defined by

$$[t^a, t^b] = 2\varepsilon^{abc} t^c \tag{390}$$

The amplitude contribution from the $\boldsymbol{B}^{(3)}$ field occurs in a second-order process using the sum over all possible fluctuations of $\boldsymbol{B}^{(3)}$ in the virtual photon that causes electron–electron interaction. The amplitude due to $\boldsymbol{B}^{(3)}$ has an ultraviolet divergence [17] described by Crowell. This may be removed by regularization techniques.

This type of process is missing from U(1) quantum field theory [6]; the $\boldsymbol{B}^{(3)}$ field produces quantum vortices [17] that interact with electrons and other charged particles. The vortices are quantized states and exist as fluctuations in the QED vacuum, fluctuations that are associated, not with an $\boldsymbol{E}^{(3)}$ field, but with the $\boldsymbol{E}^{(1)} = \boldsymbol{E}^{(2)*}$ fields:

$$\delta\boldsymbol{B}^{(3)} = i\frac{e}{\hbar}\frac{1}{\omega^2}(\delta\boldsymbol{E}^{(1)} \times \boldsymbol{E}^{(2)} + \boldsymbol{E}^{(1)} \times \delta\boldsymbol{E}^{(2)}) \tag{391}$$

Therefore quantum fluctuations in $\boldsymbol{B}^{(3)}$ are accompanied by fluctuations in the transverse electric field. The ultraviolet divergence is probably unimportant [17] because of the ω^{-2} dependence of the fluctuation. The infrared divergence is also damped statistically. The divergences in U(1) electrodynamics [6] can exist as a subset of O(3) electrodynamics and can be absorbed into integrals that involve photon loop processes associated with quantum fluctuations in $\boldsymbol{B}^{(3)}$.

Crowell [17] has argued that O(3) QED is fully renormalizable. Renormalization is necessary as in any quantum field theory because the potential and propagator become divergent as the electrons approach each other. The Heisenberg uncertainty principle $\Delta p \Delta x \geq \hbar$ means that the momentum exchanged by the electrons becomes divergent [17]. The vacuum is filled with virtual quanta, as argued by Crowell [17], with enormously high momentum fluctuations: virtual quanta that may interact with systems to contribute divergences in the short wavelength limit, the ultraviolet divergences. These divergences affect the self-energy of the electron, vacuum polarization, and vertex functions [6,15,17].

In O(3) QED, there is an additional effect from the effective photon bunching or photon interaction that emerges essentially from the photon loop generated from the $\boldsymbol{A}^{(1)}$ on one photon interacting with the $\boldsymbol{A}^{(2)}$ on the other photon. The loop is associated with quanta of the $\boldsymbol{B}^{(3)}$ field with intensity $e/\hbar$ as in Eq. (347). It will be argued, following Crowell [17], that these novel fluctuations are fully renormalizable. The virtual fluctuation of a $\boldsymbol{B}^{(3)}$ field does not lead to an ultraviolet divergence, and so O(3) QED is renormalizable by dimensional regularization.

The renormalization problem generated by O(3) is similar to the interaction of the free electron with the vacuum through the Dirac equation [6,15,17] in $c = 1, h = 1$ units:

$$(\gamma^{\mu}(\partial_{\mu} - ieA_{\mu}) - m)\psi = 0 \tag{392}$$

If there is no electromagnetic field present, the quantized vector potential fluctuates according to

$$A_{\mu} = \langle A_{\mu} \rangle + \delta A_{\mu} \tag{393}$$

and the fluctuation is present in the vacuum. This phenomenon manifests itself through the zero point energy of the harmonic oscillator expansion of the fields [17]; the electron will interact with the virtual photons, an interaction which is expanded in terms of the order $\alpha = (e^2/\hbar c)$.The divergence [17] in the first term of this series is countered by a mass term, introducing a difference between the mass and the bare mass of the electron. Similar methods can be used straightforwardly [17] to show that the loop fluctuations of photon, correlated to the virtual quanta of the $\boldsymbol{B}^{(3)}$ field, can be calculated to be finite without

divergence. The end result of this standard but complicated calculation [17] is that O(3) QED is free of intractable ultraviolet divergences. The Lamb shift calculation given already shows that O(3) QED is free of intractable infrared divergences.

In Section I, it was argued that O(3) electrodynamics on the classical level emerges from a vacuum configuration that can be described with an O(3) symmetry gauge group. On the QED level, this concept is developed by considering higher-order terms in the Hamiltonian

$$H = \frac{1}{2m}(\boldsymbol{p} - e\boldsymbol{A})^2 \tag{394}$$

and evolution operator $U = e^{-iH_0 t}$ [17], where:

$$U = e^{-iH_0 t} e^{A^{(1)} \cdot A^{(2)}} \tag{395}$$

Here, H_0 is the Hamiltonian without the quadratic term. The vector potentials are expanded as

$$A^{(1)} = \frac{A^{(0)}}{\sqrt{2}}(e_X + ie_Y)(a_k e^{i\boldsymbol{k}\cdot\boldsymbol{r} - i\omega t} - a_k^+ e^{-i\boldsymbol{k}\cdot\boldsymbol{r} + i\omega t}) \tag{396}$$

giving

$$\boldsymbol{A}^{(1)} \cdot \boldsymbol{A}^{(2)} = A^{(0)2}\left(a^+ a + \frac{1}{2} - \frac{1}{2}(a^{+2} e^{-2i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)} + a^2 e^{2i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)})\right) \tag{397}$$

The first two terms on the right-hand side [17] are precisely those obtained from the standard harmonic oscillator Hamiltonian (H_{em}) for the electromagnetic field. The evolution operator can then be written as

$$U = e^{-i(H_0 + H_{\text{em}})t} e^{(Za^{+2} + Z^* a^2)} \tag{398}$$

where $Z = te^{-2i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)}$.

The operator

$$S(Z) = \exp(Za^{+2} + Z^* a^2) \tag{399}$$

is a squeezed-state operator [17] that involves symmetries that are not precisely defined by the Hamiltonian. The quantized $\boldsymbol{B}^{(3)}$ field may correspond to such symmetries of the vacuum, coming full circle with Section I. The reason is that the $\boldsymbol{B}^{(3)}$ field is generated by writing Eq. (394) in the basis of the Pauli matrices, as discussed in Section VII.

The absence of an $\boldsymbol{E}^{(3)}$ field does not affect Lorentz symmetry, because in free space, the field equations of both O(3) electrodynamics are Lorentz-invariant, so their solutions are also Lorentz-invariant. This conclusion follows from the Jacobi identity (30), which is an identity for all group symmetries. The right-hand side is zero, and so the left-hand side is zero and invariant under the general Lorentz transformation [6], consisting of boosts, rotations, and space-time translations. It follows that the $\boldsymbol{B}^{(3)}$ field in free space Lorentz-invariant, and also that the definition (38) is invariant. The $\boldsymbol{E}^{(3)}$ field is zero and is also invariant; thus, $\boldsymbol{B}^{(3)}$ is the same for all observers and $\boldsymbol{E}^{(3)}$ is zero for all observers.

To prove the invariance of the B cyclic theorem [11–20], it is necessary only to prove the invariance of the free-space Maxwell–Heaviside equations:

$$\partial_\mu \tilde{G}^{\mu\nu(i)} = 0; \qquad i = 1, 2, 3 \tag{400}$$

Consider, for example, a Lorentz boost in the Z direction using Jackson's notation [5], and start with the 4-derivative

$$\partial'_\mu = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & i\gamma\beta \\ 0 & 0 & -i\gamma\beta & \gamma \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial X} \\ \frac{\partial}{\partial Y} \\ \frac{\partial}{\partial Z} \\ \frac{i}{c}\frac{\partial}{\partial t} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial X} \\ \frac{\partial}{\partial Y} \\ \gamma\frac{\partial}{\partial Z} - \frac{\gamma\beta}{c}\frac{\partial}{\partial t} \\ -i\left(\gamma\beta\frac{\partial}{\partial Z} - \frac{\gamma}{c}\frac{\partial}{\partial t}\right) \end{bmatrix} \tag{401}$$

where

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}; \qquad \beta = \frac{v}{c} \tag{402}$$

Using the same Z boost

$$\begin{aligned} E'_X &= \gamma(E_X - \beta B_Y) & B'_X &= B_X + \beta E_Y \\ E'_Y &= \gamma(E_Y + \beta B_X) & B'_Y &= B_Y - \beta E_X \\ E'_Z &= E_Z & B'_Z &= B_Z \end{aligned} \tag{403}$$

so

$$\begin{aligned} (\nabla\cdot\boldsymbol{E})' &= \nabla'\cdot\boldsymbol{E}' = \gamma\nabla\cdot\boldsymbol{E} = 0 = \nabla\cdot\boldsymbol{E} \\ (\nabla\cdot\boldsymbol{B})' &= \nabla'\cdot\boldsymbol{B}' = \gamma\nabla\cdot\boldsymbol{B} = 0 = \nabla\cdot\boldsymbol{B} \end{aligned} \tag{404}$$

Considering the $\boldsymbol{i}$ component of the Faraday law in frame K:

$$\frac{\partial E_Y}{\partial Z} - \frac{\partial E_Z}{\partial Y} + \frac{\partial B_X}{\partial t} = 0 \tag{405}$$

the same component in frame K' is

$$\gamma\left(\gamma\frac{\partial}{\partial Z}-\frac{\gamma\beta}{c}\frac{\partial}{\partial t}\right)(E_Y+\beta B_X)-\gamma\frac{\partial E_Z}{\partial Y}+\gamma\left(-\gamma\beta\frac{\partial}{\partial Z}+\frac{\gamma}{c}\frac{\partial}{\partial t}\right)(B_X+\beta E_Y)=0 \tag{406}$$

On the U(1) level, we can consider E_Y and B_X to be plane waves and $E_Z=0$. The following result is obtained in frame K':

$$\gamma^2\left(\frac{\partial E_Y}{\partial Z}+\frac{1}{c}\frac{\partial B_X}{\partial t}\right)-\frac{\gamma^2\beta^2}{c}\left(\frac{1}{c}\frac{\partial B_X}{\partial t}+\frac{\partial E_Y}{\partial Z}\right)=0 \tag{407}$$

This is true for all γ and β because

$$\frac{\partial E_Y}{\partial Z}+\frac{1}{c}\frac{\partial B_X}{\partial t}=0 \qquad \text{(Gaussian units)} \tag{408}$$

The result is obtained that Faraday's law of induction is invariant under a Z boost. Similarly, it can be shown to be invariant under the general Lorentz transformation, and all solutions are invariant. In general, on the U(1) level

$$(\partial_\mu\tilde{F}^{\mu\nu})'=\partial_\mu\tilde{F}^{\mu\nu}\equiv 0 \tag{409}$$

$$(\partial_\mu F^{\mu\nu})'=\partial_\mu\tilde{F}^{\mu\nu}\equiv 0 \tag{410}$$

It follows that the transverse field $\boldsymbol{B}^{(1)}=\boldsymbol{B}^{(2)*}$ is Lorentz-invariant in free space, and so is the B cyclic theorem:

$$\boldsymbol{B}^{(1)}\times\boldsymbol{B}^{(2)}=iB^{(0)}\boldsymbol{B}^{(3)*} \qquad \text{in cyclic permutation} \tag{411}$$

The general principle being followed is that, if equations of motion are the same in any Lorentz frame, that is, to any observer, then so are the solutions.

The invariance of the definition of $\boldsymbol{B}^{(3)}$ can again be illustrated on the simplest level by considering Lorentz boosts in the Z, X and Y directions of the $\boldsymbol{B}^{(3)}$ field:

$$B_Z^{(3)'}=B_Z^{(3)} \tag{412}$$

$$B_Z^{(3)'}=\gamma B_Z^{(3)}+\gamma\beta E_X^{(3)}=\gamma B_Z^{(3)} \tag{413}$$

$$B_Z^{(3)'}=\gamma B_Z^{(3)}-\gamma\beta E_Y^{(3)}=\gamma B_Z^{(3)} \tag{414}$$

In Jackson's notation, a Z boost of $\boldsymbol{A}^{(1)}$, for example, leaves it unchanged:

$$A^{(1)}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&\gamma&i\gamma\beta\\0&0&-i\gamma\beta&\gamma\end{bmatrix}\begin{bmatrix}A_X^{(1)}\\A_X^{(2)}\\0\\0\end{bmatrix}=\boldsymbol{A}^{(1)} \tag{415}$$

and since $\boldsymbol{A}^{(2)}$ is the complex conjugate of $\boldsymbol{A}^{(1)}$, a Z boost in free space results in

$$(\boldsymbol{B}^{(3)*}=-ig\boldsymbol{A}^{(1)}\times\boldsymbol{A}^{(2)})' \tag{416}$$

and leaves $\boldsymbol{B}^{(3)}$ invariant. The effect of a Y boost on $\boldsymbol{A}^{(1)}$ is as follows:

$$\boldsymbol{A}^{(1)}=\begin{bmatrix}1&0&0&0\\0&\gamma&0&i\gamma\beta\\0&0&1&0\\0&-i\gamma\beta&0&\gamma\end{bmatrix}\begin{bmatrix}A_X^{(1)}\\A_Y^{(1)}\\0\\0\end{bmatrix}=\begin{bmatrix}A_X^{(1)}\\\gamma A_Y^{(1)}\\0\\-i\gamma\beta A_Y^{(1)}\end{bmatrix} \tag{417}$$

and using

$$B_Z^{(3)*}=-ig\varepsilon_{(1)(2)(3)}A_X^{(1)}A_Y^{(2)} \tag{418}$$

it is found that

$$\gamma\boldsymbol{B}^{(3)}=-i\gamma g\boldsymbol{A}^{(1)}\times\boldsymbol{A}^{(2)} \tag{419}$$

and the definition of $\boldsymbol{B}^{(3)}$ is again invariant. Using $B^{(0)}=\kappa A^{(0)}$ [11–20] converts Eq. (416) into the B cyclic theorem, and both are self-consistently invariant. Therefore $\boldsymbol{B}^{(3)}$ is a fundamental field [11–20].

The $\boldsymbol{E}^{(3)}$ field is zero in frame K, and a Z boost means [from Eq. (403)] that it is zero in frame K'. This is consistent with the fact that $\boldsymbol{E}^{(3)}$ is a solution of an invariant equation, the Jacobi identity (30) of O(3) electrodynamics. Finally, we can consider two further illustrative example boosts of $\boldsymbol{E}^{(3)}$ in the X and Y directions, which both produce the following result:

$$E_Z^{(3)'}=\gamma E_Z^{(3)} \tag{420}$$

Therefore if $\boldsymbol{E}^{(3)}$ is null in frame K, it is null in frame K'. There is a symmetry between the Lorentz transforms of $\boldsymbol{B}^{(3)}$ and the hypothetical $\boldsymbol{E}^{(3)}$:

$$\begin{aligned} X: \quad & B_Z^{(3)'} = \gamma B_Z^{(3)}; \qquad E_Z^{(3)'} = \gamma E_Z^{(3)} \\ Y: \quad & B_Z^{(3)'} = \gamma B_Z^{(3)}; \qquad E_Z^{(3)'} = \gamma E_Z^{(3)} \\ Z: \quad & B_Z^{(3)'} = B_Z^{(3)}; \qquad E_Z^{(3)'} = E_Z^{(3)} \end{aligned} \tag{421}$$

This is self-consistent with the fact that $\boldsymbol{B}^{(3)}$ may be regarded [11–20] as dual to $[-i\boldsymbol{E}^{(3)}/c]$, so that $\boldsymbol{B}^{(3)2} + \boldsymbol{E}^{(3)2}$ contributes to a nonzero Lagrangian and so that $\boldsymbol{B}^{(3)}$ is a real physical field.

These are mathematically valid results, but physically, the Lorentz transform of $\boldsymbol{B}^{(3)}$ and the null $\boldsymbol{E}^{(3)}$ are governed by the equation

$$D_\mu \tilde{G}^{\mu\nu} = \mathbf{0}^\nu \tag{422}$$

where:

$$\mathbf{0}^\nu = 0^{\nu(1)}\boldsymbol{e}^{(1)} + 0^{\nu(2)}\boldsymbol{e}^{(2)} + 0^{\nu(3)}\boldsymbol{e}^{(3)} \tag{423}$$

is a null 12-vector, whose components are null 4-vectors. The general Lorentz transform of the null 4-vector is given by

$$0'^\mu = \Lambda^\mu_\nu 0^\mu = 0^\mu \tag{424}$$

and a null 4-vector is a null 4-vector in all Lorentz frames. This means that the left-hand side of Eq. (422) is null in all Lorentz frames and is Lorentz-invariant. Therefore its field solutions are also all Lorentz invariant, including, of course, $\boldsymbol{B}^{(3)}$ and $\boldsymbol{E}^{(3)}$. This is self-consistent with the fact that Eq. (422) is equivalent to the Jacobi identity (30) for the group O(3). Finally, when there is field–matter interaction, all field components are Lorentz covariant, and no longer invariant, on both the U(1) and O(3) levels.

In conclusion, the homogeneous field equation of O(3) electrodynamics is Lorentz-invariant, and all its classical solutions must be also Lorentz-invariant. The same result is obtained therefore in QED.

IX. NOETHER CHARGES AND CURRENTS OF O(3) ELECTRODYNAMICS IN THE VACUUM

The first example of a vacuum current was introduced by Maxwell in order to make the equations of electrostatics and magnetostatics self-consistent. The second examples were introduced in 1979 [7] by Lehnert, and O(3)

electrodynamics offers four vacuum charges and currents of topological origin as discussed already. Maxwell was led to the displacement current because the received view at the time was self-inconsistent [5]. The received view consisted of four equations

$$\nabla \cdot \boldsymbol{D} = \rho; \qquad \nabla \cdot \boldsymbol{B} = 0; \qquad \nabla \times \boldsymbol{H} = \boldsymbol{J}; \qquad \nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = \mathbf{0} \tag{425}$$

together with the continuity equation:

$$\nabla \cdot \boldsymbol{J} + \frac{\partial \rho}{\partial t} = 0 \tag{426}$$

Maxwell used the continuity equation in the Coulomb law to give

$$\nabla \cdot \left(\boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \right) = 0 \tag{427}$$

and replaced $\boldsymbol{J}$ by $\boldsymbol{J} + (\partial \boldsymbol{D}/\partial t)$. The final result is the Ampère–Maxwell law

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \tag{428}$$

which produced electromagnetic waves and is, of course, a standard part of U(1) electrodynamics. The latter asserts, in the received view [5] currently prevailing, that in the vacuum, there is a displacement current

$$\boldsymbol{J}_D = \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \tag{429}$$

using the vacuum constitutive equation $\boldsymbol{D} = \epsilon_0 \boldsymbol{E}$. The existence of Maxwell's vacuum displacement current is all-important for the theory of electromagnetic radiation. The displacement current originates in the continuity equation, which is a conservation law, similar to the laws of conservation of energy and momentum summarized in Noether's theorem [6]. The Maxwell displacement current can therefore be referred to as a "Noether current."

More than a century later, Lehnert [7] introduced and developed [7–10] the concept of vacuum charge on the classical level, and showed [7–10] that this concept leads to advantages over the Maxwell–Heaviside equations in the description of empirical data, for example, the problem of an interface with a vacuum [7–10,15]. The introduction of a vacuum charge leads to axisymmetric vacuum solutions akin to the $\boldsymbol{B}^{(3)}$ vacuum component of O(3) electrodynamics [10,15], and also leads to the Proca equation and the concept of photon mass.

The latter is therefore related to the concept of the $\boldsymbol{B}^{(3)}$ field through the Lehnert equations, which in the vacuum are

$$\nabla \cdot \boldsymbol{D} = \rho_{\text{vac}}; \qquad \nabla \cdot \boldsymbol{B} = 0; \qquad \nabla \times \boldsymbol{H} = \boldsymbol{J}_{\text{vac}} + \frac{\partial \boldsymbol{D}}{\partial t}; \qquad \nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{0} \tag{430}$$

It can be seen that these are U(1) equations, but with the addition of the vacuum charge density ρ_{vac} and the vacuum current density $\boldsymbol{J}_{\text{vac}}$. On the O(3) level, the Lehnert charge density becomes

$$\rho_{\text{vac}}^{(1)*} = ig(\boldsymbol{A}^{(2)} \cdot \boldsymbol{D}^{(3)} - \boldsymbol{D}^{(2)} \cdot \boldsymbol{A}^{(3)}) \tag{431}$$

in cyclic permutation

and the Lehnert current density becomes

$$\boldsymbol{J}_{\text{vac}}^{(1)*} = -ig(cA_0^{(2)}\boldsymbol{D}^{(3)} - cA_0^{(3)}\boldsymbol{D}^{(2)} + \boldsymbol{A}^{(2)} \times \boldsymbol{H}^{(3)} - \boldsymbol{A}^{(3)} \times \boldsymbol{H}^{(2)}) \tag{432}$$

in cyclic permutation

and O(3) electrodynamics self-consistently produces longitudinal solutions in the vacuum typified by the phaseless $\boldsymbol{B}^{(3)}$ component. However, the magnetic charge and current allowed for by O(3) electrodynamics do not appear in the Lehnert equations (430).

The Lehnert equations are consistent [10] with the continuity equation (428) of U(1) electrodynamics. Using the vacuum continuity equation in Lehnert's vacuum Coulomb law, we find

$$\begin{aligned} \boldsymbol{J} \rightarrow \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} &\equiv \boldsymbol{J}_1 \\ \nabla \times \boldsymbol{H} &= \boldsymbol{J}_1 + \frac{\partial \boldsymbol{D}}{\partial t} \\ \nabla \cdot \boldsymbol{J}_1 + \frac{\partial \rho_1}{\partial t} &= 0 \end{aligned} \tag{433}$$

Repeating this procedure gives

$$\begin{aligned} \boldsymbol{J}_1 &= \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \\ &\vdots \\ \boldsymbol{J}_n &= \boldsymbol{J} + n\frac{\partial \boldsymbol{D}}{\partial t}; \qquad n \rightarrow \infty \\ \rho_n &= \pm \int \nabla \cdot \boldsymbol{J}_n \, dt; \qquad n \rightarrow \infty \end{aligned} \tag{434}$$

and theoretically, there are two infinitely large densities in the vacuum given by

$$\begin{aligned} \rho_n &= \int \nabla \cdot \boldsymbol{J}_n \, dt; \qquad n \rightarrow \infty \\ \rho_n &= -\int \nabla \cdot \boldsymbol{J}_n \, dt; \qquad n \rightarrow \infty \end{aligned} \tag{435}$$

because charge density can either be negative or positive. In this process, $\boldsymbol{B}$ and $\boldsymbol{E}$ are unchanged, so the vector and scalar potentials defined by

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}; \qquad \boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} - \nabla \phi \tag{436}$$

remain unchanged.

Therefore the vacuum potential energy difference is given by

$$\Delta V = \pm \int \boldsymbol{J}_n \cdot \boldsymbol{A} \, d^3x \tag{437}$$

and the rate of doing work is

$$\frac{\partial W}{\partial t} = \pm \int \boldsymbol{J}_n \cdot \boldsymbol{E} \, d^3x \tag{438}$$

In thermodynamic equilibrium, the net result is zero in both cases, but locally, there may be a non-zero rate of doing work by these vacuum charges and currents on a device, creating thermal or mechanical energy. This process is unknown in the received view but conserves energy and is consistent with Noether's theorem [6].

The existence of charge density and current density in the vacuum is not consistent with the Maxwell–Heaviside equations, but leads to a description of empirical data [10,15] superior to that of the received view. Vacuum charge and current density on the classical level are therefore postulates on the same philosophical level as the existence of displacement current in the vacuum. The latter emerges from the continuity equation (426) as argued already. If a postulate leads to an improved description of empirical data, then the postulate is valid in natural philosophy, irrespective of the received view. The role of the coefficient g on the O(3) level may be discussed in a similar philosophical vein. As argued already, the existence of g is a direct consequence of the gauge principle, and it exists in the classical vacuum (or free space), on both the U(1) and O(3) levels, in the respective covariant derivatives. It follows that $e/\hbar$ exists in the vacuum in the Maxwell–Heaviside point of view itself, if this be regarded as a U(1) Yang–Mills gauge theory as is the current practice [6]. If $e/\hbar$ exists in the vacuum on the classical level, then charge density may exist in the vacuum

as argued by Lehnert, and so current density may also exist. The Lehnert equations were derived from U(1) gauge theory in Section IV. The existence of e/h in the vacuum on the O(3) level is therefore conceptually no different from its existence in the vacuum on the U(1) level.

As argued in Section III, the form of the received Maxwell–Heaviside equations in free space or classical vacuum is obtained for finite g. The factor g is a direct consequence of gauge theory [6] and is in general, a proportionality constant without which there is no gauge theory, and without which special relativity is violated. The coefficient g is present for all gauge groups in the vacuum, including U(1). The superiority of the O(3) gauge group over the U(1) gauge group in electrodynamics in no way depends on the introduction of g in O(3): g is also present in U(1). The gauge principle and special relativity therefore force the conclusion that e is itself topological in origin, and is not localized on the electron, a conclusion first reached by Frenkel [15]. The bosons (photons) obtained from a quantization of electrodynamics in any gauge group are not charged bosons, as discussed in Section VIII. The physical nature of g may be roughly summarized by noting the fact that g is a coupling constant that is a property of neither the source (electron) nor the field. As demonstrated in Section VIII, the classical O(3) electrodynamics may be extended without conceptual difficulty to quantum electrodynamics on both the nonrelativistic and relativistic levels. Similarly, the constant g exists in the vacuum in U(1) electrodynamics as a consequence of the gauge principle and special relativity, and U(1) electrodynamics quantizes to quantum electrodynamics without charged photons.

In field theory, electric charge [6] is a symmetry of action, because it is a conserved quantity. This requirement leads to the consideration of a complex scalar field ϕ. The simplest possibility [U(1)] is that ϕ have two components, but in general it may have more than two as in the internal space of O(3) electrodynamics which consists of the complex basis ((1),(2),(3)). The first two indices denote complex conjugate pairs, and the third is real-valued. These indices superimposed on the 4-vector A_μ give a 12-vector. In U(1) theory, the indices (1) and (2) are superimposed on the 4-vector A_μ in free space, so A_μ in U(1) electrodynamics in free space is considered as transverse, that is, determined by (1) and (2) only. These considerations lead to the conclusion that charge is not a point localized on an electron; rather, it is a symmetry of action dictated ultimately by the Noether theorem [6].

By way of introduction to the Noether currents and charges that exist in O(3) electrodynamics, the inhomogeneous field of Eq. (32) can be considered in the vacuum (source-free space) and split into two particular solutions:

$$\partial_\mu \boldsymbol{G}^{\mu\nu} = \boldsymbol{0} \tag{439}$$

$$\boldsymbol{J}^\nu = g\varepsilon_0 \boldsymbol{A}_\mu \times \boldsymbol{G}^{\mu\nu} \tag{440}$$

The first of these has been discussed in Section IV. The second is a vacuum charge–current 12-vector in SI units. On the O(3) level, it is a physical charge–current that gives rise to the energy

$$En^{(3)} = -\int \boldsymbol{J}^\nu \cdot \boldsymbol{A}_\nu \, dV \tag{441}$$

where V is the radiation volume. The energy term can be developed as follows

$$\begin{aligned} En^{(3)} &= -\frac{1}{\mu_0}\int g\boldsymbol{A}_\mu \times \boldsymbol{G}^{\mu\nu} \cdot \boldsymbol{A}_\nu \, dV \\ &= \frac{g}{\mu_0}\int \boldsymbol{G}^{\mu\nu} \cdot \boldsymbol{A}_\mu \times \boldsymbol{A}_\nu \, dV \\ &= \frac{g^2}{\mu_0}\int \boldsymbol{A}^\mu \times \boldsymbol{A}^\nu \cdot \boldsymbol{A}_\mu \times \boldsymbol{A}_\nu \, dV \\ &= \frac{1}{\mu_0}\int \boldsymbol{B}^{(3)} \cdot \boldsymbol{B}^{(3)} \, dV \end{aligned} \tag{442}$$

and is the energy due to the $\boldsymbol{B}^{(3)}$ component of O(3) electrodynamics. This is a concise way of demonstrating that the Noether charge–currents of O(3) electrodynamics give energy that in principle can be utilized for working devices. In analogy, the Maxwell displacement current of the vacuum gives rise to the electromagnetic field, which carries energy. The same principle is involved on the U(1) and O(3) levels, and the ultimate source of the energy is the topology of the vacuum, which manifests itself through the gauge principle and group theory (Section I). If g were zero in Eq. (440), there would be no energy due to $\boldsymbol{B}^{(3)}$, revealing the latter's topological origin. This energy can be thought of as originating in a covariant derivative with O(3) symmetry, and a covariant derivative is necessitated by special relativity and topology. So in this sense, the energy due to $\boldsymbol{B}^{(3)}$ can be thought of as energy from the vacuum, manifesting itself as part of the electromagnetic field. It is probable that devices can be constructed to take advantage of this property of the vacuum and convert energy of this nature efficiently into usable form.

The principle of taking energy from the vacuum is the gauge principle, and this is illustrated as follows on the U(1) level. The U(1) gauge equations in the vacuum are [6]

$$\left(\partial_\mu + igA_\mu\right)\tilde{G}^{\mu\nu} \equiv 0 \tag{443}$$

$$\left(\partial_\mu + igA_\mu\right)G^{\mu\nu} = 0 \tag{444}$$

where the vacuum 4-current is defined as

$$J^\nu = -ig\varepsilon_0 A_\mu G^{\mu\nu} \tag{445}$$

If we set the index $\mu = 0$ in Eq. (445), for example, the following relations are obtained:

$$\begin{aligned} J_X &= ig\varepsilon_0 E_X \\ J_Y &= ig\varepsilon_0 E_Y \end{aligned} \tag{446}$$

The average energy from this vacuum current can be defined as

$$En = c\int J^{\nu *} A_\nu dV \tag{447}$$

which is

$$En = -i\kappa c\varepsilon_0 \int (E_X^* A_X + E_Y^* A_Y) dV = \varepsilon_0 \int \kappa c E^{(0)} A^{(0)} dV \tag{448}$$

Using

$$E^{(0)} = \kappa c A^{(0)} \tag{449}$$

Eq. (448) becomes the familiar U(1) electromagnetic field energy:

$$En = \varepsilon_0 \int E^{(0)2} dV = \frac{1}{2}\int \left(\varepsilon_0 E^{(0)2} + \frac{1}{\mu_0} B^{(0)2}\right) dV \tag{450}$$

The same result is obtained from Eq. (443) using the same proportionality factor $g = \kappa/A^{(0)}$. Note carefully that without the gauge term $igA\mu$, this energy would vanish, and so the energy is due to the vacuum configuration and topology, in this case assumed to be described by the U(1) group.

Similarly, the magnitude of the linear momentum of the electromagnetic field can be obtained by using the proportionality $g = e/\hbar$ in either Eqs. (443) or (444), giving

$$\begin{aligned} \left(\partial_\mu + ie\frac{A_\mu}{\hbar}\right)\tilde{G}^{\mu\nu} &\equiv 0 \\ \left(\partial_\mu + ie\frac{A_\mu}{\hbar}\right)G^{\mu\nu} &= 0 \end{aligned} \tag{451}$$

Using the standard operator transformation of quantum mechanics

$$\rho_\mu = -i\hbar\partial_\mu \tag{452}$$

Eqs. (451) both become

$$\begin{aligned} \partial_\mu \tilde{G}^{\mu\nu} &\equiv 0 \\ \partial_\mu G^{\mu\nu} &= 0 \end{aligned} \tag{453}$$

and so we retrieve the familiar Maxwell–Heaviside equations in the vacuum. The momentum is obtained from the equivalence

$$\frac{e}{\hbar} = \frac{\kappa}{A^{(0)}} \tag{454}$$

giving the magnitude of the linear momentum as

$$p = \hbar\kappa = eA^{(0)} \tag{455}$$

which is again a topological or vacuum property. Using $En = \hbar\omega$, the energy is given from Eq. (455) by

$$En = ecA^{(0)} \tag{456}$$

and is again topological in origin; that is, it originates from energy inherent in a vacuum configuration described by the non-singly connected group U(1).

The principle behind this derivation is the gauge principle, and so is the same for all gauge groups. The equivalence (456) was first demonstrated on the O(3) level [15], but evidently exists for all gauge group symmetries. The gauge principle in electrodynamics therefore leads to the energy and momentum of the photon and classical field. The 4-current J_μ appears in both Eqs. (443) and (444) and is self-dual, a result that is echoed in the self-duality of the vacuum field equations:

$$\partial_\mu \tilde{G}^{\mu\nu} = \partial_\mu G^{\mu\nu} \tag{457}$$

Another advantage of this principle is that the coupling constant g is always present implicitly in the calculation, meaning that the energy and momentum have a cause, or source. This source is not the charge on the electron, but rather the structure or configuration of the vacuum itself, obtained as a direct result of the gauge principle taken to its logical conclusion.

If the procedure is repeated for the rate of doing work by the vacuum 4-current J^ν

$$\frac{dW}{dt} = c\int \boldsymbol{J}^* \cdot \boldsymbol{E}\, dV \tag{458}$$

it is found that

$$\frac{dW}{dt} = c\int \left(J_X^* E_X + J_Y^* E_Y\right) dV \tag{459}$$

which is zero if $\boldsymbol{E}$ is a transverse plane wave. This result means that the energy corresponding to J^ν is conserved in the vacuum because the rate of doing work is energy per unit time. Therefore the field momentum is also conserved in the vacuum. And therefore J^ν is a Noether current in the vacuum.

On the O(3) level, several new sources of energy from the vacuum emerge as follows. First, define the charge and potential 12-vectors:

$$\boldsymbol{J}^{\mu(i)} \equiv \left(\rho, \frac{\boldsymbol{J}^{(i)}}{c}\right) \tag{460}$$

$$\boldsymbol{A}^{\mu(i)} \equiv \left(\phi, c\boldsymbol{A}^{(i)}\right) \tag{461}$$

so that the energy from a vacuum configuration considered to have O(3) gauge group symmetry is

$$En = -\int \left(\boldsymbol{J}^{\nu(1)} \cdot \boldsymbol{A}_\nu^{(2)} + \boldsymbol{J}^{\nu(2)} \cdot \boldsymbol{A}_\nu^{(1)} + \boldsymbol{J}^{\nu(3)} \cdot \boldsymbol{A}_\nu^{(3)}\right) dV \tag{462}$$

(*En* is used here to denote energy, not to be confuse with E as an electrical field). The 12-vector is a spinor in which the Greek indices in covariant contravariant notation are 0, 1, 2, and 3 and the numerical index (i) runs from 1 to 3, representing the circular basis ((1),(2),(3)). For example [11–20]), $A^{\mu(1)}$ is the 4-vector, $(\phi^{(1)}, c\boldsymbol{A}^{(1)})$, $A^{\mu(2)}$ is the 4-vector, $(\phi^{(2)}, c\boldsymbol{A}^2)$, and $A^{\mu(3)}$ is the 4-vector $(\phi^{(3)}, c\boldsymbol{A}^{(3)})$. Each of the three 4-vectors has four components, making a 12-vector. This must not be confused with a vector of 12 components. The field 12-vector is defined as

$$\boldsymbol{G}_{\mu\nu} \equiv G_{\mu\nu}^{(1)} \boldsymbol{e}^{(1)} + G_{\mu\nu}^{(2)} \boldsymbol{e}^{(2)} + G_{\mu\nu}^{(3)} \boldsymbol{e}^{(3)} \tag{463}$$

where each component in indices (1), (2), and (3) have the structure:

$$G^{\mu\nu} \equiv \begin{bmatrix} 0 & -\frac{E^1}{c} & -\frac{E^2}{c} & -\frac{E^3}{c} \\ \frac{E^1}{c} & 0 & -B^3 & B^2 \\ \frac{E^2}{c} & B^3 & 0 & -B^1 \\ \frac{E^3}{c} & -B^2 & B^1 & 0 \end{bmatrix}; \qquad G_{\mu\nu} \equiv \begin{bmatrix} 0 & \frac{E_1}{c} & \frac{E_2}{c} & \frac{E_3}{c} \\ -\frac{E_1}{c} & 0 & -B_3 & B_2 \\ -\frac{E_2}{c} & B_3 & 0 & -B_1 \\ -\frac{E_3}{c} & -B_2 & B_1 & 0 \end{bmatrix} \tag{464}$$

The field equations in the vacuum are (31) and (32), and there are two possible vacuum charge current 12-vectors:

$$\boldsymbol{J}_H^\nu = -g\varepsilon_0 \boldsymbol{A}_\mu \times \tilde{\boldsymbol{G}}^{\mu\nu} \tag{465}$$

$$\boldsymbol{J}^\nu = -g\varepsilon_0 \boldsymbol{A}_\mu \times \boldsymbol{G}^{\mu\nu} \tag{466}$$

which, from Eq. (462), are sources of energy from energy inherent in a vacuum configuration as a direct result of the gauge principle. These two 12-vectors provide several more sources of energy, a result that can be illustrated with Eq. (466) by developing it as follows in the ((1),(2),(3)) basis:

$$\begin{aligned} \boldsymbol{J}^{(1)*} &= -ig\varepsilon_0 \boldsymbol{A}_\mu^{(2)} \times \boldsymbol{G}^{\mu\nu(3)} \\ \boldsymbol{J}^{(2)*} &= -ig\varepsilon_0 \boldsymbol{A}_\mu^{(3)} \times \boldsymbol{G}^{\mu\nu(1)} \\ \boldsymbol{J}^{(3)*} &= -ig\varepsilon_0 \boldsymbol{A}_\mu^{(1)} \times \boldsymbol{G}^{\mu\nu(2)} \end{aligned} \tag{467}$$

This result follows because of the negative sign in Eqs. (465) and (466). Equation (462) for the energy is therefore

$$\begin{aligned} En = -ig\varepsilon_0 \Bigg(&\int \boldsymbol{A}_\mu^{(3)} \times \boldsymbol{G}^{\mu\nu(1)} \cdot \boldsymbol{A}_\nu^{(2)} dV + \int \boldsymbol{A}_\mu^{(2)} \times \boldsymbol{G}^{\mu\nu(2)} \cdot \boldsymbol{A}_\nu^{(3)} dV \\ &+ \int \boldsymbol{A}_\mu^{(1)} \times \boldsymbol{G}^{\mu\nu(2)} \cdot \boldsymbol{A}_\nu^{(3)} dV \Bigg) \end{aligned} \tag{468}$$

Now use the 3-vector identity:

$$\boldsymbol{F} \cdot \boldsymbol{G} \times \boldsymbol{H} = \boldsymbol{G} \cdot \boldsymbol{H} \times \boldsymbol{F} \tag{469}$$

to obtain

$$\begin{aligned} En = -ig\varepsilon_0 \Bigg(&\int \boldsymbol{G}^{\mu\nu(1)} \cdot \boldsymbol{A}_\nu^{(2)} \times \boldsymbol{A}_\mu^{(3)} dV + \int \boldsymbol{G}^{\mu\nu(3)} \cdot \boldsymbol{A}_\nu^{(1)} \times \boldsymbol{A}_\mu^{(2)} dV \\ &+ \int \boldsymbol{G}^{\mu\nu(2)} \cdot \boldsymbol{A}_\nu^{(3)} \times \boldsymbol{A}_\mu^{(1)} dV \Bigg) \end{aligned} \tag{470}$$

The definition (461) implies that we can write

$$c^2 \boldsymbol{B}_{\nu\mu}^{(1)*} = c^2 \boldsymbol{B}_{\nu\mu}^{(2)} \equiv -ig\boldsymbol{A}_\nu^{(2)} \times \boldsymbol{A}_\mu^{(3)} \tag{471}$$

The energy terms in Eq. (470) can therefore be developed as follows:

1.

$$
\begin{aligned}
En_1 &= \varepsilon_0 c^2 \int \boldsymbol{G}^{\mu\nu(1)} \cdot \boldsymbol{B}^{(2)}_{\nu\mu}\, dV \\
&= \frac{1}{\mu_0}\int \boldsymbol{B}^{\mu\nu(1)} \cdot B^{(2)}_{\nu\mu}\, dV \\
&= \frac{1}{\mu_0}\int \boldsymbol{B}^{3\nu(1)} \cdot \boldsymbol{B}^{(2)}_{\nu 3}\, dV \\
&= \frac{1}{\mu_0}\int B^{31(1)}B^{(2)}_{13} + B^{32(1)}B^{(2)}_{23}\, dV \\
&= \frac{1}{\mu_0}\int -B^{2(1)}B^{(2)}_{2} - B^{1(1)}B^{(2)}_{1}\, dV \\
&= \frac{1}{\mu_0}\int B^{(1)}_{Y}B^{(2)}_{Y} + B^{(1)}_{X}B^{(2)}_{X}\, dV
\end{aligned}
\tag{472}
$$

2.

$$
\begin{aligned}
En_2 &= \varepsilon_0 c^2 \int \boldsymbol{G}^{\mu\nu(3)} \cdot \boldsymbol{B}^{(3)}_{\nu\mu}\, dV \\
&= \varepsilon_0 c^2 \int G^{12(3)}B^{(3)}_{21} + G^{21(3)}B^{(3)}_{12}\, dV \\
&= \frac{1}{\mu_0}\int B^{(3)}_{Z}B^{(3)}_{Z} + B^{(3)}_{Z}B^{(3)}_{Z}\, dV
\end{aligned}
\tag{473}
$$

3.

$$
\begin{aligned}
En_3 &= \varepsilon_0 c^2 \int \boldsymbol{G}^{\mu\nu(2)} \cdot \boldsymbol{B}^{(1)}_{\nu\mu}\, dV \\
&= \varepsilon_0 c^2 \int \boldsymbol{G}^{\mu 3(2)} \cdot \boldsymbol{B}^{(1)}_{3\mu}\, dV \\
&= \frac{1}{\mu_0}\int B^{(2)}_{Y}B^{(1)}_{Y} + B^{(2)}_{X}B^{(1)}_{X}\, dV
\end{aligned}
\tag{474}
$$

These derivations are given in full detail to show that the O(3) gauge principle leads to several more terms than in U(1), where the same gauge principle leads to Eq. (450).

The overall result for the vacuum energy in U(1) is

$$
En = \frac{1}{\mu_0}\int B^{(0)2}\, dV \tag{475}
$$

and the corresponding result in O(3) is

$$
En = En_1 + En_2 + En_3 \tag{476}
$$

If we adopt a gauge group of higher symmetry than O(3), there will be more terms and so on, and this is a general principle. Electromagnetic charge current and electromagnetic energy depend on the configuration of the vacuum, and ultimately on the topology of the vacuum as represented in the language of gauge and group theory (Section I). Charge current is a property of the vacuum, and charge is not localized to a point as in the conventional view. On both U(1) and O(3) levels, the field equations can be expressed in terms solely of potentials that, in the language of general relativity, are connections. The constant e becomes a scaling factor and both $g = \frac{e}{\hbar} = \frac{\kappa}{A^{(0)}}$ and all field potentials are consequences of the gauge principle for all gauge groups, including U(1).

We can begin to think of the electromagnetic field in the same terms as the gravitational field, and the former is not an entity superimposed on the vacuum irrespective of the vacuum structure. This conclusion is reminiscent of Faraday's concept, as adopted by Maxwell [4], of charge as being the result of the field. In gauge theory, g is a property of neither electron nor field, but a property of the structure of the vacuum itself. The energy and charge current also come from the vacuum. These concepts are further developed in Section XII. Finally, the energy momentum of the field on the O(3) level is a 12-vector:

$$
\boldsymbol{p}_\mu \equiv p^{(1)}_\mu \boldsymbol{e}^{(1)} + p^{(2)}_\mu \boldsymbol{e}^{(2)} + p^{(3)}_\mu \boldsymbol{e}^{(3)} \tag{477}
$$

giving a new view of field momentum. This view is quite different from the problematic [4] view of electromagnetic energy proposed by Poynting.

Electromagnetic theory in the vacuum at the O(3) level begins to look like the theory of gravitation, the electromagnetic field can be replaced by physical potential differences, and these are primary. Analogously, mass in general relativity is a curvature of spacetime, and the gravitational field is the coordinate system itself. On the O(3) level, the potentials are connection coefficients, and charge is the result of topology expressed through gauge theory and group theory. It has been shown that the topology of the vacuum can produce energy, and that charge–current emanates from the same source. If the potential is a connection, then the field can be expressed in terms of the potential and therefore wholly in terms of the connection, and therefore in terms of topology. The view presented here of the field particle dualism of de Broglie is that all particles are pseudo particles and the vacuum electromagnetic field is the topology of the vacuum itself. This point of view rejects action at a distance, as did Newton himself. It is clear that particles result from the gauge principle, for example, photons and quarks, as the result of quantization of the potential.

The potential is again primary in canonical quantization, and it has been shown in Section IX that quantization of O(3) electrodynamics does not lead to charged photons.

X. SCALAR INTERFEROMETRY AND CANONICAL QUANTIZATION FROM WHITTAKER'S POTENTIALS

Whittaker's early work [27,28] is the precursor [4] to twistor theory and is well developed. Whittaker showed that a scalar potential satisfying the Laplace and d'Alembert equations is structured in the vacuum, and can be expanded in terms of plane waves. This means that in the vacuum, there are both propagating and standing waves, and electromagnetic waves are not necessarily transverse. In this section, a straightforward application of Whittaker's work is reviewed, leading to the feasibility of interferometry between scalar potentials in the vacuum, and to a trouble-free method of canonical quantization.

Whittaker [27,28] derived equations defining the electromagnetic field in the vacuum in terms of functions $\boldsymbol{f}$ and $\boldsymbol{g}$ with the units of magnetic flux directed longitudinally in the axis of propagation (Z)

$$[\boldsymbol{f} = F\boldsymbol{k}; \qquad \boldsymbol{g} = G\boldsymbol{k} \tag{478}$$

and defined all field components in terms of $\boldsymbol{f}$ and $\boldsymbol{g}$. The electric and magnetic field vectors in the vacuum, in SI units, are defined by

$$\boldsymbol{E} = c\nabla \times (\nabla \times \boldsymbol{f}) + \nabla \times \dot{\boldsymbol{g}} \tag{479a}$$

$$\boldsymbol{B} = \frac{1}{c}\nabla \times \dot{\boldsymbol{f}} - \nabla \times (\nabla \times \boldsymbol{g}) \tag{479b}$$

If we use the Stratton potential defined by

$$B^{\mu} \equiv (cP, \boldsymbol{S}) \tag{480}$$

where

$$\boldsymbol{E} = -\nabla \times \boldsymbol{S}; \qquad \boldsymbol{B} = -\frac{\partial \boldsymbol{S}}{\partial t} - \nabla P \tag{481}$$

and the 4-potential defined by

$$A^{\mu} \equiv (\phi, c\boldsymbol{A}) \tag{482}$$

where

$$\boldsymbol{B} = -\nabla \times \boldsymbol{A}; \qquad \boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} - \nabla\phi \tag{483}$$

it is deduced that

$$\boldsymbol{A} = -\nabla \times \boldsymbol{g} + \frac{1}{c}\dot{\boldsymbol{f}} \tag{484}$$

$$\boldsymbol{S} = -c\nabla \times \boldsymbol{f} - \dot{\boldsymbol{g}} \tag{485}$$

in the vacuum. So, in general, the Maxwell potential $\boldsymbol{A}$ and the Stratton potential $\boldsymbol{S}$ both have longitudinal components in the vacuum. Both $\boldsymbol{A}$ and $\boldsymbol{S}$ are generated from the more fundamental $\boldsymbol{f}$ and $\boldsymbol{g}$, and their longitudinal components in the vacuum are

$$A_Z = \frac{1}{c}\dot{F}; \qquad S_Z = -\dot{G} \tag{486}$$

The longitudinal magnetic and electric field components are [27,28]:

$$B_Z = \frac{\partial^2 G}{\partial X^2} + \frac{\partial^2 G}{\partial Y^2}; \qquad E_Z = \frac{\partial^2 F}{\partial Z^2} - \frac{1}{c^2}\frac{\partial^2 F}{\partial t^2} \tag{487}$$

It is now known that these equations correspond to twistor contour integral solutions for a particle with zero rest mass, and lead to an O(3) symmetry gauge group for electromagnetism in the vacuum because the Whittaker solution is a spinor formalism. Electrodynamics on the O(3) level is also a spinor, and ultimately a twistor, formalism. Using the Penrose transform [4], the full significance of the Whittaker solution becomes apparent. Later in this section, the $\boldsymbol{B}^{(3)}$ field is expressed in terms of $\boldsymbol{f}$ and $\boldsymbol{g}$, which are therefore physical. It is this property that leads to the possibility of interferometry between scalar potentials. In the received view [U(1) level], the scalar potential in the vacuum is zero or unphysical, and so the received view loses a great deal of information. The work of Whittaker therefore anticipates much of contemporary non-Abelian gauge theory applied to electrodynamics in the vacuum. In the original equations of J. C. Maxwell [78], Faraday's electrotonic state is a physical *vector potential*, a term that was introduced by Maxwell himself [79]. It is the later interpretation of Maxwell's original intent by Heaviside [80] that relegates the U(1) vector potential to a mathematical subsidiary with no physical meaning. Several refutations of Heaviside's opinion have been given in this chapter already. It is also incompatible with electromagnetism as a twistor theory, where Maxwell's original intent is realized, and vector potentials are physical on the classical level. To be precise, vector and scalar potential differences can be measured experimentally on the classical level.

Without loss of generality, it can be assumed that plane waves can be used for the transverse parts of $\boldsymbol{S}$ and $\boldsymbol{A}$, resulting in

$$\boldsymbol{S} = ic\boldsymbol{A} \tag{488}$$

We obtain, self-consistently

$$\boldsymbol{f} = i\boldsymbol{g}; \quad \dot{\boldsymbol{f}} = i\dot{\boldsymbol{g}} \tag{489}$$

The following scalar magnetic flux gives transverse plane waves for $\boldsymbol{A}$ and $\boldsymbol{S}$

$$G = \frac{A^{(0)}}{\sqrt{2}}(X - iY)e^{i(\omega t - \kappa Z)} \tag{490}$$

so that

$$\begin{aligned} \boldsymbol{A} &= -\nabla \times \boldsymbol{g} = \frac{A^{(0)}}{\sqrt{2}}(i\boldsymbol{i} + \boldsymbol{j})e^{i(\omega t - \kappa Z)} \\ \boldsymbol{B} &= -\nabla \times \boldsymbol{A} = \frac{B^{(0)}}{\sqrt{2}}(i\boldsymbol{i} + \boldsymbol{j})e^{i(\omega t - \kappa Z)} \end{aligned} \tag{491}$$

Importantly, there also exists a longitudinal propagating part of the vector potential

$$\boldsymbol{A}_L = \frac{i}{c}\dot{G}\boldsymbol{k} = -\kappa\frac{A^{(0)}}{\sqrt{2}}(X - iY)e^{i(\omega t - \kappa Z)}\boldsymbol{k} \tag{492}$$

that is not present in the received view [6]. For example, $\boldsymbol{A}_L$ is zero in the radiation and Coulomb gauges, and is considered in the received view to be unphysical in the Lorenz gauge [6]. The longitudinal vector potential gives rise to the transverse magnetic plane wave

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}_L = -\frac{B^{(0)}}{\sqrt{2}}(\boldsymbol{i} + i\boldsymbol{j})e^{i(\omega t - \kappa Z)} \tag{493}$$

and to the electric field:

$$\boldsymbol{E}_L = -\frac{\partial \boldsymbol{A}_L}{\partial t} - \nabla\phi = i\frac{\kappa^2}{c}\frac{A^{(0)}}{\sqrt{2}}(X - iY)e^{i(\omega t - \kappa Z)}\boldsymbol{k} - \nabla\phi \tag{494}$$

In general, therefore, there is a longitudinal propagating component of the electric field in the vacuum. However, in the plane-wave approximation used here, there occurs the relation

$$\nabla\phi = \nabla \times \boldsymbol{S} - \frac{\partial \boldsymbol{A}}{\partial t} \tag{495}$$

and the longitudinal part of $\nabla\phi$ is

$$(\nabla\phi)_L = -\frac{\partial \boldsymbol{A}}{\partial t} \tag{496}$$

so the net longitudinal propagating electric field vanishes. Similarly, the longitudinal magnetic field is

$$\boldsymbol{B}_L = -ic\frac{\partial \boldsymbol{A}_L}{\partial t} - \nabla P = \omega^2\frac{A^{(0)}}{\sqrt{2}}(X - iY)e^{i(\omega t - \kappa Z)} - \nabla P \tag{497}$$

and using

$$\nabla P = \nabla \times \boldsymbol{A} + \frac{\partial \boldsymbol{S}}{\partial t} \tag{498}$$

the longitudinal part of ∇P is

$$(\nabla P)_L = \frac{\partial \boldsymbol{S}}{\partial t} \tag{499}$$

and the longitudinal magnetic field vanishes. These results are consistent with Whittaker's

$$\begin{aligned} E_Z &= \frac{\partial^2 F}{\partial Z^2} - \frac{1}{c^2}\frac{\partial^2 F}{\partial t^2} = 0 \\ B_Z &= \frac{\partial^2 G}{\partial X^2} + \frac{\partial^2 G}{\partial Y^2} = 0 \end{aligned} \tag{500}$$

when F and G correspond to plane waves. The presence of a longitudinal vector potential and longitudinal $\boldsymbol{f}$ and $\boldsymbol{g}$ potentials in Whittaker's theory demonstrate that it is not a U(1) theory of electromagnetism. On the simplest level, Whittaker's theory defines the $\boldsymbol{B}^{(3)}$ field as

$$\boldsymbol{B}^{(3)*} = -i\frac{\boldsymbol{\kappa}}{A^{(0)}}(\nabla \times \boldsymbol{g}) \times (\nabla \times \boldsymbol{g}^*) \tag{501}$$

so $\boldsymbol{g}$ is a physical and measurable quantity, a result that is consistent with Whittaker's own result that G and F can be expanded in terms of plane waves and are structured and physical quantities, and with the fact that Whittaker reduces the U(1) equations in the vacuum to two d'Alembert equations

$$\Box F = \Box G = 0 \tag{502}$$

which are Lorentz- and gauge-invariant. Canonical quantization can therefore proceed through consideration of F and G, giving the photon straightforwardly as demonstrated later in this section. This type of canonical quantization is free of the difficulties associated with canonical quantization [6] in the Coulomb and Lorenz gauges.

In the plane-wave approximation, all electromagnetic effects are derived from the structured time-like potential difference

$$\phi = \dot{F} = i\dot{G} = -\omega \frac{A^{(0)}}{\sqrt{2}}(X - iY)e^{i(\omega t - \kappa Z)} \tag{503}$$

which is thereby a physical observable in effects such as those observed reproducibly and repeatedly by Priore and others [81–85]. These effects have no explanation in the received view, but may be highly beneficial if properly developed. The entities known as electric and magnetic fields are double differentials of ϕ in the plane-wave approximation in the vacuum, a result that is consistent with the ontology developed in Section IX, that the topology of the vacuum is primary, and that potential differences are the result of the vacuum topology. Whittaker uses the usual Lorenz condition, and it is easily verified that

$$\nabla \cdot \boldsymbol{A}_L + \frac{1}{c^2}\frac{\partial \phi_L}{\partial t} = 0 \tag{504}$$

If gauge freedom is lost, however, the Lorenz condition is no longer valid, and a far more comprehensive view of the electromagnetic entity would be obtained by solving the O(3) equations numerically. On the O(3) level, there is no gauge freedom, and no Lorenz condition.

As discussed by Frauendiener and Tsun in Ref. 4, gauge field theory is a form of twistor theory, and as discussed in Section IX, the covariant derivative must always be used in a gauge field theory, even on the U(1) level. The covariant derivative must be used in curved spacetime, and in gauge theory when used with ordinary flat spacetime. These authors also point out that the phase on the U(1) level has no physical significance: it can be redefined by an arbitrary rotation at any point in spacetime. The role of the covariant derivative, or connection, is to compare phases at two neighboring points [4]. This property leads directly to the conclusion that electromagnetism in the vacuum is not a U(1) theory, but a Yang–Mills theory of higher symmetry. We have seen, for example, that the U(1) covariant derivative does not describe the Sagnac effect, whereas O(3) theory describes it accurately because of the nontrivial self interaction [4] resulting in the $\boldsymbol{B}^{(3)}$ field concept. Since O(3) electrodynamics is a Yang–Mills theory, it is also a spinor theory and also a twistor theory [4], which takes us full circle to the fact that Whittaker's theory is a twistor theory.

Potential differences are primary in gauge theory, because they define both the covariant derivative and the field tensor. In Whittaker's theory [27,28], potentials can exist without the presence of fields, but the converse is not true. This conclusion can be demonstrated as follows. Equation (479b) is invariant under

$$\boldsymbol{g} \rightarrow \boldsymbol{g} + \nabla \boldsymbol{a}; \qquad \nabla \times \boldsymbol{g} \rightarrow \nabla \times \boldsymbol{g} + \nabla \boldsymbol{b} \tag{505a}$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are arbitrary. This invariance implies that:

$$\nabla \times \boldsymbol{g} \rightarrow \nabla \times \boldsymbol{g} \tag{505b}$$

The transverse part of the vector potential is therefore invariant under the transformations (505), because of the definition

$$\boldsymbol{A}_T = -\nabla \times \boldsymbol{g} \tag{506}$$

and this is a clear sign of the fact that Whittaker's theory contains something contrary to the received view that the transverse $\boldsymbol{A}_T$ is always unphysical. The gauge invariance of $\boldsymbol{A}_T$ does not occur at the U(1) level, but on the O(3) level, the vector potential is gauge covariant and physical, as in the Sagnac effect with rotating platform.

The magnetic fluxes F and G obey the Klein–Gordon equation for a massless particle in the vacuum:

$$\Box F = \Box G = 0 \tag{507}$$

and if we apply Eq. (505), we obtain

$$\Box(\nabla a) = \Box(\nabla c) = 0 \tag{508}$$

indicating that $\boldsymbol{a}$ and $\boldsymbol{c}$ are not arbitrary. Therefore $\boldsymbol{f}$ and $\boldsymbol{g}$ are physical and observable, $\boldsymbol{A}_T$ is physical and observable, and the transverse part of $\boldsymbol{A}^{\mu}$ is physical. These conclusions refute U(1) electrodynamics.

This result is consistent with Whittaker's main conclusion [27,28], that the scalar potential ϕ is structured and physical in the vacuum, leading to the possibility of interferometry between different scalar potentials, without the presence of fields. To reinforce this conclusion, we can differentiate Eq. (484)

$$\dot{\boldsymbol{A}} = -\nabla \times \dot{\boldsymbol{g}} + \frac{1}{c}\ddot{\boldsymbol{f}} \tag{509}$$

and use the Lorenz condition (also used by Whittaker)

$$\nabla \cdot \boldsymbol{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0 \tag{510}$$

to give the following expression for the scalar potential:

$$\dot{\phi} = c^2 \nabla \cdot (\nabla \times \boldsymbol{g}) - c \nabla \cdot \dot{\boldsymbol{f}} \tag{511}$$

This results in the following expression for the potential 4-vector

$$\begin{aligned} A^{\mu} &= (\phi, c\boldsymbol{A}) \\ &= \left(c^2 \int \nabla \cdot (\nabla \times \boldsymbol{g}) dt - c \nabla \cdot \boldsymbol{f}, -c \nabla \times \boldsymbol{g} + \dot{\boldsymbol{f}} \right) \\ &= \left(-c^2 \int \nabla \cdot \boldsymbol{A} \, dt - c \nabla \cdot \boldsymbol{f}, -c \nabla \times \boldsymbol{g} + \dot{\boldsymbol{f}} \right) \\ &= (\phi - c \nabla \cdot \boldsymbol{f}, c\boldsymbol{A} + \dot{f}) \\ &= (\phi_T, c\boldsymbol{A}_T) + (\phi_L, c\boldsymbol{A}_L) \end{aligned} \tag{512}$$

where it is split into its transverse and longitudinal components in the vacuum. The longitudinal component is

$$A_L^{\mu} = (\phi_L, c\boldsymbol{A}_L) = (-c \nabla \cdot \boldsymbol{f}, \dot{\boldsymbol{f}}) \tag{513}$$

and is physical because $\boldsymbol{f}$ is physical. On canonical quantization, therefore, there exist physical longitudinal photons and time-like photons. By definition

$$A_L^{\mu} = \left(-c \frac{\partial F}{\partial Z}, \frac{\partial F}{\partial t} \boldsymbol{k} \right) \tag{514}$$

and in the special case where the transverse A_T^{μ} consists of plane waves, $F = iG$ and

$$A_L^{\mu} = -\frac{A^{(0)}}{\sqrt{2}} \omega (X - iY) e^{i(\omega t - \kappa Z)} (1, \boldsymbol{k}) \tag{515}$$

The vacuum longitudinal potential is light-like

$$A_{L\mu} A_L^{\mu} = 0 \tag{516}$$

and may be written as

$$A_L^{\mu} = (\phi_L, c\phi_L \boldsymbol{k}) \tag{517}$$

The potential ϕ_L obeys the massless Klein–Gordon equation

$$\Box \phi_L = 0 \tag{518}$$

and it is well known that canonical quantization of this equation is straightforward [6]. This result is consistent with Whittaker's main result that ϕ_L is physical and made up of a sum of plane waves and standing waves in the vacuum [27,28].

The Lagrangian for Eq. (518) is well known [6] to be

$$\mathscr{L} = \frac{1}{2} g^{\kappa\lambda} (\partial_{\kappa} \phi_L)(\partial_{\lambda} \phi_L^*) \tag{519}$$

from which is obtained the energy momentum tensor

$$\theta_{\nu}^{\mu} = \partial \frac{\mathscr{L}}{\partial(\partial_{\mu} \phi_L)} \partial_{\nu} \phi_L^* - \delta_{\nu}^{\mu} \mathscr{L} \tag{520}$$

and the Hamiltonian

$$H_L = \int \theta^{00} d^3 x \tag{521}$$

In SI units, the Hamiltonian is the positive definite

$$H_L = \frac{1}{\mu_0 R^2} \int (\partial_0 \phi_L^* \partial_0 \phi_L + \nabla \phi_L^* \cdot \nabla \phi_L) dV \tag{522}$$

where the beam radius is $R^2 = X^2 + Y^2$. Using the relations

$$\begin{aligned} \phi_L &= -\frac{A^{(0)}}{\sqrt{2}} \omega (X - iY) e^{i(\omega t - \kappa Z)} \\ \partial_0 \phi_L &= -\frac{1}{c} i\omega^2 (X - iY) e^{i(\omega t - \kappa Z)} \frac{A^{(0)}}{\sqrt{2}} \\ \nabla \phi_L^* &= i \frac{A^{(0)}}{\sqrt{2}} \kappa \omega (X + iY) e^{-i(\omega t - \kappa Z)} \\ \nabla \phi_L &= -i \frac{A^{(0)}}{\sqrt{2}} \kappa \omega (X - iY) e^{i(\omega t - \kappa Z)} \end{aligned} \tag{523}$$

the Hamiltonian reduces to

$$H_L = \frac{1}{\mu_0} \int B^{(0)2} dV \tag{524}$$

which is identical with Eq. (450) of Section IX. This result proves that ϕ_L is physical because the result (524) is a physical vacuum electromagnetic energy.

Whittaker theory refutes U(1) theory in several ways, so it may be more appropriate to describe the result (524) as a component at the O(3) level:

$$H_L = \frac{1}{\mu_0}\int \boldsymbol{B}^{(3)} \cdot \boldsymbol{B}^{(3)} dV \tag{525}$$

It may also be argued as follows that $\boldsymbol{f}$ and $\boldsymbol{g}$ are physical. If an attempt is made to apply the usual U(1) gauge transform rule to A_L^μ

$$A_L^\mu \rightarrow A_L^\mu + \partial^\mu \chi; \quad A_L \rightarrow A_L - \nabla\chi; \quad \phi \rightarrow \phi + \frac{\partial \chi}{\partial t} \tag{526}$$

it follows that

$$\dot{\boldsymbol{f}} \rightarrow \dot{\boldsymbol{f}} - c\nabla\chi; \qquad \nabla \cdot \boldsymbol{f} \rightarrow \nabla \cdot \boldsymbol{f} - \frac{1}{c}\frac{\partial \chi}{\partial t} \tag{527}$$

and

$$\int \nabla\chi \, dt = \frac{1}{c^2}\int \frac{d\chi}{dt} dZ \tag{528}$$

It follows from Eq. (528) that the quantity χ is not random, contrary to the U(1) rule that χ must be random. Euation (528) implies solutions of the type

$$\chi = \chi_0 e^{i(\omega t - \kappa Z)} \tag{529}$$

so that

$$A_L^\mu \rightarrow A_L^\mu + A_L^{\mu'} \tag{530}$$

where

$$A_L^{\mu'} = i\omega\chi_0 e^{i\theta}(1, \boldsymbol{k}) = (\phi_L', c\boldsymbol{A}_L') \tag{531}$$

and

$$\Box A_L^{\mu'} = 0; \qquad \phi_L' = 0 \tag{532}$$

The net result is

$$\begin{aligned} &\phi_L \rightarrow \phi_L + i\omega\chi_0 e^{i\Phi} \\ &\Phi = \omega t - \kappa Z \end{aligned} \tag{533}$$

If, for example

$$i\chi_0 \equiv -\frac{A^{(0)}}{\sqrt{2}}(X - iY) \tag{534}$$

then

$$\phi_L \rightarrow 2\phi_L; \qquad F \rightarrow 2F; \qquad G \rightarrow 2G \tag{535}$$

and a field such as [27,28]

$$E_X = \frac{\partial^2 F}{\partial X \partial Z} + \frac{1}{c}\frac{\partial^2 G}{\partial Y \partial t} \tag{536}$$

doubles in magnitude. The field is not invariant, contrary to the requirements of U(1) theory. The only possibility is that $\chi = 0$, and that is physical and observable.

Physical potentials are present in Whittaker's theory without fields. This is demonstrated as follows in the special case of a plane wave for the transverse parts of $\boldsymbol{E}$ and $\boldsymbol{B}$. In this special case

$$\boldsymbol{f} = i\boldsymbol{g} \tag{537}$$

and from Eqs. (479a) and (479b)

$$\boldsymbol{E} = ic\nabla \times (\nabla \times \boldsymbol{g}) + a\nabla \times \dot{\boldsymbol{g}} \tag{538a}$$

$$\boldsymbol{B} = \frac{i}{c}\nabla \times \dot{\boldsymbol{g}} - \nabla \times (\nabla \times \boldsymbol{g}) \tag{538b}$$

Under the condition

$$\nabla \times (\nabla \times \boldsymbol{g}) = \frac{i}{c}\frac{\partial}{\partial t}(\nabla \times \boldsymbol{g}) \tag{539}$$

all the components of $\boldsymbol{E}$ and $\boldsymbol{B}$ vanish. The condition (539) is satisfied by

$$\nabla \times \boldsymbol{A}_T = \frac{i}{c}\frac{\partial \boldsymbol{A}_T}{\partial t} \tag{540}$$

whose solution is

$$\boldsymbol{A}_T = \frac{A^{(0)}}{\sqrt{2}}(i\boldsymbol{i} + \boldsymbol{j})e^{-i(\omega t - \kappa Z)} \tag{541}$$

The overall result is

$$\begin{aligned} \boldsymbol{E} &= \boldsymbol{B} = \boldsymbol{0} \\ \boldsymbol{A}_L &= -\kappa \frac{A^{(0)}}{\sqrt{2}}(X - iY)e^{-i(\omega t - \kappa Z)}\boldsymbol{k} \\ \phi_L &= -\omega \frac{A^{(0)}}{\sqrt{2}}(X - iY)e^{-i(\omega t - \kappa Z)} \\ G &= \frac{F}{i} = \frac{A^{(0)}}{\sqrt{2}}(X - iY)e^{-i(\omega t - \kappa Z)} \end{aligned} \tag{542}$$

so there can be both transverse and longitudinal physical potentials, or connections. Electromagnetism can be described entirely without fields, and in terms of the vacuum topology.

Whittaker also argued [27,28] that longitudinal standing waves occur in the vacuum. These can be illustrated by the choice of flux

$$G = \frac{A^{(0)}}{\sqrt{2}}(X - iY)(e^{i(\omega t - \kappa Z)} + e^{-i(\omega t - \kappa Z)}) \tag{543}$$

a choice that obeys the d'Alembert equation:

$$\Box G = 0 \tag{544}$$

The real part of Eq. (543) is

$$\mathrm{Re}(G) = \frac{2}{\sqrt{2}}A^{(0)}(X \cos \omega t \cos \kappa Z + Y \cos \omega t \sin \kappa Z) \tag{545}$$

which is a standing wave in the vacuum, directed along the propagation axis. Such waves do not exist in the received U(1) theory. The magnetic flux

$$\boldsymbol{g} = \frac{2}{\sqrt{2}}A^{(0)}(X \cos \omega t \cos \kappa Z + Y \cos \omega t \sin \kappa Z)\boldsymbol{k} \tag{546}$$

is a solution to the vibrating-string problem, and the idea that electromagnetism must be described in the vacuum by transverse plane waves of $\boldsymbol{E}$ and $\boldsymbol{B}$ is clearly erroneous. Fluxes of the type (546) give rise to scalar potential interferometry where there are no detectible fields.

It has been shown that the electromagnetic field in Whittaker's view originates in the vacuum, and in the plane wave approximation, in the equation

$$\phi_L = \dot{F} = i\dot{G} = -\omega \frac{A^{(0)}}{\sqrt{2}}(X - iY)e^{i(\omega t - \kappa Z)} \tag{547}$$

under conditions of circular polarization. The scalar potential ϕ_L is time-like, physical, and structured, and it propagates. An experimental design can be used to test experimentally whether $\boldsymbol{f}$ and $\boldsymbol{g}$ are physical. The principle of the design is very simple. Two dipole antennae are set up in close proximity so that the vector potentials from each antenna cancel:

$$\boldsymbol{A}_1 = -i\frac{\kappa e^{i\kappa r}}{4\pi c\varepsilon_0 r}\boldsymbol{p}_1 \tag{548}$$

Here $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ are the dipole moments of each antenna and r is the magnitude of the radius vector

$$\boldsymbol{A}_2 = i\frac{\kappa e^{i\kappa r}}{4\pi c\varepsilon_0 \boldsymbol{r}}\boldsymbol{p}_2 \tag{549}$$

in spherical coordinates [86]. It follows that

$$E = \boldsymbol{0}; \qquad \boldsymbol{B} = \boldsymbol{0}; \qquad \boldsymbol{A} = \boldsymbol{A}_1 + A_2 = \boldsymbol{0} \tag{550}$$

so there are no vector potentials or fields radiated into the vacuum by this antenna arrangement. Whittaker's $\boldsymbol{f}$ and $\boldsymbol{g}$ magnetic flux vectors are defined as follows by this arrangement:

$$\boldsymbol{g}_1 = -\boldsymbol{g}_2; \qquad \boldsymbol{f}_1 = -\boldsymbol{f}_2 \tag{551}$$

However, the scalar magnitudes of $\boldsymbol{g}$ and $\boldsymbol{f}$ from both antennas (G and F) are the same, because the scalar magnitude of a vector is the square root of the vector squared. Thus the following quantity is radiated into the vacuum:

$$2G = \frac{2}{\sqrt{2}}A^{(0)}(X - iY)e^{i(\omega t - \kappa Z)} \tag{552}$$

and the scalar potential

$$\phi_L \equiv 2\dot{G} \tag{553}$$

is also present in the vacuum. On canonical quantization, this scalar potential gives an ensemble of massless photons from the Klein–Gordon equation. This property will be proved later in this section. These are physical time-like photons each with the Planck energy $\hbar\omega$. The energy from these photons is therefore Eq. (524), and is phase-free. For a large number of frequencies, the photons are distributed according to the Planck distribution for blackbody radiation [69],

which is radiated heat detectible by a bolometer. There are no vector potentials or fields present, so the heat is due entirely to the physical F and G. In the received view, such photons are unphysical and no heat should be detected. An improvement on this design, due to Labounsky [87], is shown in Fig. 2, which illustrates how fieldless G waves can be generated.

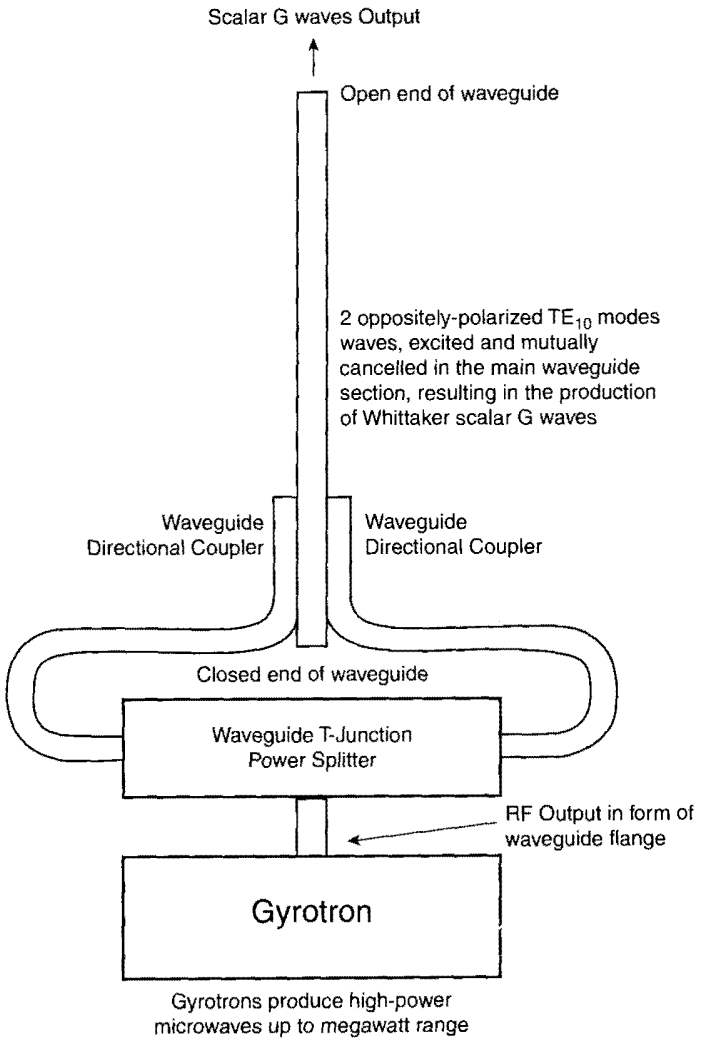


Figure 2. Practical conception for a source of scalar G waves.

Scalar interferometry is possible in this view if F and G are physical in th vacuum. When two scalar beams of the type

$$G_1 = \frac{A^{(0)}}{\sqrt{2}}(X - iY)e^{i(\omega t - \kappa Z_1)}$$
$$G_2 = \frac{A^{(0)}}{\sqrt{2}}(X - iY)e^{i(\omega t - \kappa Z_2)} \tag{554}$$

interfere, an interferogram is generated, as usual, and their combined energy density in the zone of interference is

$$\frac{En}{V} = \frac{cI}{R^2\omega^2}(1 + \cos(\kappa(Z_1 - Z_2))) \tag{555}$$

where I is the combined power density of the two beams in watts per square meter. Here, $Z_1 - Z_2$ is the path difference as usual, that is, the difference in distance traversed by each beam from source (the design in Fig. 2) to interference zone. If we now define

$$G_3 \equiv \frac{1}{G^{(0)}}(G_1 + G_2)(G_1^* + G_2^*) \tag{556}$$

then

$$\Box G_3 = B \neq 0 \tag{557}$$

and a fluctuating magnetic flux density B appears in the zone of interference even though no field is radiated by either source. The presence of a magnetic field indicates the presence of an electric field. There are magnetic and electric fields in the zone of interference but none outside. Equation (557) is a gauge-invariant construct, and the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields in the zone of interference are real and physical, and so interact with matter in the zone of interference. The energy density within this zone is also gauge-invariant and physical

$$\frac{En}{V} = \frac{B^{(0)2}}{\mu_0} = \frac{GG^*}{R^4\mu_0} \tag{558}$$

where R^2 is the beam area, assumed to be the same for each beam. The lateral extent of the radiated beams from the device in Fig. 2 is constrained by the inverse fourth-power dependence on R.

It has been proved that F and G of Whittaker are physical and gauge-invariant, and it follows, as shown next, that there exist physical time-like and

longitudinal photons. These have an independent existence and appear from canonical quantization of the classical, physical, and time-like scalar potential difference in vacuo [Eq. (547)]. Canonical quantization follows straightforwardly from the massless Klein–Gordon equation:

$$\Box\phi_L = 0 \tag{559}$$

The potential ϕ_L is treated as usual [6] as an operator subject to the commutator relation of quantum mechanics. This procedure gives the positive definite Hamiltonian (521) and vacuum energy (524) self-consistently. The scalar potential ϕ_L is Fourier expanded as

$$\phi_L = \int \frac{\partial^3\kappa}{(2\pi)^3 2\omega_\kappa}(a(\kappa)e^{-i\omega Z} + a^+(\kappa)e^{i\omega Z}) \tag{560}$$

a procedure that is self-consistent with Whittaker's original demonstration [27,28] that ϕ_L can be expanded in a Fourier series in the argument denoted by Whittaker in his general solution for ϕ_L. Equation (560) has frequencies $\omega_\kappa = \kappa c$ generated by the Fourier expansion. So many different photons emerge, each corresponding to a different frequency; quantization results in an ensemble [6] of physical time-like photons, each of energy $\hbar\omega$. This is consistent with the Planck quantization of energy momentum

$$p^\mu = \hbar\kappa^\mu \tag{561}$$

where the time-like component has energy $\hbar\omega$.

The coefficients a and a^+ in the expansion (560) are operators defined by the commutators [6]:

$$\begin{aligned}[a(\kappa), a(\kappa')] &= [a^+(\kappa), a^+(\kappa)] = 0 \\ [a(\kappa), a^+(\kappa')] &= (2\pi)^3 2\omega_\kappa \delta^3(\kappa - \kappa')\end{aligned} \tag{562}$$

The operator:

$$N(\kappa) = a^+(\kappa)a(\kappa) \tag{563}$$

represents the number of particles with energy $\hbar\omega$ and longitudinal momentum $\hbar\kappa$. The Hamiltonian, after quantization, takes the form

$$H = \frac{1}{2}P^2(\kappa) + \frac{\omega_\kappa^2}{2}Q^2(\kappa) \tag{564}$$

where

$$\begin{aligned}P(\kappa) &= \left(\frac{\omega_\kappa}{2}\right)^{1/2}(a(\kappa) + a^+(\kappa)) \\ Q(\kappa) &= \frac{1}{(2\omega_\kappa)^{1/2}}(a(\kappa) - a^+(\kappa))\end{aligned} \tag{565}$$

and ϕ_L, after quantization, is an infinite sum of oscillators, that is, an ensemble of time-like photons with energy $\hbar\omega$. The operators a and a^+ respectively are therefore the annihilation and creation operators for the quanta of ϕ_L and the energy of the quantized ϕ_L is rigorously positive. The photons obtained after this type of quantization obey Bose–Einstein statistics [6], and any number of particles (photons) can exist in the same quantization state. These photons are spin zero and massless and, because they are spin zero, are not absorbed by an atom or molecule, in contrast to physical space-like photons carrying angular momentum. The received view of canonical quantization asserts [6] that these photons are unphysical. Paradoxically, the received view also asserts that the vector (561) is physical. This paradox is seen in the Compton and photoelectric effects as argued already in Section III. There are insurmountable difficulties [6] in the received methods of canonical quantization. In the radiation gauge, for example, the scalar and longitudinal parts of the 4-vector A^μ are missing, so A^μ is not fully covariant at the outset. In the Lorenz gauge, there are several difficulties well summarized by Ryder [6]. For example, there is an indefinite metric where the Lorenz condition has to be used and then discarded, then a gauge fixing term has to be used, and the final result is paradoxical in that an admixture of time-like and longitudinal photons are physical [6], but each component is unphysical. The procedure of canonical quantization in the Lorenz gauge gives photons with spin, and these are asserted to be physical transverse photons.

It is far simpler to introduce spin into the assumed massless photon by following the little group method of Wigner [6], that is, by examining the most general type of Lorentz transform possible for a particle without mass. This produces the normalized helicities -1 and 1 through parity considerations. These correspond, in the received view, to physical right and left circularly polarized photons. If the photon is massive, as implied by O(3) electrodynamics, there occurs in addition the helicity zero, corresponding to a physical longitudinal space-like photon without spin and corresponding to a physical O(3) symmetry little group. [The little group of the massless photon is the unphysical [6,15] E(2), another paradox of the received view.] As argued already, there also occurs a time-like photon that is a scalar and that is purely energetic in nature.

These various considerations point toward the O(3) definition of the energy-momentum 4-vector:

$$\boldsymbol{p}^\mu = p^{\mu(1)}\boldsymbol{e}^{(1)} + p^{\mu(2)}\boldsymbol{e}^{(2)} + p^{\mu(3)}\boldsymbol{e}^{(3)} \tag{566}$$

There are therefore three energy-momentum 4-vectors present:

$$p^{\mu(3)} = (En, cp^{(3)}); \qquad p^{\mu(2)} = (En, cp^{(2)}); \qquad p^{\mu(1)} = (En, cp^{(1)}) \tag{567}$$

Energy is a scalar and so does not carry an internal gauge index. There are three momenta; $p^{(3)}$ is longitudinal, and $p^{(1)}$ and $p^{(2)}$ are circularly polarized conjugates. Applying Planck quantization gives immediately a time-like photon $\hbar\omega$ without spin, a longitudinal photon $\hbar k^{(3)}$ without spin and with energy $\hbar\omega$, and right and left circularly polarized photons $\hbar k^{(1),(2)}$, each of energy $\hbar\omega$.

Therefore Whittaker's theory points toward the existence of O(3) electrodynamics. This conclusion is reinforced by the fact that Eqs. (479a) and (479b) are invariant under the duality transform:

$$\begin{aligned} f &\rightarrow -g \\ g &\rightarrow f \end{aligned} \tag{568}$$

and Eqs. (484) and (485) can be written as

$$\nabla \times f + \frac{1}{c}\frac{\partial g}{\partial t} = -\frac{1}{c}S \tag{569}$$

$$\nabla \times g - \frac{1}{c}\frac{\partial f}{\partial t} = -A \tag{570}$$

These equations are invariant under the transform:

$$f \rightarrow -g; \qquad g \rightarrow f; \qquad S \rightarrow -cA \tag{571}$$

Special relativity then dictates that there exists the set of equations

$$\begin{aligned} \nabla \cdot g &= \frac{\phi}{c} \\ \nabla \times g &= \frac{1}{c}\frac{\partial f}{\partial t} - A \\ \nabla \cdot f &= -P \\ \nabla \times f + \frac{1}{c}\frac{\partial g}{\partial t} &= -\frac{1}{c}S \end{aligned} \tag{572}$$

which can be written as

$$\begin{aligned} \partial_\mu \tilde{g}^{\mu\nu} &= A^\nu \\ \partial_\mu g^{\mu\nu} &= -S^\nu \end{aligned} \tag{573}$$

where

$$\tilde{g}^{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & -g^3 \\ 0 & 0 & f^3 & 0 \\ 0 & -f^3 & 0 & 0 \\ g^3 & 0 & 0 & 0 \end{bmatrix} \tag{574}$$

$$g^{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & -f^3 \\ 0 & 0 & -g^3 & 0 \\ 0 & g^3 & 0 & 0 \\ f^3 & 0 & 0 & 0 \end{bmatrix} \tag{575}$$

Equations (573) have overall O(3) symmetry, and have the same structure as the Maxwell–Heaviside equations with magnetic charge and current [3,4]. From Eqs. (573), we obtain the wave equation

$$\Box \tilde{g}^{\mu\nu} = \frac{1}{2}F^{\mu\nu} = 0 \tag{576}$$

which is consistent with Whittaker's starting point:

$$\Box G = \Box F = 0 \tag{577}$$

The received view asserts that A^μ is always random, but in this section, several counter arguments have been given. Several more counterarguments appear throughout this chapter and elsewhere in the literature [3].

XI. PREPARING FOR COMPUTATION

In this section, the field equations (31) and (32) are considered in free space and reduced to a form suitable for computation to give the most general solutions for the vector potentials in the vacuum in O(3) electrodynamics. This procedure shows that Eqs. (86) and (87) are true in general, and are not just particular solutions. On the O(3) level, therefore, there exist no topological monopoles or magnetic charges. This is consistent with empirical data—no magnetic monopoles of any kind have been observed in nature.

If consideration is restricted to the vacuum, the field equations (86) and (90) apply. The Jacobi identity (86) is first considered and written in the following form [6]:

$$D_\lambda G_{\mu\nu} + D_\mu G_{\nu\lambda} + D_\nu G_{\lambda\mu} \equiv \mathbf{0} \tag{578}$$

This reduces in general to the form

$$\partial_\lambda \boldsymbol{G}_{\mu\nu} + \partial_\mu \boldsymbol{G}_{\nu\lambda} + \partial_\nu \boldsymbol{G}_{\lambda\mu} \equiv \boldsymbol{0} \tag{579}$$

because

$$\boldsymbol{A}_\mu \times \tilde{\boldsymbol{G}}^{\mu\nu} \equiv \boldsymbol{0} \tag{580}$$

is identically zero. The proof of this latter result proceeds by using the definitions

$$\begin{aligned} \boldsymbol{G}^{(1)*}_{\mu\nu} &= \partial_\mu \boldsymbol{A}^{(1)*}_\nu - \partial_\nu \boldsymbol{A}^{(1)*}_\mu - ig\boldsymbol{A}^{(2)}_\mu \times \boldsymbol{A}^{(3)}_\nu \\ \boldsymbol{G}^{(2)*}_{\mu\nu} &= \partial_\mu \boldsymbol{A}^{(2)*}_\nu - \partial_\nu \boldsymbol{A}^{(2)*}_\mu - ig\boldsymbol{A}^{(3)}_\mu \times \boldsymbol{A}^{(1)}_\nu \\ \boldsymbol{G}^{(3)*}_{\mu\nu} &= \partial_\mu \boldsymbol{A}^{(3)*}_\nu - \partial_\nu \boldsymbol{A}^{(3)*}_\mu - ig\boldsymbol{A}^{(1)}_\mu \times \boldsymbol{A}^{(2)}_\nu \end{aligned} \tag{581}$$

and Jacobi identities such as:

$$\boldsymbol{A}^{(2)}_\lambda \times (\boldsymbol{A}^{(1)}_\mu \times \boldsymbol{A}^{(2)}_\nu) + \boldsymbol{A}^{(2)}_\mu \times (\boldsymbol{A}^{(1)}_\nu \times \boldsymbol{A}^{(2)}_\lambda) + \boldsymbol{A}^{(2)}_\nu \times (\boldsymbol{A}^{(1)}_\lambda \times \boldsymbol{A}^{(2)}_\mu) \equiv \boldsymbol{0} \tag{582}$$

The terms

$$\begin{aligned} \boldsymbol{A}^{(1)}_\lambda \times (\partial_\mu \boldsymbol{A}^{(2)}_\nu - \partial_\nu \boldsymbol{A}^{(2)}_\mu) &\equiv \boldsymbol{0} \\ \boldsymbol{A}^{(2)}_\lambda \times (\partial_\mu \boldsymbol{A}^{(3)}_\nu - \partial_\nu \boldsymbol{A}^{(3)}_\mu) &\equiv \boldsymbol{0} \\ \boldsymbol{A}^{(3)}_\lambda \times (\partial_\mu \boldsymbol{A}^{(1)}_\nu - \partial_\nu \boldsymbol{A}^{(1)}_\mu) &\equiv \boldsymbol{0} \\ \cdots & \end{aligned} \tag{583}$$

vanish individually as follows:

$$\begin{aligned} \boldsymbol{A}^{(1)}_\lambda \times (\partial_\mu \boldsymbol{A}^{(2)}_\nu - \partial_\nu \boldsymbol{A}^{(2)}_\mu) &= \varepsilon_{(1)(2)(3)} A^{(1)}_\lambda \partial_\mu A^{(2)}_\nu - \varepsilon_{(1)(2)(3)} A^{(1)}_\lambda \partial_\nu A^{(2)}_\mu \\ &= A^{(1)}_\lambda F^{(2)}_{\mu\nu} - A^{(2)}_\lambda F^{(1)}_{\mu\nu} \\ &= A_X B_X (\boldsymbol{e}^{(2)} \cdot \boldsymbol{e}^{(1)} - \boldsymbol{e}^{(1)} \boldsymbol{e}^{(2)}) \equiv 0 \\ &\cdots \end{aligned} \tag{584}$$

Equation (584) implies that the topological magnetic charge–current

$$\boldsymbol{J}^\nu_m \propto \boldsymbol{A}_\mu \times \tilde{\boldsymbol{G}}^{\mu\nu} \equiv \boldsymbol{0} \tag{585}$$

vanishes in the vacuum, while $\boldsymbol{B}^{(3)}$ is nonzero in the vacuum, a result that is consistent with empirical data, which show the existence of $\boldsymbol{B}^{(3)}$ and the nonexistence of a magnetic monopole and magnetic current.

The computational problem reduces therefore to a numerical solution of three differential equations:

$$\partial_\lambda \boldsymbol{G}^{(1)}_{\mu\nu} + \partial_\mu \boldsymbol{G}^{(1)}_{\nu\lambda} + \partial_\nu \boldsymbol{G}^{(1)}_{\lambda\mu} \equiv \boldsymbol{0} \tag{586}$$

$$\partial_\lambda \boldsymbol{G}^{(2)}_{\mu\nu} + \partial_\mu \boldsymbol{G}^{(2)}_{\nu\lambda} + \partial_\nu \boldsymbol{G}^{(2)}_{\lambda\mu} \equiv \boldsymbol{0} \tag{587}$$

$$\partial_\lambda \boldsymbol{G}^{(3)}_{\mu\nu} + \partial_\mu \boldsymbol{G}^{(3)}_{\nu\lambda} + \partial_\nu \boldsymbol{G}^{(3)}_{\lambda\mu} \equiv \boldsymbol{0} \tag{588}$$

using the definitions (581). There are three equations in three unknowns, so the problem can be solved for given boundary conditions.

The work of Whittaker described in the previous section can be summarized by the potential

$$A^{(3)}_\mu \equiv (A^{(3)}_0, c\boldsymbol{A}^{(3)}) \tag{589}$$

where the magnitude of $\boldsymbol{A}^{(3)}$ is $A^{(3)}_0/c$. The O(3) theory allows $A^{(3)}_0$ and $\boldsymbol{A}^{(3)}$ to be structured, constant or zero. The $\boldsymbol{B}^{(3)}$ field exists in all three cases. If, however, $\boldsymbol{A}^{(3)}$ is zero, so is $A^{(3)}_0$ and there is no scalar potential. The conclusion reached is that there can be an infinite number of components of the 4-vector $A^{(3)}_\mu$ for a given phaseless $\boldsymbol{B}^{(3)}$. In other words, the scalar potential can be expanded in a Fourier series, or some other suitable series that includes the terms $\boldsymbol{A}^{(3)} = 0$ and $\boldsymbol{A}^{(3)} =$ constant.

The disappearance of the magnetic charge–current (585) means that the topological terms on the right-hand sides of Eqs. (95)–(100) vanish identically in the vacuum. The only topological charges and currents present are therefore those introduced by Lehnert [7–10]. There is no empirical evidence for the existence of an $\boldsymbol{E}^{(3)}$ field, so we are left with

$$\nabla \times \boldsymbol{E}^{(1)} + \frac{\partial \boldsymbol{B}^{(1)}}{\partial t} \equiv \boldsymbol{0} \tag{590}$$

$$\nabla \times \boldsymbol{E}^{(2)} + \frac{\partial \boldsymbol{B}^{(2)}}{\partial t} \equiv \boldsymbol{0} \tag{591}$$

$$\frac{\partial \boldsymbol{B}^{(3)}}{\partial t} \equiv \boldsymbol{0} \tag{592}$$

as first proposed some time ago [11–20]. Equation (592) has been verified empirically by Raja et al. [88] and Compton et al. [89]. The most general type of solution must be found, however, by solving Eqs. (586)–(588) numerically, so that potentials are primary and fields are derived from potentials. The mathematical structure of O(3) Yang–Mills theory applied to electrodynamics

allows for $A^{(3)} = 0$ as one of many possible solutions. However, if $A^{(3)} = 0$, then the scalar potential is also zero, while the $B^{(3)}$ field remains nonzero.

The vanishing of the topological magnetic current in Eqs. (98)–(100) leads to two components of the B cyclic theorem as follows. In Eq. (98)

$$A_0^{(2)} = 0; \qquad E^{(3)} = 0 \tag{593}$$

and so

$$-cA_0^{(3)}B^{(1)} = E^{(1)} \times A^{(3)} \tag{594}$$

$$B^{(1)} = \frac{1}{c}k \times E^{(1)} \tag{595}$$

for any $A_0^{(3)} = cA^{(3)}$. This result is self-consistent with the left-hand side of Eq. (98), because Eq. (595) is a solution of Eq. (596):

$$\nabla \times E^{(1)} + \frac{\partial B^{(1)}}{\partial t} \equiv 0 \tag{596}$$

The B cyclic component emerges as follows:

$$B^{(1)} \times B^{(2)} = \frac{1}{c}(k \times E^{(1)}) \times B^{(2)} = iB^{(0)}B^{(3)*} \tag{597}$$

$$\ldots$$

Therefore all is self-consistent.

These calculations show that $B^{(3)}$ is not dependent on the existence of a vacuum magnetic monopole [11–20]. Therefore the explanation of phenomena based on $B^{(3)}$ is not dependent on a topological magnetic charge or monopole. The fundamental reason for this is that $B^{(3)}$ is defined in terms of quantities that are not dependent on a magnetic monopole, namely, g, $A^{(1)}$, and $A^{(2)}$. Furthermore, the structure of O(3) Yang–Mills theory forces us to conclude that $E^{(3)}$ is zero through the structure of Eqs. (98)–(100) [11–20]. The existence of a phaseless $E^{(3)}$ has never been observed empirically. Action at a distance in electrodynamics is obviously denied by the fact that we are working with a gauge theory, and there is no convincing evidence for superluminal phenomena in electrodynamics. It should also be clear that $B^{(3)}$ is not a static magnetic field; rather, it is a radiated field, propagating with the third Stokes parameter.

The three equations (586)–(588) can be written in condensed form

$$\partial_\mu \tilde{G}^{\mu\nu(i)} = 0; \qquad i = 1, 2, 3 \tag{598}$$

which is self-dual to another set of three simultaneous equations suitable for computation and derivable from Eq. (90):

$$D_\mu H^{\mu\nu(i)} = 0; \qquad i = 1, 2, 3 \tag{599}$$

where $G^{\mu\nu}$ of Eq. (90) has been replaced by $H^{\mu\nu}$ for greater clarity and to indicate the presence of vacuum polarization Therefore

$$\partial_\mu \tilde{G}^{\mu\nu(i)} = D_\mu H^{\mu\nu(i)} = 0 \tag{600}$$

represents the O(3) wave equation, which has a much richer structure than its U(1) counterpart, and many more solutions. The charge current 12-vector in vacuo, Eq. (91), is nonzero. This can be demonstrated by writing it out in component form:

$$\begin{aligned} J^{\nu(1)*} &= -igA_\mu^{(2)} \times H^{\mu\nu(3)} \\ J^{\nu(2)*} &= -igA_\mu^{(3)} \times H^{\mu\nu(1)} \\ J^{\nu(3)*} &= -igA_\mu^{(1)} \times H^{\mu\nu(2)} \end{aligned} \tag{601}$$

Terms such as

$$J^{\nu(2)} = -igA_\mu^{(2)} \times (\partial^\mu A^{\nu(3)} - \partial^\nu A^{\mu(3)} - igA^{\mu(1)} \times A^{\nu(2)}) \tag{602}$$

are obtained. The first part can be expanded as

$$\begin{aligned} A_\mu^{(2)} \times \partial^\mu A^{\nu(3)} - A_\nu^{(2)} \times \partial^\nu A^{\mu(3)} &= \varepsilon_{(2)(3)(1)} A_\mu^{(2)} \partial^\mu A^{\nu(3)} - \varepsilon_{(2)(3)(1)} A_\mu^{(2)} \partial^\nu A^{\mu(3)} \\ &= -A_\mu^{(2)} F^{\mu\nu(3)} - A_\mu^{(3)} F^{\mu\nu(2)} \end{aligned} \tag{603}$$

which is nonzero in general. The second part can be expanded as

$$\begin{aligned} A_\mu^{(2)} \times (A^{\mu(1)} \times A^{\nu(2)}) &= A^{\mu(1)}(A_\mu^{(2)} \cdot A^{\nu(2)}) - A^{\nu(2)}(A_\mu^{(2)} \cdot A^{\mu(1)}) \\ &= -A^{\nu(2)}(A_\mu^{(2)} \cdot A^{\mu(1)}) \end{aligned} \tag{604}$$

which is also nonzero in general.

Therefore we reach the important overall conclusion that the structure of the O(3) equations is a development into O(3) symmetry of the Lehnert field equations [7–10], which are written in U(1) form. The Lehnert field equations have been extensively developed and tested empirically and theoretically [7–10].

The O(3) Coulomb and Ampère–Maxwell laws in the vacuum are therefore written in terms of displacement and magnetic field strength, and are as follows. The Coulomb Law in the vacuum is

$$\begin{aligned}\nabla\cdot\boldsymbol{D}^{(1)*} &= ig(\boldsymbol{D}^{(2)}\cdot\boldsymbol{D}^{(3)} - \boldsymbol{D}^{(2)}\cdot\boldsymbol{A}^{(3)})\\ \nabla\cdot\boldsymbol{D}^{(2)*} &= ig(\boldsymbol{A}^{(3)}\cdot\boldsymbol{D}^{(1)} - \boldsymbol{D}^{(3)}\cdot\boldsymbol{A}^{(1)})\\ \nabla\cdot\boldsymbol{D}^{(3)*} &= ig(\boldsymbol{A}^{(1)}\cdot\boldsymbol{D}^{(2)} - \boldsymbol{D}^{(1)}\cdot\boldsymbol{A}^{(2)})\end{aligned} \tag{605}$$

and the Ampère–Maxwell law in the vacuum is

$$\begin{aligned}\nabla\times\boldsymbol{H}^{(1)*} - \frac{\partial\boldsymbol{D}^{(1)*}}{\partial t} &= -ig(cA_0^{(2)}\boldsymbol{D}^{(3)} - cA_0^{(3)}\boldsymbol{D}^{(2)} + \boldsymbol{A}^{(2)}\times\boldsymbol{H}^{(3)} - \boldsymbol{A}^{(3)}\times\boldsymbol{H}^{(2)})\\ \nabla\times\boldsymbol{H}^{(2)*} - \frac{\partial\boldsymbol{D}^{(2)*}}{\partial t} &= -ig(cA_0^{(3)}\boldsymbol{D}^{(1)} - cA_0^{(1)}\boldsymbol{D}^{(3)} + \boldsymbol{A}^{(3)}\times\boldsymbol{H}^{(1)} - \boldsymbol{A}^{(1)}\times H^{(3)})\\ \nabla\times\boldsymbol{H}^{(3)*} - \frac{\partial\boldsymbol{D}^{(3)*}}{\partial t} &= -ig(cA_0^{(1)}\boldsymbol{D}^{(2)} - cA_0^{(2)}\boldsymbol{D}^{(1)} + \boldsymbol{A}^{(1)}\times\boldsymbol{H}^{(2)} - \boldsymbol{A}^{(2)}\times H^{(1)})\end{aligned} \tag{606}$$

The displacement $\boldsymbol{D}^{(3)}$ for example can be developed as

$$\boldsymbol{D}^{(3)} = \varepsilon_0\boldsymbol{E}^{(3)} + \boldsymbol{P}^{(3)} \tag{607}$$

and since $\boldsymbol{E}^{(3)}$ is zero, we obtain

$$\boldsymbol{D}^{(3)} = \boldsymbol{P}^{(3)}$$

indicating the presence of classical vacuum polarization $\boldsymbol{P}^{(3)}$ due to the topology of the vacuum as represented by a gauge field theory with an assumed O(3) gauge group symmetry. Therefore the energy inherent in the vacuum is obtained entirely from the electric charge current (91), as discussed in Section IV. The magnetic charge-current (585) vanishes, and so there is no energy inherent in the vacuum from the magnetic charge–current for an internal O(3) gauge group symmetry. On the O(3) level, there can therefore be classical vacuum polarization, whose analog in quantum electrodynamics is the photon self-energy [6].

The constitutive equations in the vacuum in O(3) electrodynamics are not the same as those of U(1) electrodynamics, and in general

$$\boldsymbol{D}^{(i)} = \varepsilon_0\boldsymbol{E}^{(i)} + \boldsymbol{P}^{(i)}; \qquad i = 1,2,3 \tag{609}$$

where $\boldsymbol{P}^{(i)}$ are vacuum polarizations.

To summarize, there are three equations [Eqs. (586)–(588)] in three unknowns (indices of the vector potential appropriate to $\tilde{G}^{\mu\nu}$) and another three equations [Eq. (599)] in three unknowns (indices of the vector potential appropriate to $\boldsymbol{H}^{\mu\nu}$ in the vacuum). Simple vacuum constitutive relations such as

$$\boldsymbol{D} = \varepsilon_0\boldsymbol{E}; \qquad \boldsymbol{H} = \frac{1}{\mu_0}\boldsymbol{B} \tag{610}$$

of U(1) electrodynamics no longer apply, because of the existence of classical vacuum polarization. The latter also occurs in the Lehnert equations [7–10], which are known to give axisymmetric solutions similar to $\boldsymbol{B}^{(3)}$, to indicate photon mass, and to be superior in ability to the Maxwell–Heaviside equations.

To put the O(3) equations into the form of the Lehnert equations, we use the definitions

$$\begin{aligned}\boldsymbol{D} &= \boldsymbol{D}^{(1)} + \boldsymbol{D}^{(2)} + \boldsymbol{D}^{(3)}\\ \boldsymbol{H} &= \boldsymbol{H}^{(1)} + \boldsymbol{H}^{(2)} + \boldsymbol{H}^{(3)}\\ \boldsymbol{E} &= \boldsymbol{E}^{(1)} + \boldsymbol{E}^{(2)} + \boldsymbol{E}^{(3)}\\ \boldsymbol{B} &= \boldsymbol{B}^{(1)} + \boldsymbol{B}^{(2)} + \boldsymbol{B}^{(3)}\\ &\cdots\end{aligned} \tag{611}$$

to obtain

$$\begin{aligned}\nabla\cdot\boldsymbol{B} &= 0\\ \nabla\times\boldsymbol{E} + \frac{\partial\boldsymbol{B}}{\partial t} &= \boldsymbol{0}\\ \nabla\cdot\boldsymbol{B} &= \rho_{\text{vac}}\\ \nabla\times\boldsymbol{H} - \frac{\partial\boldsymbol{D}}{\partial t} &= \boldsymbol{J}_{\text{vac}}\end{aligned} \tag{612}$$

which are mathematically identical to the Lehnert equations. The O(3) gauge theory, however, shows that the origin of the vacuum charge and current postulated phenomenologically by Lehnert [7–10] is the topology of the vacuum described by an O(3) gauge group. The O(3) theory also shows, self-consistently, that there is a vacuum polarization, so that the simple constitutive relations (610) used by Lehnert do not hold. The O(3) gauge theory also reveals that the presence of the $\boldsymbol{B}^{(3)*}$ component, through its definition, is proportional to the conjugate product of potentials, $\boldsymbol{A}^{(1)}\times\boldsymbol{A}^{(2)}$. However, the mathematical form of the O(3) equations (612) is identical with that of the Lehnert equations.

Formally, the O(3) equations are written most generally as

$$\begin{aligned} \nabla \cdot \boldsymbol{H} &= \rho_{m,\mathrm{vac}} \\ \nabla \times \boldsymbol{D} + \frac{\partial \boldsymbol{H}}{\partial t} &= \boldsymbol{J}_{m,\mathrm{vac}} \\ \nabla \cdot \boldsymbol{D} &= \rho_{\mathrm{vac}} \\ \nabla \times \boldsymbol{H} - \frac{\partial \boldsymbol{D}}{\partial t} &= \boldsymbol{J}_{\mathrm{vac}} \end{aligned} \tag{613}$$

which are identical in mathematical structure with the Harmuth equations [21,22] and Barrett equations [3,4]. However, in O(3) electrodynamics, there is no magnetic monopole or magnetic current as argued already. The structure (612) in the vacuum is identical with the structure of the Maxwell–Heaviside equations as used for field-matter interaction.

The complete computational problem in the vacuum is therefore as follows:

1. Use eqs. (586) to (588) to obtain $A_\mu^{(1)}, A_\mu^{(2)}, A_\mu^{(3)}; \mu = 0, ..., 3$ with the simplifying definitions

$$\begin{aligned} A_0^{(i)} &= A_3^{(i)} = 0; \qquad i = 1, 2 \\ A_\mu^{(3)} &= (A_0^{(3)}, cA^{(3)}) = (A_0^{(3)}, cA_3^{(3)}\boldsymbol{k}) \\ A_0^{(1)} &= A_0^{(2)} = 0; \\ A_1^{(3)} &= A_2^{(3)} = 0 \end{aligned} \tag{614}$$

2. Use Eqs. (101)–(103) to obtain $\boldsymbol{D}^{(1)}$, $\boldsymbol{D}^{(2)}$, and $\boldsymbol{D}^{(3)}$.
3. Use Eqs. (104)–(106) to obtain $\boldsymbol{H}^{(1)}$, $\boldsymbol{H}^{(2)}$, and $\boldsymbol{H}^{(3)}$.
4. The complete displacement and magnetic field strength vectors in the vacuum are then

$$\begin{aligned} \boldsymbol{D} &= \boldsymbol{D}^{(1)} + \boldsymbol{D}^{(2)} + \boldsymbol{D}^{(3)} \\ &= D_X\boldsymbol{i} + D_Y\boldsymbol{j} + D_Z\boldsymbol{k} \end{aligned} \tag{615}$$

$$\boldsymbol{H} = \boldsymbol{H}^{(1)} + \boldsymbol{H}^{(2)} + \boldsymbol{H}^{(3)} \tag{616}$$

5. Use

$$\begin{aligned} \boldsymbol{B}^{(1)} &= \nabla \times \boldsymbol{A}^{(1)} \\ \boldsymbol{B}^{(2)} &= \nabla \times \boldsymbol{A}^{(2)} \\ \boldsymbol{B}^{(3)} &= -ig\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} \end{aligned} \tag{617}$$

to obtain $\boldsymbol{B}^{(1)}$, $\boldsymbol{B}^{(2)}$, and $\boldsymbol{B}^{(3)}$.

6. Use Eqs. (590)–(592) to obtain $\boldsymbol{E}^{(1)}$ and $\boldsymbol{E}^{(2)}$.
7. Simplify the code with

$$\begin{aligned} \boldsymbol{B}^{(1)} &= \boldsymbol{B}^{(2)*} \\ \boldsymbol{E}^{(1)} &= \boldsymbol{E}^{(2)*} \\ \boldsymbol{A}^{(1)} &= \boldsymbol{A}^{(2)*} \\ &\cdots \end{aligned} \tag{618}$$

8. Finally, find $\boldsymbol{P}^{(1)}$, $\boldsymbol{P}^{(2)}$, and $\boldsymbol{P}^{(3)}$ and, if they exist, $\boldsymbol{M}^{(1)}$, $\boldsymbol{M}^{(2)}$, and $\boldsymbol{M}^{(3)}$ in the vacuum.

The computational problem for the vacuum involves the definition of vacuum boundary conditions, which, for example, may be a volume of radiation or a beam radius. The computational method assumes no Lorenz condition, and gives a vast number of solutions. Having obtained these solutions, we can next check whether the non-Abelian Stokes theorem (153) is obeyed numerically. Essentially, everything is obtained from potentials in the vacuum, and everything is expressible in terms of these potentials, including the charge and the current. In evaluating the coupling constant $\kappa/A^{(0)}$, the denominator is the magnitude of $\boldsymbol{A}^{(1)} = \boldsymbol{A}^{(2)*}$, defined by

$$A^{(0)} = \left(\boldsymbol{A}^{(1)} \cdot \boldsymbol{A}^{(2)}\right)^{1/2} \tag{619}$$

This is then a computational solution of a classical problem in the vacuum. If g is defined as $\kappa/A^{(0)}$, then e is never used.

In field–matter interaction, the fields $\boldsymbol{B}$ and $\boldsymbol{E}$ remain unchanged. The fields $\boldsymbol{D}$ and $\boldsymbol{H}$ change because $\boldsymbol{P}$ and $\boldsymbol{M}$ change. Equations (612) have precisely the same structure as Eqs. (9-7) of Panofsky and Phillips [86] with the following identifications:

$$\rho_{\mathrm{vac}} \equiv \rho_{\mathrm{true}}; \qquad \boldsymbol{J}_{\mathrm{vac}} \equiv \boldsymbol{J}_{\mathrm{true}} \tag{620}$$

The ρ_{true} and $\boldsymbol{J}_{\mathrm{true}}$ of Ref. 86 are therefore identified as being due to the topology of the vacuum, a topology that gives rise to potential energy inherent in the vacuum. The potential energy appears in O(3) electrodynamics through the connections $\boldsymbol{A}_\mu^{(i)}$, and so the connections are regarded as physical entities. Fields, currents, and charges are obtained from the potentials, or more precisely, potential energy differences that are dictated by the topology of the vacuum itself. On the classical level, $g = \kappa/A^{(0)}$ so the constant e does not appear in the vacuum. As demonstrated already in this review, the equivalent of the Poynting theorem can be obtained by considering the energy inherent in the vacuum, on both the U(1) and on the O(3) levels.

In dealing with Eqs. (612), the vacuum is treated as if it were a material, and the equations are solved with stipulated boundary conditions and constitutive relations. The ontology behind Eqs. (612) is that charge–current is the result of spacetime. Similarly, in general relativity, matter is the result of spacetime. A complete theory would obviate the need for constitutive relations and be based on grand unified field theory with an O(3) electromagnetic sector. Equations (612) deal only with the electromagnetic sector on a classical level and still utilize the concept of field as a matter of convenience. So we still write in terms of field–matter interaction, although the ontology dictates that field–matter interaction is dictated solely by the topology of spacetime.

The computational problem in the vacuum has to be solved first, to obtain the vacuum polarizations. To simulate the interaction with matter, the polarization changes in the medium must be modeled using constitutive relations, and boundary conditions defined according to the problem being solved. Integral forms of Eqs. (612) may be useful, and integral forms must be obtained through the non-Abelian Stokes theorem using O(3) covariant derivatives. For example, the integral form of Eqs. (590)–(592) is

$$\oint \boldsymbol{E}^{(1)} \cdot d\boldsymbol{r} + \frac{\partial}{\partial t} \oint \boldsymbol{B}^{(1)} \cdot d\boldsymbol{A}\boldsymbol{r} = 0 \tag{621}$$

$$\oint \boldsymbol{E}^{(2)} \cdot d\boldsymbol{r} + \frac{\partial}{\partial t} \oint \boldsymbol{B}^{(2)} \cdot d\boldsymbol{A}\boldsymbol{r} = 0 \tag{622}$$

$$\frac{\partial}{\partial t} \oint \boldsymbol{B}^{(3)} \cdot d\boldsymbol{A}\boldsymbol{r} = 0 \tag{623}$$

and the integral form of $\nabla \cdot \boldsymbol{B}^{(i)} = 0; i = 1, 2, 3$ is

$$\oint \boldsymbol{B}^{(i)} \cdot d\boldsymbol{r} = 0; \qquad i = 1, 2, 3 \tag{624}$$

A simple example of a computational problem on the U(1) level is the numerical solution of the equation

$$\nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A} + \frac{1}{c^2} \frac{\partial}{\partial t} \nabla \phi + \frac{1}{c^2} \frac{\partial^2 \boldsymbol{A}}{\partial t^2} = \boldsymbol{0} \tag{625}$$

which is equivalent to solving the following equations simultaneously:

$$\begin{aligned} \nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} &= \boldsymbol{0} \\ \nabla \times \boldsymbol{B} - \frac{1}{c^2} \frac{\partial \boldsymbol{E}}{\partial t} &= \boldsymbol{0} \end{aligned} \tag{626}$$

In the received opinion [5], these are the vacuum Faraday law and Ampère–Maxwell law, respectively. The vacuum charges and currents are missing in the received opinion. Nevertheless, solving Eq. (625) numerically is a useful computational problem with boundary conditions stipulated in the vacuum. The potentials and fields are related as usual by

$$\begin{aligned} \boldsymbol{B} &= \nabla \times \boldsymbol{A} \\ \boldsymbol{E} &= -\frac{\partial \boldsymbol{A}}{\partial t} - \nabla \phi \end{aligned} \tag{627}$$

In the received view, it is customary to simplify the problem of solving Eq. (625) with the Lorenz condition

$$\nabla \cdot \boldsymbol{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} = 0 \tag{628}$$

to give the d'Alembert equation in vacuo

$$\nabla^2 \boldsymbol{A} - \frac{1}{c^2} \frac{\partial^2 \boldsymbol{A}}{\partial t^2} = \boldsymbol{0} \tag{629}$$

an equation that has analytical solutions such as plane waves. The Lorenz condition (628) is asserted to be the result of gauge freedom. The computational problem therefore consists in solving Eq. (625) with and without Eq. (628) for different boundary conditions.

Regardless of whether the Lorenz gauge is used, the equation $\Box \chi = 0$ is obtained. So χ is not random after being assumed to be random (a reduction to absurdity) proof of the self-consistency of the U(1) gauge ansatz. Ludwig V. Lorenz introduced the idea of the Lorenz gauge or condition (often misattributed to Henrik Anton Lorentz) in 1867, so we can write the structured scalar potential as $\phi = \phi_0 e^{i\omega(t)}$, where (t) is the retarded time. So in this sense, we can have pure time-like potentials (something that apparently was discussed between Bearden and Wigner) in the context of a pure time-like photon. Whittaker's work depends on the Lorenz condition on the U(1) level.

Plane waves have infinite lateral extent and, for this reason, cannot be simulated on a computer because of floating-point overflow. If the lateral extent is constrained, as in Problem 6.11 of Jackson [5], longitudinal solutions appear in the vacuum, even on the U(1) level without vacuum charges and currents. This property can be simulated on the computer using boundary conditions, for example, a cylindrical beam of light. It can be seen from a comparison of Eqs. (625) and (629) that if the Lorenz condition is not used, there is no increase

in the number of variables. Therefore Eq. (625) is one equation in two unknowns, ϕ and $\boldsymbol{A}$. If we use the Lorenz condition

$$\nabla \cdot \boldsymbol{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0 \tag{630}$$

we still have one equation in two unknowns. Making use of the vacuum Coulomb and Gauss laws in the received view

$$\begin{aligned} \nabla \cdot \boldsymbol{E} &= 0 \\ \nabla \cdot \boldsymbol{B} &= 0 \end{aligned} \tag{631}$$

we obtain two more equations:

$$\nabla \cdot \left(\frac{\partial \boldsymbol{A}}{\partial t} + \nabla \phi \right) = 0 \tag{632}$$

$$\nabla \cdot \nabla \times \boldsymbol{A} = 0 \tag{633}$$

So there are three equations, (625), (632), and (633), in two unknowns $\boldsymbol{A}$ and ϕ. These are enough to solve for the components of $\boldsymbol{A}$ and for ϕ for any boundary condition. For any physical boundary condition, there will be longitudinal as well as transverse components of $\boldsymbol{A}$ in the vacuum, and ϕ will in general be phase-dependent and structured. This computational exercise shows that the Lorenz condition is arbitrary and, if it is discarded, the values of $\boldsymbol{A}$ and ϕ from Eqs. (625), (632), and (633) change.

Under the U(1) gauge transform

$$A^\mu \rightarrow A^\mu + \partial^\mu \chi; \quad A^\mu \equiv (\phi, c\boldsymbol{A}); \quad \left(\text{i.e. } \phi \rightarrow \phi + \frac{1}{c}\frac{\partial \chi}{\partial t}; \quad \boldsymbol{A} \rightarrow \boldsymbol{A} - \frac{1}{c}\nabla \chi \right) \tag{634}$$

we see that $\boldsymbol{E}$ and $\boldsymbol{B}$ do not change:

$$\begin{aligned} \boldsymbol{E} &\rightarrow \boldsymbol{E} + \frac{1}{c}\frac{\partial}{\partial t}\nabla \chi - \frac{1}{c}\nabla \frac{\partial \chi}{\partial t} = \boldsymbol{E} \\ \boldsymbol{B} &\rightarrow \boldsymbol{B} - \frac{1}{c}\nabla \times (\nabla \chi) = \boldsymbol{B} \end{aligned} \tag{635}$$

and Eqs. (625), (632), and (633) do not change. This means that for any given boundary condition, we can find the solutions

$$\phi' \equiv \phi + \frac{1}{c}\frac{\partial \chi}{\partial t} \tag{636}$$

$$\boldsymbol{A}' \equiv \boldsymbol{A} - \frac{1}{c}\nabla \chi \tag{637}$$

from Eqs. (625), (632), and (633) numerically. The solutions ϕ' and $\boldsymbol{A}'$, however, are *not* arbitrary for a given boundary condition, indicating another self-inconsistency in U(1) gauge theory (Section II). Furthermore, under the same gauge transform (634), Eq. (625) indicates that χ must obey the equation

$$\Box \chi = 0 \tag{638}$$

whose general solution has been given by Whittaker [27] and is *not* arbitrary. If we arbitrarily decouple Eq. (625) into

$$\begin{aligned} \Box \boldsymbol{A} &= 0 \\ \nabla \cdot \boldsymbol{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} &= 0 \end{aligned} \tag{639}$$

then Eq. (638) is obtained again, indicating that the Lorenz condition and d'Alembert equation in vacuo are arbitrary constructs, that is, particular solutions of Eq. (625). The Lorenz condition has no physical meaning, nor does the vacuum d'Alembert equation. The function χ is not arbitrary, contrary to the U(1) gauge transform ansatz, Eq. (634). In other words, the gauge transformed ϕ' and $\boldsymbol{A}'$ are not arbitrary, as they are solutions of two differential equations, (625) and (632), in two unknowns, ϕ' and $\boldsymbol{A}'$, for a given boundary condition. We conclude that ϕ' and $\boldsymbol{A}'$ are physical, not arbitrary, thus refuting Heaviside's point of view and supporting that of Maxwell and Faraday. For a self-consistent picture of electrodynamics, we have to go to the O(3) level, as discussed earlier in this section.

The same conclusion regarding the Lorenz gauge is reached by Jackson [5], who shows that:

$$\partial_\mu A^{\mu'} = \partial_\mu A^\mu + \Box \chi \tag{640}$$

However, Jackson follows the received opinion and forces

$$\Box \chi = -\partial_\mu A^\mu \tag{641}$$

through the arbitrary assumption:

$$\partial_\mu A^{\mu'} = ? \quad 0 \tag{642}$$

The latter merely reinforces the conclusion that χ is not arbitrary.

By discarding the Lorenz condition, a vacuum current $\boldsymbol{J}_{\text{vac}}$ is introduced. The vacuum current $\boldsymbol{J}_{\text{vac}}$ is conceptually similar to the one introduced by Lehnert and Roy [10]. Relativity then indicates the presence of a vacuum charge, so the

field equations in vacuo become identical with those of Panofsky and Phillips [86] and those of O(3) electrodynamics [11–20] i.e., [Eqs. (612)]. Phipps [90] has also derived the same structure and describes it as "neo-Hertzian." There is therefore a remarkable degree of agreement in the literature that the structure of the Heaviside–Maxwell equations in vacuo is such that the overall symmetry is O(3). This conclusion is consistent with the fact that there is no Lorenz condition on the O(3) level, necessitating numerical solution as described earlier in this section.

The source of Eq. (625), however, is the set of vacuum Maxwell–Heaviside equations

$$\begin{aligned} \nabla \cdot \boldsymbol{B} &= 0 \\ \nabla \cdot \boldsymbol{E} &= 0 \\ \nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} &= \boldsymbol{0} \\ \nabla \times \boldsymbol{B} - \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} &= \boldsymbol{0} \end{aligned} \tag{643}$$

and to identify Eqs. (612) with Eqs. (643), it is necessary to write the vacuum displacement as

$$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P} \tag{644}$$

and to introduce the vacuum polarization. This result is self-consistent with our constitutive equations (607)–(609) on the O(3) level. The vacuum polarization gives rise to a polarization current

$$\boldsymbol{J}_P \equiv -\frac{\partial \boldsymbol{P}}{\partial t} \tag{645}$$

and exists if and only if we discard the Lorenz condition. It therefore becomes clear that use of the Lorenz condition prohibits the evolution of U(1) into O(3) electrodynamics and arbitrarily asserts a zero vacuum polarization. The existence of vacuum charge and currents means the existence of vacuum energy, as argued already. The experimental challenge is how to tap this energy, which is theoretically infinite, that is, extends throughout the universe.

The vacuum charge density and current density are

$$\begin{aligned} \rho_{\text{vac}} &= \frac{1}{\mu_0}\Box\phi \\ \boldsymbol{J}_{\text{vac}} &= \frac{1}{\mu_0}\Box\boldsymbol{A} \end{aligned} \tag{646}$$

and so it becomes clear that Whittaker's theory [27] is restricted severely by his adoption of the Lorenz condition. The received view is similarly restricted. The new paradigm introduced here is that the vacuum itself is the source of charge–current, including, of course, Maxwell's displacement current. The latter has nothing to do with charged electrons, and similarly, the Noether currents of O(3) electrodynamics have nothing to do with charged electrons. The received view asserts that the Maxwell displacement current is the origin of the electromagnetic field, which carries energy and momentum; the new paradigm asserts that the vacuum itself is the source of energy and momentum through the intermediary of entities labeled charge, current, and field. The topology of the vacuum is described by physical $\boldsymbol{A}$ and ϕ , which, in turn, originate in the gauge principle and group theory. We have argued that the notion of unphysical $\boldsymbol{A}$ and ϕ is untenable. It is this idea that leads to the Lorenz condition, which is, in turn, untenable.

Therefore electric and magnetic fields do not emanate from a point charge, as in the received view; both charge and field are outcomes of the topology of the vacuum. In the new paradigm, the energy that is said to be transmitted by the electromagnetic field in the received opinion is inherent in the vacuum structure; all is determined by the nature of the connection in gauge theory, and by the physical nature of the potential, which is more precisely described as potential energy difference. An intense electromagnetic field in the received view corresponds in the new paradigm to a warping of space-time by the gauge connection inherent in the covariant derivative. On the classical level, the proportionality constant g is $\kappa/A^{(0)}$, and $e/\hbar$ is not necessary. Curvature or warping of spacetime determines the process of radiation and of detection of radiation. Causality implies that the cause precedes the effect in time. This new view of electromagnetism as being essentially the vacuum itself is similar to general relativity. The major implication is that the vacuum carries an unknown amount of electromagnetic energy; the electromagnetic field is far stronger than the gravitational field, so the amount of electromagnetic energy in the vacuum is commensurably greater.

> The vast paradox inherent in the concept of field is vividly summarized by Koestler [91, p. 502ff.]: a steel cable of a thickness equaling the diameter of the earth would not be strong enough to hold the earth in its orbit. Yet the gravitational force which holds the earth in its orbit is transmitted from the sun across 93 million miles of space without any material medium to carry that force. The paradox is further illustrated by Newton's own words, which I have quoted before, but which bear repeating: It is inconceivable, that inanimate brute matter should, without the mediation of something else, which is not material, operate upon, and affect other matter without mutual contact, ··· And this is one reason why I desired you would not ascribe innate gravity to me. That gravity should be innate, inherent, and essential to matter, so that one body may act upon another, at a

distance through a vacuum, without the mediation of anything else, by and through which their action and force may be conveyed from one to another, is to me so great an absurdity, that I believe no man who has in philosophical matters a competent faculty of thinking, can ever fall into it. Gravity must be caused by an agent acting constantly according to certain laws; but whether this agent be material or immaterial, I have left to the consideration of my readers.

The paradox is compounded greatly in electrodynamics, where, in the received view, the field is superimposed on spacetime. In the new view, both the gravitational and electromagnetic fields are the results of topology, or vacuum structure. The enormous amount of energy inherent in the vacuum is metaphorically apparent in Koestler's steel cable. The electromagnetic energy from the same source is orders of magnitude greater. Thus a few simple computational trials are needed.

XII. SU(2) × SU(2) ELECTROWEAK THEORY WITH AN O(3) ELECTROMAGNETIC SECTOR

It has been demonstrated conclusively that classical electrodynamics is not a U(1) gauge theory; therefore, the continued use of a U(1) sector in unified field theory is misleading. In this section, a first attempt is made to unify the electromagnetic and weak fields with an O(3) electromagnetic sector. The theory has SU(2) × SU(2) symmetry instead of the usual U(1) × SU(2) symmetry. The change in symmetry has several ramifications, including the appearance of a novel massive boson that has been detected empirically [92]. The use of an O(3) electromagnetic sector will also have ramifications in grand unified field theory, a paradigm shift that extends throughout field and particle physics and challenges the standard model at a fundamental level. In the new view of grand unified field theory, all four fields are manifestations of non-Abelian gauge theory. If we go a step further and drop the word "field," then all physics becomes a manifestation of vacuum topology.

The extension of U(1) × SU(2) electroweak theory to SU(2) × SU(2) electroweak theory succeeds in describing the empirically measured masses of the weakly interacting vector bosons, and predicts a novel massive boson that was been detected in 1999 [92]. The SU(2) × SU(2) theory is developed initially with one Higgs field for both parts of the twisted bundle [93], and is further developed later in this section.

The physical vacuum is assumed to be defined by the Higgs mechanism, and the SU(2) × SU(2) covariant derivative is

$$D_\mu = \partial_\mu + ig'\boldsymbol{\sigma}\cdot\boldsymbol{A}_\mu + ig\boldsymbol{\tau}\cdot\boldsymbol{b}_\mu \tag{647}$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ are the generators for the two SU(2) gauge fields represented as Pauli matrices, and where $\boldsymbol{A}$ and $\boldsymbol{b}$ are the gauge connections defined on the two

SU(2) principal bundles. There is an additional Lagrangian for the ϕ^4 scalar field [93]:

$$\mathscr{L}_\phi = \frac{1}{2}|D_\mu(\phi)^2| - \frac{1}{2}\mu^2|\phi|^2 + \frac{1}{4}|\lambda|(|\phi|^2)^2 \tag{648}$$

The expectation value for the scalar field is then

$$\langle\phi_0\rangle = \left(0, \frac{v}{\sqrt{2}}\right) \tag{649}$$

for $v = (-\mu^2/\lambda)^{1/2}$. The generators for the theory on the broken vacuum are

$$\begin{aligned}\langle\phi_0\rangle_{\sigma_1} &= \left(\frac{v}{\sqrt{2}}, 0\right)\\ \langle\phi_0\rangle_{\sigma_2} &= \left(i\frac{v}{\sqrt{2}}, 0\right)\\ \langle\phi_0\rangle_{\sigma_3} &= \left(0, -\frac{v}{\sqrt{2}}\right)\end{aligned} \tag{650}$$

These are the same for the other SU(2) sector of the theory. The hypercharge formula of Nishijima, if applied directly, would lead to an electric charge

$$\begin{aligned}Q(\phi_0) &= \frac{1}{2}\langle\phi_0\rangle(\boldsymbol{\sigma}_3 + \boldsymbol{\tau}_1)\\ &= \left(0, -\frac{v}{\sqrt{2}}\right) + \left(0, \frac{v}{\sqrt{2}}\right)\end{aligned} \tag{651}$$

implying two unphysical oppositely charged photons. The equation for the hypercharge must therefore be modified to

$$Q(\phi_0) = \frac{1}{2}\langle\phi_0\rangle(\boldsymbol{n}_2\cdot\boldsymbol{\tau}_3 + \boldsymbol{n}_1\cdot\boldsymbol{\sigma}_1) = 0 \tag{652}$$

where $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ are unit vectors on the doublet defined by the two eigenstates of the vacuum. This projection on to σ_1 and τ_3 is required because we are using a single Higgs field on both bundles on both SU(2) connections. This requirement can be relaxed as discussed later in this section. At this stage of the development, the generators of the theory have a broken symmetry on the physical vacuum. Therefore, the photon is defined according to the σ_1 generator in one SU(2) sector of the theory, while the charged neutral current of the weak interaction is defined on the τ^3 generator.

The fundamental Lagrangian contains the electro-weak Lagrangians and the ϕ^4 scalar field:

$$\mathscr{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} + |D_\mu\phi|^2 - \frac{1}{2}\mu^2|\phi|^2 + \frac{1}{4}\lambda(|\phi|^2)^2 \tag{653}$$

where $G^a_{\mu\nu}$ and $F^a_{\mu\nu}$ are elements of the field strength tensors for the two SU(2) principal bundles. In order to develop the theory further, it would be necessary to include the Dirac and Yukawa Lagrangians that couple the Higgs field to the leptons and quarks. The ϕ^4 field may be developed as a small displacement in the vacuum energy:

$$\phi' = \phi + \langle\phi_0\rangle \approx \frac{(v + \xi + i\chi)}{\sqrt{2}} \tag{654}$$

The fields ξ and χ are orthogonal components in the complex phase plane for the oscillations due to the small displacement of the scalar field, which is thereby characterized completely. The scalar field Lagrangian becomes

$$\begin{aligned}\mathscr{L}_\phi = \frac{1}{2}(\partial_\mu\xi\,\partial^\mu\xi - 2\mu^2\xi^2) \quad &+\frac{1}{2}v^2\left(g'\boldsymbol{A}_\mu + g\boldsymbol{b}_\mu + \left(\frac{1}{gv} + \frac{1}{g'v}\right)\partial_\mu\chi\right) \\ &\times \left(g'\boldsymbol{A}_\mu + g\boldsymbol{b}_\mu + \left(\frac{1}{gv} + \frac{1}{g'v}\right)\partial_\mu\chi\right)\end{aligned} \tag{655}$$

where Lie algebraic indices are implied. The Higgs field is described by the harmonic oscillator equation where the field has the mass $M_H \approx 7.0\ \text{TeV}/c^2$.

On the physical vacuum the gauge fields are:

$$g'\boldsymbol{A}_\mu + g\boldsymbol{b}_\mu \rightarrow g'\boldsymbol{A}'_\mu + g\boldsymbol{b}'_\mu \tag{656}$$

which corresponds to a phase rotation induced by the transition of the vacuum to the physical vacuum. The Lagrangian is now decomposed into components by expanding about the minimum of the scalar potential

$$\begin{aligned}\mathscr{L}_\phi = \frac{1}{2}(\partial_\mu\xi\,\partial^\mu\xi - 2\mu^2\xi^2) + \frac{1}{8}v^2(g'|\boldsymbol{b}^{(3)}|^3 + g'^2(|\boldsymbol{W}^+|^2 + |\boldsymbol{W}^-|^2) \\ + g^2|\boldsymbol{A}^{(1)}|^2 + g^2|\boldsymbol{A}^{(3)} + i\boldsymbol{A}^{(2)}|^2)\end{aligned} \tag{657}$$

where the charged weak fields are identified as

$$W^\pm_\mu = \frac{1}{\sqrt{2}}(\boldsymbol{b}^{(1)}_\mu \pm i\boldsymbol{b}^{(2)}_\mu) \tag{658}$$

with mass $gv/2$. The other parts of the Lagrangian define the fields:

$$A_\mu = \frac{(g\boldsymbol{A}^{(3)}_\mu + g'\boldsymbol{b}^{(3)}_\mu - g\boldsymbol{A}^{(1)}_\mu)}{(g^2 + g'^2)^{1/2}} \tag{659a}$$

$$Z^0_\mu = \frac{(g'\boldsymbol{A}^{(3)}_\mu + g\boldsymbol{b}^{(3)}_\mu - g'\boldsymbol{A}^{(1)}_\mu)}{(g^2 + g'^2)^{1/2}} \tag{659b}$$

On scales larger than unification, the requirement $A^{(3)}_\mu = 0$ is needed [94] because otherwise Z_0 would have a mass greater than empirically measured, or there would be an additional massive boson along with the Z_0 neutral boson. A more complete discussion of $A^{(3)}_\mu$ is given later in this chapter. The additional massive boson predicted by the theory has been observed empirically [92]. The considerations thus far lead to the standard result that the mass of the photon vanishes, and that the mass of the Z_0 particle is

$$\begin{aligned}M_{Z_0} &= \frac{v}{2}(g^2 + g'^2)^{1/2} \\ &= M_W\left(1 + \left(\frac{g'}{g}\right)^2\right)^{1/2}\end{aligned} \tag{660}$$

The weak angles are defined trigonometrically by the terms $g/(g^2 + g'^2)$ and $g'/(g^2 + g'^2)$. This means that the field strength tensor satisfies

$$\begin{aligned}F^{(3)}_{\mu\nu} &= \partial_\nu A^{(3)}_\mu - \partial_\mu A^{(3)}_\nu - ig[A^{(1)}_\nu, A^{(2)}_\mu] \\ &= -ig[A^{(1)}_\nu, A^{(2)}_\mu]\end{aligned} \tag{661}$$

and that the $\boldsymbol{B}^{(3)}$ field is defined, in this notation, by

$$B^{(3)}_j = \varepsilon^{\mu\nu}_j F^{(3)}_{\mu\nu} = -ig\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} \tag{662}$$

The $\boldsymbol{E}^{(3)}$ field, however, is zero, as we have seen, so that the Lagrangian is satisfactorily nonzero. The $\boldsymbol{E}^{(3)}$ field vanishes by definition [Eqs. (581)]. Specifically [11–20]

$$G^{03(3)*} = \partial^0 A^{3(3)*} - \partial^3 A^{0(3)*} - ig(A^{0(1)}A^{3(2)} - A^{3(2)}A^{0(1)}) \equiv 0 \tag{663}$$

a result that is consistent with the B cyclic theorem and with the fact that there are no magnetic monopoles or currents in O(3) electrodynamics. The $\boldsymbol{E}^{(3)}$ field

also vanishes if $\boldsymbol{A}^{(3)}$ is a constant, or is structured. Therefore an SU(2) × SU(2) electroweak theory can be constructed that self-consistently describes the empirically observed Z_0, $W^{\pm}$ bosons, and the $\boldsymbol{B}^{(3)}$ field in the electromagnetic sector. The theory of electromagnetism on the physical vacuum that emerges is

$$\begin{aligned}\mathscr{L} = &-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{4}G^{a\mu\nu}G^{a}_{\mu\nu} - \frac{1}{2}B^{(3)2} \\ &+ M_0|Z_0|^2 + M_W|W^{\pm}|^2 + \frac{1}{2}(|\partial\xi|^2 - \partial_\mu^2|\xi|^2) \\ &+ \text{Dirac Lagrangian} + \text{Yukawa/Fermi/Higgs}\end{aligned} \tag{664}$$

where $F_{\mu\nu}$ and $G^{a}_{\mu\nu}$ are the field tensor components for standard electromagnetism and the weak interaction, and the cyclic magnetic fields define the Lagrangian in the third term. The occurrence of the massive Z_0 and $W^{\pm}$ particles breaks the gauge symmetry of the SU(2) weak interactions.

The longitudinal field $\boldsymbol{B}^{(3)}$ therefore results from the breaking of gauge invariance. There is no $\boldsymbol{E}^{(3)}$ field by definition [Eq. (663)]. Under the gauge transform

$$A^{(1)} \rightarrow UA^{(1)}U^{-1} + U\partial U^{-1} \tag{665}$$

the $\boldsymbol{B}^{(3)}$ field is invariant [11–20]:

$$B^{(3)i} = \varepsilon^{ijk}U[A_j^{(1)}, A_k^{(2)}]U^{-1} \tag{666}$$

The condition $\boldsymbol{A}_\mu^3 = \boldsymbol{0}$ is, however, restrictive, and can be removed by the inclusion in the theory of massive fermions. This makes the SU(2) × SU(2) theory consistent with the fact that $\boldsymbol{A}^{(3)}$ is phase-dependent and structured from Eqs. (586)–(588) and with the fact that there can be many solutions for $\boldsymbol{A}_\mu^{(3)}$ in the vacuum. The condition is therefore a first step in the development of SU(2) × SU(2) theory. If the condition $\boldsymbol{A}_\mu^{(3)} = \boldsymbol{0}$ is relaxed, the currents will contain vector and axial components that obey SU(2) × SU(2)_C algebra, and on the physical vacuum, fields acquire masses that violate the current conservation of the axial vector current.

The theory so far is incomplete, however, because it has two SU(2) algebras that both act on the same Fermi spinor fields, and only one Higgs mechanism is used to compute the vacuum expectations for both fields. To improve the theory, consider that each SU(2) acts on separate spinor field doublets and that there are two Higgs fields that compute separate physical vacua for each SU(2) sector independently. The Higgs fields will give 2 × 2 vacuum diagonal expectations. If two entries in each of these matrices are equal, the resulting massive fermions in each of the two spinor doublets are identical. If the spin in one doublet

assumes a very large mass, then at low energies, the doublet will appear as a singlet and the gauge theory that acts on it will be O(3), with the algebra of singlets:

$$\boldsymbol{e}_i = \varepsilon_{ijk}[\boldsymbol{e}_j, \boldsymbol{e}_k] \tag{667}$$

The theory on the physical vacuum will involve transformations on a singlet according to a broken O(3) gauge theory, and transformations on a doublet according to a broken SU(2) gauge theory. The broken O(3) theory signals the existence of a very massive $\boldsymbol{A}^{(3)}$ boson, which has been observed empirically [92], and massless $\boldsymbol{A}^{(1)}$ and $\boldsymbol{A}^{(2)}$ bosons. This broken O(3) gauge theory reduces to electromagnetism with the cyclicity condition. The broken SU(2) theory reflects the occurrence, as usual, of a massive charged and neutral weak bosons. The theory can be taken further by embedding it into an SU(4) gauge theory where the gauge potentials are described by 4 × 4 traceless Hermitian matrices and the Dirac spinor has 16-components. The neutrality of the photon is then given by a sum over charges, a sum that vanishes because the theory is traceless. The Higgs field is described by a 4 × 4 matrix of entries.

By invoking the condition $A_\mu^{(3)} = 0$ in the above development, what is meant is that the *transverse* components of $A_\mu^{(3)}$ are zero. This is always the case in pure electromagnetism, because (3) is the longitudinal index. The longitudinal

$$A_\mu^{(3)} \equiv (\phi, c\boldsymbol{A}) \tag{668}$$

is evidently nonzero from the arguments of Section XI. In general, in electroweak theory, however, the indices (1), (2), and (3) denote isospin, and not the circular complex space ((1),(2),(3)). So if we take $A_\mu^{(3)}$ to denote a 4-vector with isospin index (3), it may have a transverse component that is nonzero. This would mean that the current for this gauge boson is not highly conserved with a very large mass so that the interaction scale is far smaller than that for the electromagnetic field.

If we take (1), (2), and (3) to denote isospin indices, we have in general

$$\begin{aligned}A_\mu^{\prime(1)} &= \frac{(gA_\mu^{(3)} + g'b_\mu^{(3)} - gA_\mu^{(1)})}{(g^2 + g'^2)^{1/2}} \\ Z_\mu^0 &= \frac{(gb_\mu^{(3)} + g'A_\mu^{(3)})}{(g^2 + g'^2)^{1/2}} \\ \omega_\mu^{(3)} &= \frac{g'}{(g^2 + g'^2)^{1/2}}A_\mu^{(3)}\end{aligned} \tag{669}$$

The $\omega_\mu^{(3)}$ connection has a chiral component that seems to imply that $\boldsymbol{B}^{(3)}$ has a chiral component, or is mixed with the chiral component of the other SU(2) chiral field of the electroweak theory. This is what happens to SU(2) electromagnetism at very high energies. It becomes very similar in formal structure to the theory of weak interactions and has implications for the theory of leptons. The electromagnetic interaction acts on a doublet that can be treated as an element of a Fermi doublet of charged leptons and their neutrinos in the SU(2) theory of the weak interaction.

Let ψ be a doublet that describes an electron according to the (1) field and the (3) field, where the indices (1) and (3) are isospin indices in general. The free-particle Dirac Lagrangian is ($c = 1$; $\hbar = 1$)

$$\begin{aligned}\mathscr{L} &= \bar{\psi}(i\gamma^\mu D_\mu - m)\psi = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - gA_\mu^b \bar{\psi}\gamma^\mu \sigma_b \psi \\ &= \mathscr{L}_{\text{free}} + A_\mu^b J_b^\mu \end{aligned} \tag{670}$$

where $\bar{\psi} = \psi^+ \gamma_4$. We decompose the current J_μ^b into vector and chiral components

$$J_\mu^b = \psi^+ \gamma_4 \gamma_\mu (1 + \gamma_5)\sigma^{(3)}\psi = V_\mu^b + \chi_\mu^b \tag{671}$$

a procedure that is analogous to the current algebra for weak and electromagnetic interactions between fermions. There are two vector current operators

$$V_\mu^a = \frac{i}{2}\bar{\psi}\gamma_\mu \sigma^a \psi \tag{672}$$

and two axial current operators

$$\chi_\mu^b = \frac{i}{2}\bar{\psi}\gamma_\mu \gamma_5 \tau^b \psi \tag{673}$$

where $\gamma_5 = i\gamma_1\gamma_2\gamma_3\gamma_4$ and where τ^b are Pauli matrices. These define an algebra of equal time commutators:

$$\begin{aligned}[V_4^a, V_\mu^b] &= it^{abc} V_\mu^c \\ [V_4^a, \chi_\mu^b] &= -it^{abc}\chi_\mu^c \end{aligned} \tag{674}$$

The index $\mu = 4$ implies the following algebra:

$$\begin{aligned}[V_4^a, V_4^b] &= it^{abc} V_4^c \\ [V_4^a, \chi_4^b] &= -it^{abc}\chi_4^c \end{aligned} \tag{675}$$

The definition

$$Q_\pm^a = \frac{1}{2}(V_4^a \pm \chi_4^a) \tag{676}$$

gives the algebra

$$\begin{aligned}[Q_+^a, Q_+^b] &= it^{abc} Q_+^c \\ [Q_-^a, Q_-^b] &= it^{abc} Q_-^c \\ [Q_+^a, Q_-^b] &= 0 \end{aligned} \tag{677}$$

which defines the SU(2) × SU(2) algebra. The parity operator P acts as follows:

$$\begin{aligned} PV_4^b P^+ &= V_4^b \\ P\chi_4^b P^+ &= -\chi_4^b \end{aligned} \tag{678}$$

and one SU(2) group differs from the other. The total group is therefore the chiral group SU(2) × SU(2)_P.

On the physical vacuum, the above theory becomes a vector gauge theory where the indices (1), (2), and (3) are now defined in the complex circular basis ((1),(2),(3)) described by

$$\begin{aligned} \boldsymbol{e}^{(1)} \times \boldsymbol{e}^{(2)} &= i\boldsymbol{e}^{(3)*} \\ &\cdots \end{aligned} \tag{679}$$

On the physical vacuum, therefore, there are no transverse components of $A_\mu^{(3)}$, and its longitudinal components are structured as in Section XI. On the physical vacuum, there is a mixture of vector and chiral gauge components within both the electromagnetic and weak-field sectors. This means that any transverse component of $\boldsymbol{A}^{(3)}$ will vanish identically at low energies, and any transverse component of $\boldsymbol{A}^{(3)}$ can exist only if (3) is regarded as an isospin index. If so, any transverse $\boldsymbol{A}^{(3)}$ will be massive and short-ranged and will quantize to the massive boson detected in Ref. 92. Clearly, a transverse component of $A^{(3)}$ in the pure electromagnetic sector vanishes by definition, and can exist only as a result of the mixing of the electromagnetic and weak field, and then only if (3) is generalized to an isospin index from a purely spatial index (3) = k.

If there exists a very high energy massive $\boldsymbol{A}^{(3)}$, as the data in Ref. 92 appear to indicate, there exists the nonconserved current

$$\partial^\mu J_\mu^{(3)} = im_\psi \psi^+ \gamma_4 \gamma_5 \sigma^{(3)} \psi \tag{680}$$

where inhomogeneous terms correspond to quark–antiquark and lepton–antilepton pairs that are formed from the decay of these particles. This breaks the chiral symmetry of the theory. The action of this current on the physical vacuum is such that when projected on a massive eigenstate for any 3-photon with transverse modes, for instance

$$\langle 0|\partial^\mu J_\mu^{(3)}|X_b\rangle = \frac{m^2}{(\omega(k)\omega(k'))^{1/2}}\langle X_{k'}\,|\,X_b\rangle e^{ikx} \tag{681}$$

the mass of the chiral bosons will vanish, while the mass of the chiral 3-boson will be m. Therefore $\boldsymbol{A}^{(3)}$ is a separate chiral gauge field that obeys axial vector field that does not obey axial vector conservation and occurs only at short ranges. Therefore $\boldsymbol{A}^{(3)}$ must not be confused with a transverse component of the low-energy electromagnetic $A_\mu^{(3)}$, which is zero by definition. Furthermore, the condition $\boldsymbol{A}^{(3)} = \boldsymbol{0}$ must not be taken to imply that the scalar and longitudinal vector parts of $\boldsymbol{A}^{(3)}$ are zero.

Therefore the electroweak theory is chiral at high energies, but is vector and chiral in separate sectors on the physical vacuum of low energies. The high-energy chiral field combines with the other chiral field in the twisted bundle to produce a vector field plus a broken chiral field at low energy. There are independent fields that are decoupled on the physical vacuum at low energies.

Consider two fermion fields, ψ and χ, each consisting of the two component right- and left-handed fields $R_\psi, L_\phi, R_\chi, L_\chi$. These Fermi doublets have the masses m_1 and m_2. The two gauge potentials $\boldsymbol{A}_\mu$ and $\boldsymbol{B}_\mu$ interact respectively with the ψ and χ fields. In general, these Fermi fields are degeneracies that split into the multiplet of known fermions, so that there are four possible masses for these fields in the physical vacuum. The masses originate in Yukawa couplings with the Higgs field on the physical vacuum, which give Lagrangian terms of the form $Y_\phi R_\psi^+ \phi L_\chi$ + H.C. and $Y_\eta L_\psi^+ \eta R_\chi$ + H.C. where there are two component ϕ^4 fields for the Higgs mechanism. (H.C. = higher contributions). These components assume the minimal expectation values $\langle\phi_0\rangle$ and $\langle\eta_0\rangle$ on the physical vacuum with the Lagrangian:

$$\begin{aligned}\mathscr{L} = {} & \bar\psi(i\gamma^\mu(\partial_\mu + igA_\mu) - m_1)\psi \\ & + \bar\chi(i\gamma^\mu(\partial_\mu + igB_\mu) - m_2)\chi - Y_\phi R_\psi^+ L_\chi + \text{H.C.} \\ & - Y_\eta L_\psi^+ \eta R_\chi + \text{H.C.}\end{aligned} \tag{682}$$

that can be further broken into the left and right two component spinors

$$\begin{aligned}\mathscr{L} = {} & R_\psi^+ i\sigma^\mu(\partial_\mu + igA_\mu)R_\psi + L_\psi^+ i\sigma^\mu(\partial_\mu + igA_\mu)L_\psi \\ & + R_\chi^+ i\sigma^\mu(\partial_\mu + igB_\mu)R_\chi + L_\chi^+ \sigma^\mu(\partial_\mu + igB_\mu)L_\chi \\ & - m_1 R_\psi^+ L_\psi - m_1 L_\psi^+ R_\psi - m_2 R_\chi^+ L_\chi - m_2 L_\chi^+ R_\chi \\ & - Y_\phi R_\psi^+ \phi L_\chi + Y_\phi^* L_\chi^+ \phi^* R_\psi - Y_\eta L_\psi^+ \eta R_\chi + Y_\eta^* R_\chi^+ \eta^* L_\psi\end{aligned} \tag{683}$$

The gauge potentials A_μ and B_μ are 2×2 Hermitian traceless matrices, and the Higgs fields ϕ and χ are also 2×2 matrices. These expectations are real-valued, and the nonzero contributions of the Higgs field on the physical vacuum are given by the diagonal matrix entries [95]:

$$\langle\phi\rangle = \begin{bmatrix}\langle\phi^{(1)}\rangle & 0 \\ 0 & \langle\phi^{(2)}\rangle\end{bmatrix}; \qquad \langle\chi\rangle = \begin{bmatrix}\langle\chi^{(1)}\rangle & 0 \\ 0 & \langle\chi^{(2)}\rangle\end{bmatrix} \tag{684}$$

The values of the vacuum expectations are such that, at high energy, the left-handed fields R_χ and the right-handed doublet field L_ψ couple to the SU(2) vector boson field B_μ, while at low energy, the theory is one with a left-handed SU(2) doublet R_ψ that interacts with the right-handed doublet L_χ through the massive gauge fields A_μ. The mass terms from the Yukawa coupling Lagrangians will give

$$m' = Y_\eta\langle\chi^{(1)}\rangle \gg m'' = Y_\eta\langle\chi^{(2)}\rangle \gg m''' = Y_\phi\langle\phi^{(1)}\rangle \gg m''' = Y_\phi\langle\phi^{(2)}\rangle \tag{685}$$

If the SU(2) theory for B_μ potentials are right-handed chiral and the SU(2) theory for A_μ potentials are left-handed chiral, a chiral theory at high energies can become a vector theory at low energies.

This is a broken gauge theory at low energy, which can be expressed as in Eq. (686) as a gauge theory accompanied by a broken gauge symmetry. Assume a simple Lagrangian that couples the left-handed fields ψ_l to the right-handed boson A_μ and the right-handed fields ψ_r to the left-handed boson B_μ:

$$\begin{aligned}\mathscr{L} = {} & \bar\psi_l(i\gamma^\mu(\partial_\mu + igA_\mu) - m_1)\psi_l + \bar\psi_r(i\gamma^\mu(\partial_\mu + igB_\mu) - m_2)\psi_r \\ & - Y_\phi\psi_l^+\phi\psi_r - Y_\phi\psi_r^+\psi_l\end{aligned} \tag{686}$$

If the coupling constant Y_ϕ is comparable with the coupling constant g, then the Fermi expectation energies of the fermions occur at the mean expectation value for the Higgs field $\langle\phi_0\rangle$. In this case, the vacuum expectation value is proportional to the identity matrix, meaning that the masses acquired by the right chiral plus left chiral gauge bosons $A_\mu + B_\mu$ are zero, while the right chiral minus left chiral gauge bosons $A_\mu - B_\mu$ acquire masses approximately equal to $Y_\phi\langle\phi_0\rangle$. The theory at low energies is one with an unbroken vector gauge theory plus a broken chiral gauge theory [95]. The additive charges $A^{(1)(2)}, B^{(1),(2)}$ of the two chiral fields are opposite so that of the resulting vector gauge bosons are chargeless. Therefore gauge theories can change their vector and chiral character, and so also can the doublets of the theory. In so doing, this will give rise to the doublets of leptons and quarks plus doublets of very massive fermions that should be observable in the multi-TeV range.

The two parts of the twisted bundle are copies of SU(2) with a doublet fermion structure. One of the fermions has a very large mass, $m' = Y_\eta \langle \chi^{(1)} \rangle$, which is assumed to be unstable and not observed at low energies. So one sector of the twisted bundle is left with the same Abelian structure, but with a singlet fermion, meaning that the SU(2) gauge theory becomes defined by the algebra over the basis elements

$$[\hat{e}_i, \hat{e}_j] = i\varepsilon_{ijk}\hat{e}_k \tag{687}$$

To calculate the photon masses, define the Higgs field by a small expansion around the vacuum expectations

$$\begin{aligned} \eta^{(1)} &= \xi^{(1)} + \langle \eta_0^{(1)} \rangle \\ \eta^{(2)} &= \xi^{(2)} + \langle \eta_0^{(2)} \rangle \end{aligned} \tag{688}$$

The contraction of the generators $\sigma^{(1)}$ and $\sigma^{(2)}$ with the Higgs field matrix and right and left fields gives

$$\boldsymbol{\sigma}^{(1)} \cdot \eta R + \boldsymbol{\sigma}^{(2)} \cdot \eta L = 0 \tag{689}$$

so that the charges of the $\boldsymbol{A}^{(1)}$ and $\boldsymbol{A}^{(2)}$ fields are zero. On the low-energy vacuum, these fields can be thought of as massless fields composed of two gauge bosons, with masses $(m' + m'')^{1/2} \gg M_Z$ and with opposite charges. These electrically charged fields can be thought of as $\boldsymbol{A}^{\pm} = \boldsymbol{A}^{(1)} \pm \boldsymbol{A}^{(2)}$, giving rise to particles that cancel each other and massless vector photon gauge fields. The $\boldsymbol{A}^{(3)}$ field has an unstable mass that decays into particle pairs.

Therefore the more massive Higgs field acts to give the gauge theory $SU(2) \times O(3)$, where the first gauge group acts on singlets. On a lower energy scale, or longer timescale, $\boldsymbol{A}^{(3)}$ has decayed and vanished. The second gauge group is then represented by $O(3)_P$, a notation that implies "partial group." The latter describes Maxwell's equations, and the $\boldsymbol{B}^{(3)}$ field is defined through $-ig\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$. Evidently, in this scale, the isospin indices become identified with the space indices (1), (2), and (3) of the circular basis.

The second Higgs field acts in such a way that if the vacuum expectation value is zero, $\langle \phi^{(2)} \rangle = 0$, then the symmetry breaking mechanism effectively collapses to the Higgs mechanism of the standard $SU(2) \times U(1)$ electroweak theory. The result is a vector electromagnetic gauge theory $O(3)_P$ and a broken chiral SU(2) weak interaction theory. The mass of the vector boson sector is in the $\boldsymbol{A}^{(3)}$ boson plus the $W^{\pm}$ and Z^0 particles.

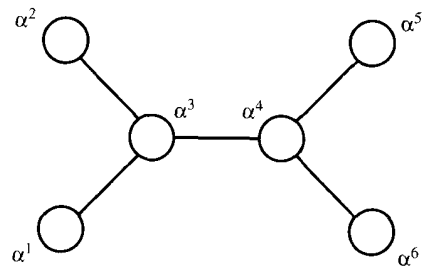
The two SU(2) theories can be represented as the block diagonals of the SU(4) gauge theory. The Lagrangian density for the system is then

$$\mathscr{L} = \bar{\psi}(i\gamma^\mu(\partial_\mu + igA_\mu) - m_1)\psi - Y\bar{\psi}\phi\psi \tag{690}$$

and the gauge potentials A_μ now have 4×4 traceless representations. The scalar field theory that describes the vacuum will satisfy field equations that involve all 16 components of the gauge potential. By selectively coupling these fields to the fermions, it might be possible to construct a theory that recovers a low energy theory that is the standard model with the $O(3)_P$ gauge theory for electromagnetism. We arrive at the important conclusion that the electroweak theory can be constructed with an O(3) electrodynamic sector to provide additional physical details at high energy.

The prediction of a heavy boson $\boldsymbol{A}^{(3)}$ has received preliminary empirical support [92,96] from an anomaly in Z decay widths that points toward the existence of Z bosons with a mass of 812 GeV^{+339}_{-152} [92,96] within the SO(1) grand unified field model, and a Higgs mechanism of 145 GeV^{+103}_{-61}. This suggests that a new massive neutral boson has been detected. Analysis of the hadronic peak cross sections obtained at LEP [96] implies a small amount of missing invisible width in Z decays. The effective number of massless neutrinos is 2.985 ± 0.008, which is below the prediction of 3 by the standard model of electroweak interactions. The weak charge Q_W in atomic parity violation can be interpreted as a measurement of the S parameter. This indicates a new $Q_W = -72.06 \pm 0.44$, which is found to be above the standard model prediction, an effect interpreted as being due to the occurrence of the Z' particle, which is referred to hereinafter as the Z_γ particle.

SO(10) has the six roots $\alpha^i, i = 1, \ldots, 6$. The angle between the connected roots are all 120°, where the roots α^3, α^4 are connected to each other and two other roots. The Dynkin diagram is



The decomposition of SO(10) to $SU(5) \times U(1)$ is performed by removing the circles representing the roots $\alpha^{1,2,5,6}$ connected by a single branch. The remaining connected graph describes the SU(5) group, and the isolated circle is the U(1) group. However, by removing either of the circles $\alpha^{3,4}$ connected by three branches forces SO(10) to decompose into $SU(2) \times SU(2) \times SU(4)$. Here, we have an SU(2) and a mirror SU(2) that describe opposite-handed chiral gauge fields, plus an SU(4) gauge field. The chiral fields are precisely the sort of electroweak structure proposed in this section and elsewhere [17,94]. Since

SU(4) can be represented by a 4, that is, $3 \oplus 1$ and $\bar{4}$ as $\bar{3} \oplus \bar{1}$, SU(4) can be decomposed into SU(3) × U(1). The neutrino short fall is furthermore a signature of the opposite chiralities of the two "mirrored" gauge fields [17,94].

The mechanism $SU(2) \times SU(2) \rightarrow SU(2) \times O(3)$ discussed in this section predicts the occurrence of a massive $\boldsymbol{A}^{(3)}$, so it is possible that the LEP data could corroborate the work outlined in this section, with an extended electromagnetic sector. Quantum chromodynamics (QCD) and the standard model of the electroweak theory are understood empirically. There is reasonable empirical corroboration in the TeV range and ideas about quantum gravity at 10^{19} GeV, but nothing in between. The LEP data therefore give some confidence that O(3) electrodynamics is a valid theory, and the data suggest that at high energy, electrodynamics and the weak interactions are dual-field theories in the TeV range of energy, which is expected to be accessible to the CERN heavy hadron collider.

The LEP data could be the first indication that the universe is dual according to the Olive Montonen construct [97], which asserts that coupling constants have inverse relationships. One field is weak, and the other is strong at high energy. The experimental finding [96] of the massive $\boldsymbol{A}^{(3)}$ might bring a basic change in the foundations of physics. For example, it may be conjectured that there is a dual field theory to the SU(3) nuclear interaction of QCD with a chiral SU(2) × SU(2) electroweak theory, implying the existence of an additional weak field in nature. The problem with such a program is that supergravity and superstring theories imply that, at very high energies, the universe is one of 10 or 11 dimensions [98]. The minimal grand unified field theory is the SU(5) theory that breaks into SU(3) × SU(2) × U(1) at lower energy. This is a gauge theory in six dimensions that fits into the Calabi–Yau construction of compactified manifolds. These spaces leave the four-dimensional spacetime left over and uncompactified from the 10 dimensions at high energy. A Calabi–Yau manifold of seven dimensions would accommodate an SU(3) × SU(2) × SU(2) bundle. The-low energy SU(2) × SU(2) electroweak theory would then suggest a superstring theory of 11 dimensions, which appears to preclude any SU(3) field dual to QCD because this would demand a Calabi–Yau space that can subsume an SU(3) × SU(3) × SU(2) × SU(2) bundle of 10 dimensions, and a supergravity theory of 14 dimensions.

The theory of gravitation, however, need not involve four dimensions; information [99] may exist on a two dimensional surface, such as the event horizon of a black hole. If the symmetries relevant to gravitation involve the evolution of a two-dimensional surface, then an SU(2) × SU(2) × SU(3) gauge theory plus gravity would be 11-dimensional, and duality between the two surfaces that construct spacetime would reduce this to nine dimensions. However, the issue of duality with nuclear interactions would still increase the dimensionality required to 12 or 14, and supergravity requires a total space of 11 dimensions. Strings exist at 10 dimensions.

If the nature of spacetime involves the interference of dual wave fronts of two dimensions, then there are two wave fronts, each of two dimensions, that constructively and destructively interfere, but that are determined by the same symmetry space. Gravitation can be described by the set of diffeomorphisms of a two-dimensional surface and SU(2) × SU(2) × SU(3) plus gravity involving a space of nine dimensions. The additional dimensions to spacetime are purely virtual in nature. A field dual to QCD would require a large space of 12 dimensions, and an additional constraint is required in order for this theory to satisfy current models of supergravity.

Gravitation is described by the Lie group $SO(3,1) \sim SL(2,C)/Z_2$. It can be seen that the relevant symmetries are contained in the SL(2,C) component of two dimensions, and the Lie group has a hyperbolic metric structure. The Euclidean group for gravity is $SO(4) \sim (SU(2) \times SU(2))/Z_2$. In effect, these two groups are related by a rotation $t \rightarrow it$, which might suggest that the electroweak interaction and gravitation can be regarded as two states of a single symmetry that may manifest itself by the action of a U(1) rotation on the Cartan center of SU(2), $\sigma'_{(3)} = e^{i\theta}\sigma^{(3)}$. At low energy, the circle associated with this rotation is reduced to a point and the direction of the angle θ determines the coupling constant for the electroweak and gravitational fields, implying a superstring theory in 11 dimensions.

If there is a field dual to the SU(3) QCD field, and if the theory is similar in form to the electroweak unification scheme outlined in this section, there may be a right–left chiral SU(3) bundle that, at low energy, combines into a right − left chiral and right + left chiral field. This result would indicate that QCD is a vector theory but associated with another field that is chiral or that has a broken chirality. Since QCD is the strongest force in the universe with $g = 1$, its putative dual field is one with a very weak coupling constant. For example, there may be slight chiral couplings between quarks. This would, in turn, imply the discovery of chirality with gluons, usually regarded as vector bosons.

In the absence of data, it seems best to proceed on the assumption that gauge theory at low energy is SU(2) × SU(2) × SU(3) and that the inclusion of gravity gives a space of 11 dimensions at high energy, fitting in with supergravity models. These thoughts [7,94] indicate the major impact on physics of the $\boldsymbol{B}^{(3)}$ field.

XIII. RELATIVISTIC HELICITY

In this section, we extend consideration from the Lorentz to the Poincaré group within the structure of O(3) electrodynamics, by introducing the generator of spacetime translations along the axis of propagation in the normalized (unit 12-vector) form:

$$\varepsilon_\mu \equiv \varepsilon_\mu^{(1)} \boldsymbol{e}^{(1)} + \varepsilon_\mu^{(2)} \boldsymbol{e}^{(2)} + \varepsilon_\mu^{(3)} \boldsymbol{e}^{(3)} \tag{691}$$

The relativistic helicity is then the product

$$\tilde{G}_\nu \equiv \tilde{G}^{(1)}_{\mu\nu}\varepsilon^{\mu(2)} + \tilde{G}^{(2)}_{\mu\nu}\varepsilon^{\mu(1)} + \tilde{G}^{(3)}_{\mu\nu}\varepsilon^{\mu(3)} \tag{692}$$

which, for $Z = (3)$ axis propagation, is the Pauli–Lubanski pseudo vector (PL vector):

$$\tilde{G}_\nu = \tilde{G}^{(3)}_{\mu\nu}\varepsilon^{\mu(3)} = \frac{1}{2}\varepsilon_{\mu\nu\sigma\rho}G^{\sigma\rho(3)}\varepsilon^{\mu(3)} \tag{693}$$

Evidently, this vanishes on the U(1) level, a basic paradox, because the photon has helicity after quantization. By using the Poincaré group, a fundamental geometric proof can be given for the existence of $\boldsymbol{B}^{(3)}$ in the vacuum, and helicity defined entirely through $\boldsymbol{B}^{(3)}$. This proof proceeds by constructing the PL vector from the geometric 3-manifold in 4-space, a 3-manifold that is in general a tensor of rank 3 in four dimensions, antisymmetric in all 3 indices. The PL vector is dual to this 3-tensor and has the same magnitude. The 3-tensor $S^{\mu\nu\sigma}$ is in general the following product:

$$S^{\nu\sigma\mu} \equiv \tilde{G}^{\nu\sigma}\varepsilon^{\mu} \tag{694}$$

This approach is therefore based in rigorous and general geometric tensor theory. The PL vector dual to $S^{\nu\sigma\mu}$ turns out to be the light-like invariant:

$$\tilde{B}^{\mu} = (B^{(3)}, 0, 0, B^{(3)}) \tag{695}$$

In the Lorentz group, this concept is missing, and in the Poincaré group, the relativistic helicity vanishes if $\boldsymbol{B}^{(3)}$ is not zero. Therefore $\boldsymbol{B}^{(3)}$can be regarded as the fundamental field component representing spin in the classical electromagnetic field. If $\boldsymbol{B}^{(3)}$ were zero, the PL vector would be a null vector, meaning that the space part of the equivalent hypersurface element is null. This result is a paradox, because a physical beam of light must always have a finite cross section or area perpendicular to the propagation axis of the beam, the Z or (3) axis. So if $\boldsymbol{B}^{(3)}$ vanishes, reduction to absurdity occurs, and the beam of light vanishes. This result, in turn, is self-consistent with the fact that if $\boldsymbol{B}^{(3)}$ were zero in the B cyclic theorem, $\boldsymbol{B}^{(1)}$ and $\boldsymbol{B}^{(2)}$ would also vanish, and electromagnetism would vanish.

The unit 12-vector ε_μ acts essentially as a normalized spacetime translation on the classical level. The concept of spacetime translation operator was introduced by Wigner, thus extending [100] the Lorentz group to the Poincaré group. The PL vector is essential for a self-consistent description of particle spin.

The dual pseudotensor of any antisymmetric tensor in 4-space arises from the integral over a two-dimensional surface in 4-space [101], in which the infinitesimal element of surface is given by the antisymmetric tensor:

$$df^{\mu\nu} = dx^{\mu}dx^{\nu'} - dx^{\nu}dx^{\mu'} \tag{696}$$

The components of this tensor are projections of the element of area on the coordinate planes. In 3-space, it is always possible to define an axial pseudovector element $d\tilde{f}_i$, dual to the antisymmetric tensor df_{jk}:

$$d\tilde{f}_i \equiv \frac{1}{2}\varepsilon_{ijk}df_{jk} \tag{697}$$

The pseudovector element $d\tilde{f}_i$ represents the same surface element as df_{jk}, and, geometrically, is a pseudovector normal to the surface element and equal in magnitude to the area of the element. In 4-space, such a pseudovector cannot be constructed from an antisymmetric tensor such as $df_{\mu\nu}$. However, the dual pseudotensor can be defined by [10]:

$$d\tilde{f}^{\mu\nu} \equiv \frac{1}{2}\varepsilon^{\mu\nu\sigma\rho}df_{\sigma\rho} \tag{698}$$

where $\varepsilon^{\mu\nu\sigma\rho}$ is the totally symmetric unit pseudotensor in four dimensions, with the property

$$\varepsilon^{0123} = -\varepsilon_{0123} = 1 \tag{699}$$

in cyclic permutation of indices. In geometric terms, $d\tilde{f}^{\mu\nu}$ is an element of surface equal and normal to the element $df_{\sigma\rho}$. All segments in it [101] are orthogonal to all segments in $df_{\sigma\rho}$, leading to the following result:

$$d\tilde{f}^{\mu\nu}df_{\mu\nu} = 0 \tag{700}$$

In general, therefore, an antisymmetric 4-tensor is an element of surface in 4-space. There are three of these elements of surface in the 12-vector $\tilde{G}^{\mu\nu}$.

Equation (700) means that $\tilde{G}^{\mu\nu}$ is orthogonal to $G_{\mu\nu}$ in free space

$$\tilde{G}^{\mu\nu}G_{\mu\nu} = 0 \tag{701}$$

where

$$\tilde{G}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\sigma\rho}G_{\sigma\rho} \tag{702}$$

$$G_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\sigma\rho}\tilde{G}^{\sigma\rho} \tag{703}$$

In contravariant covariant notation, the field tensors are defined by [101]

$$G_{\sigma\rho} = \begin{bmatrix} 0 & \frac{E_1}{c} & \frac{E_2}{c} & \frac{E_3}{c} \\ \frac{-E1}{c} & 0 & -B_3 & B_2 \\ \frac{-E_2}{c} & B_3 & 0 & -B_1 \\ \frac{-E_3}{c} & -B_2 & B_1 & 0 \end{bmatrix}; \quad G^{\sigma\rho} = \begin{bmatrix} 0 & \frac{-E^1}{c} & \frac{-E^2}{c} & \frac{-E^3}{c} \\ \frac{E_1}{c} & 0 & -B^3 & B^2 \\ \frac{E^2}{c} & B^3 & 0 & -B^1 \\ \frac{E^3}{c} & -B^2 & B^1 & 0 \end{bmatrix} \tag{704}$$

and the dual tensors by

$$\tilde{G}^{\mu\nu} = \begin{bmatrix} 0 & -B^1 & -B^2 & -B^3 \\ B^1 & 0 & \frac{E^3}{c} & \frac{-E^2}{c} \\ B^2 & \frac{-E^2}{c} & 0 & \frac{E^1}{c} \\ B^3 & \frac{E^2}{c} & \frac{-E^1}{c} & 0 \end{bmatrix}; \quad \tilde{G}_{\mu\nu} = \begin{bmatrix} 0 & B_1 & B_2 & B_3 \\ -B_1 & 0 & \frac{E_3}{c} & \frac{-E_2}{c} \\ -B_2 & \frac{-E_3}{c} & 0 & \frac{E_1}{c} \\ -B_3 & \frac{E_2}{c} & \frac{-E_1}{c} & 0 \end{bmatrix} \tag{705}$$

It follows that

$$\tilde{G}^{\mu\nu(3)} G^{(3)}_{\mu\nu} = 0 = 4\boldsymbol{B}^{(3)} \cdot \boldsymbol{E}^{(3)} \tag{706}$$

and that $\boldsymbol{B}^{(3)}$ is zero. This result is self-consistent with earlier arguments and with the fact that the light like products of PL vectors are null:

$$B^\mu B^*_\mu = E^\mu E^*_\mu = 0 \tag{707}$$

The only nonzero components of the PL vectors $\tilde{B}^\mu$ and $\tilde{B}_\mu$ are the longitudinal and time-like components. It follows that since $\boldsymbol{B}^{(3)}$ is null, its magnitude is zero, and so $\tilde{E}^\mu$ and $\tilde{E}_\mu$ are null. This result is, in turn, consistent with the fact that the PL vector is a pseudovector, whereas $\tilde{E}^\mu$ is a null vector whose dual is null.

The dual axial vector in 4-space is constructed geometrically from the integral over a hypersurface, or manifold, a rank 3-tensor in 4-space antisymmetric in all three indices [101]. In three-dimensional space, the volume of the parallelepiped spanned by three vectors is equal to the determinant of the third rank formed from the components of the vectors. In four dimensions, the projections can be defined analogously of the volume of the parallelepiped (i.e., areas of the hypersurface) spanned by three vector elements: dx^μ, dx'^μ and dx''^μ. They are given by the determinant

$$dS^{\mu\nu\sigma} = \begin{vmatrix} dx^\mu & dx'^\mu & dx''^\mu \\ dx^\nu & dx'^\nu & dx''^\nu \\ dx^\sigma & dx'^\sigma & dx''^\sigma \end{vmatrix} \tag{708}$$

which forms a tensor of rank 3, antisymmetric in all three indices. The axial 4-vector element $d\tilde{S}^\mu$ dual to the tensor element $dS^{\mu\nu\sigma}$ is the element of integration over a hypersurface in four dimensions:

$$\begin{aligned} d\tilde{S}^\mu &= -\frac{1}{6}\varepsilon^{\mu\nu\sigma\rho} dS_{\nu\sigma\rho} \\ dS_{\nu\sigma\rho} &= \varepsilon_{\mu\nu\sigma\rho} d\tilde{S}^\mu \end{aligned} \tag{709}$$

so that $d\tilde{S}^0 = dS^{123}$, $d\tilde{S}^1 = dS^{023}$, and so on. The S^0 component of S^μ is therefore equivalent to the S^{123} component of $S^{\nu\sigma\rho}$, normal to it and equal to it in magnitude. The PL vector is an example of a 4-vector dual to the 3-manifold in 4-space. This is a rigorous geometric result, and if the PL vector were null, it would represent a null hypersurface in four dimensions. This, as follows, is a rigorous geometric proof of the fact that $\boldsymbol{B}^{(3)}$ is nonzero within the Poincaré group. The dual-vector S^μ is a 4-vector equal in magnitude to the area of the hypersurface to which it is dual, and is normal to this hypersurface. It is therefore perpendicular to all lines drawn in the hypersurface. In particular, the element $dS^0 = dXdYdZ$ is an element of three-dimensional volume, dV, the projection of the hypersurface on to the hyperplane $x^0 =$ constant.

In classical electromagnetic theory, the PL vector is defined through the 3-manifold

$$S^{\mu\nu\sigma} \equiv \begin{vmatrix} \partial^\mu & A^\mu & \varepsilon^\mu \\ \partial^\nu & A^\nu & \varepsilon^\nu \\ \partial^\sigma & A^\sigma & \varepsilon^\sigma \end{vmatrix} \tag{710}$$

defining the fully antisymmetric rank 3-tensor

$$S^{\nu\sigma\mu} = (\partial^\nu A^\sigma - \partial^\sigma A^\nu)\varepsilon^\mu + \cdots \tag{711}$$

which consists of three terms, the first of which is the product of ϵ^μ with the antisymmetric tensor $G^{\nu\sigma}$, a component in internal gauge space of Eq. (22). This product gives the PL vector through

$$\tilde{S}^\mu = \frac{1}{2}\varepsilon^{\mu\nu\sigma\rho} S_{\nu\sigma\rho} \tag{712}$$

The second two terms of the sum (711) can be eliminated using a combination of the free-photon minimal prescription and the quantum hypothesis

$$\partial_\mu = -i\frac{e}{\hbar} A_\mu \tag{713}$$

and the manifold defined in Eq. (711) reduces precisely to

$$S^{\nu\sigma\mu} = G^{\nu\sigma}\varepsilon^\mu \tag{714}$$

It is now possible to adopt the standard definition [6] of the PL vector to the problem at hand to give

$$\tilde{G}^\mu = \frac{1}{2}\varepsilon^{\mu\nu\sigma\rho} G_{\sigma\rho}\varepsilon_\nu \tag{715}$$

where

$$G_{\sigma\rho} = \partial_\sigma A_\rho - \partial_\rho A_\sigma \tag{716}$$

In Eq. (715), $\tilde{G}^\mu$ is dual to the third rank $G_{\sigma\rho}\varepsilon_\nu$ in four dimensions and normal to it with the same magnitude. In the received view, there is nothing normal to the purely transverse $G_{\sigma\rho}$ on the U(1) level, and therefore $\tilde{G}^\mu$ cannot be consistently dual with $G_{\sigma\rho}\varepsilon_\nu$. This result is inconsistent with the four-dimensional algebra of the Poincaré group. If we adopt the notation $\tilde{G}_\nu \equiv \tilde{B}_\nu$, we obtain

$$\tilde{B}_\nu = \tilde{\boldsymbol{G}}_{\mu\nu} \cdot \varepsilon^\mu \tag{717}$$

and the complete PL vector in consequence is

$$\begin{aligned} \tilde{B}_\nu &= \tilde{G}^{(3)}_{\mu\nu}\varepsilon^{\mu(3)} \\ &= \frac{1}{2}\varepsilon_{\mu\nu\sigma\rho}G^{\sigma\rho(3)}\varepsilon^{\mu(3)} \\ &= (B^{(3)}, 0, 0, -B^{(3)}) \end{aligned} \tag{718}$$

Similarly

$$\tilde{B}^\nu = (B^{(3)}, \boldsymbol{B}^{(3)}) \tag{719}$$

which is orthogonal to $\tilde{B}_\nu$.

The PL vector was originally constructed for particles from the generators of the Poincaré group. The PL vector corresponding to the photon's angular momentum corresponds in free space and in $c = 1$ units to

$$\tilde{J}^\mu = (J^{(3)}, 0, 0, J^{(3)}) \tag{720}$$

and the light-like momentum in $c = 1$ units is

$$p^\mu = (p^{(3)}, 0, 0, p^{(3)}) \tag{721}$$

If the mass of the photon is identically zero, its normalized helicity takes the values $+1$ and -1 because $\tilde{J}^\mu$ is proportional to p^μ [6]. The 0 component, which usually appears for a boson, is not considered but reappears if the photon has identically nonzero mass. In this case, the Wigner little group becomes O(3). The $\boldsymbol{B}^{(3)}$ field corresponds to $\tilde{\boldsymbol{J}}^{(3)}$ for the photon with a tiny but nonzero mass because, as argued earlier, the structure of the O(3) field equations is identical with that of the Lehnert equations [Eqs. (612)], which imply photon mass. Therefore p^μ and $\tilde{J}^\mu$ in the laboratory are infinitesimally different from light-like,

but on an astronomical scale, they may become substantially different from light-like [11–20].

A complete consideration of relativistic helicity in the electromagnetic field therefore requires consideration of the Poincaré group. It is not sufficient to consider the Lorentz group. The vector dual to the antisymmetric field tensor introduced by Lorentz, Poincaré and Einstein could not have been defined prior to the introduction of the Pauli–Lubanski vector and Wigner's work of 1939 [100]. This work characterized all particles in terms of two Casimir invariants: one for mass and one for spin. The photon and electromagnetic field are linked by quantization, so the Wigner method must also be applied to the field. When this is done, as in this section, the relativistic helicity in O(3) electrodynamics is defined entirely by $\boldsymbol{B}^{(3)}$. U(1) electrodynamics can be described in terms of the Lorentz group, in which relativistic helicity is incompletely defined. A full understanding of $\boldsymbol{B}^{(3)}$ therefore requires the Poincaré group [11–20]. Furthermore, Noether's theorem is reduced to energy-momentum conservation only with the use of the spacetime translation generator, which within a factor $\hbar$, is the energy-momentum 4-vector itself. In the received view of the classical field [5], energy momentum is defined only through transverse components, whereas in O(3) electrodynamics, it is straightforwardly defined through $\boldsymbol{A}^{(3)}$, which is purely longitudinal at low energies.

The nature of the dual vector $(\tilde{B}^\mu)$ can be deduced without using any equation of motion, but the dual 4-vector is a fundamental geometric property in the four dimensions of spacetime. The complete description of the electromagnetic field in O(3) electrodynamics must therefore involve boosts, rotations, and spacetime translations, meaning that $\tilde{B}^\mu$ is a fundamental geometric property of spacetime. The unit 4-vector ε_μ is orthogonal to the unit 4-vector $\tilde{B}^\mu$:

$$\varepsilon_\mu \tilde{B}^\mu \equiv 0 \tag{722}$$

and this is a fundamental property of the Poincaré group. The Casimir invariants of the electromagnetic field are therefore

$$\begin{aligned} \varepsilon_\mu \varepsilon^\mu &\equiv 0 \\ \tilde{B}_\mu \tilde{B}^\mu &\equiv 0 \\ \varepsilon_\mu \tilde{B}^\mu &\equiv 0 \end{aligned} \tag{723}$$

The homogeneous O(3) equations in the vacuum are obtained by considering the helicities:

$$\begin{aligned} \varepsilon^{(3)}_\mu \tilde{G}^{\mu\nu(3)} &= (B^{(3)}, 0, 0, B^{(3)}) \\ \varepsilon^{(3)}_\mu \tilde{G}^{\mu\nu(1)} &= 0 \\ \varepsilon^{(3)}_\mu \tilde{G}^{\mu\nu(2)} &= 0 \end{aligned} \tag{724}$$

The first of these gives the vector $\tilde{B}^{\mu}$, and the second and third give terms such as

$$-B_X^{(1)} + \frac{E_Y^{(1)}}{c} = 0 \tag{725}$$

The three relativistic helicities (724) therefore give Eqs. (590)–(592) with the addition of the following equation:

$$\nabla \cdot \boldsymbol{B}^{(3)} = 0 \tag{726}$$

In arriving at this conclusion, we have used antisymmetric tensor definitions such as

$$\tilde{G}^{\mu\nu(1)} \equiv \begin{bmatrix} 0 & -B_X^{(1)} & -B_Y^{(1)} & 0 \\ B_X^{(1)} & 0 & 0 & \frac{-E_Y^{(1)}}{c} \\ B_Y^{(1)} & 0 & 0 & \frac{E_X^{(1)}}{c} \\ 0 & \frac{E_Y^{(1)}}{c} & \frac{-E_X^{(1)}}{c} & \end{bmatrix} \tag{727}$$

By considering the conserved quantity $\tilde{B}^{\mu(3)}$, we arrive at

$$\partial_{\mu}\tilde{B}^{\mu(3)} = 0 \tag{728}$$

a solution of which is

$$\frac{\partial \boldsymbol{B}^{(3)}}{\partial t} = \boldsymbol{0}; \qquad \nabla \cdot \boldsymbol{B}^{(3)} = 0 \tag{729}$$

The overall structure of the O(3) equations in the vacuum is therefore

$$\partial_{\mu}\tilde{G}^{\mu\nu} \equiv 0 \tag{730}$$

This is the same structure as the homogenous Maxwell–Heaviside equations in the vacuum, which can therefore be obtained by a consideration of relativistic helicity.

We have seen that the overall structure of the inhomogeneous O(3) equations in the vacuum is [Eqs. (612)]

$$\partial_{\mu}H^{\mu\nu} = J^{\nu}_{\text{vac}} \tag{731}$$

where the vacuum charge density is defined by

$$\rho_{\text{vac}} = ig(\boldsymbol{A}^{(2)} \cdot \boldsymbol{D}^{(3)} - \boldsymbol{D}^{(2)} \cdot \boldsymbol{A}^{(3)} + \boldsymbol{A}^{(3)} \cdot \boldsymbol{D}^{(1)} - \boldsymbol{D}^{(3)} \cdot \boldsymbol{A}^{(1)} + \boldsymbol{A}^{(1)} \cdot \boldsymbol{D}^{(2)} - \boldsymbol{D}^{(1)} \cdot \boldsymbol{A}^{(2)}) \tag{732}$$

and the vacuum current density by

$$\begin{aligned} \boldsymbol{J}_{\text{vac}} = -ig(&cA_0^{(2)}\boldsymbol{D}^{(3)} - cA_0^{(3)}\boldsymbol{D}^{(2)} + \boldsymbol{A}^{(2)} \times \boldsymbol{H}^{(3)} - \boldsymbol{A}^{(3)} \times \boldsymbol{H}^{(2)} \\ &+ cA_0^{(3)}\boldsymbol{D}^{(1)} - cA_0^{(1)}\boldsymbol{D}^{(3)} + \boldsymbol{A}^{(3)} \times \boldsymbol{H}^{(1)} - \boldsymbol{A}^{(1)} \times \boldsymbol{H}^{(3)} \\ &+ cA_0^{(1)}\boldsymbol{D}^{(2)} - cA_0^{(2)}\boldsymbol{D}^{(1)} + \boldsymbol{A}^{(1)} \times \boldsymbol{H}^{(1)} - \boldsymbol{A}^{(2)} \times \boldsymbol{H}^{(1)}) \end{aligned} \tag{733}$$

Therefore, the vacuum charge and current densities of Panofsky and Phillips [86], or of Lehnert and Roy [10], are given a topological meaning in O(3) electrodynamics. In this condensed notation, the vacuum O(3) field tensor is given by

$$H^{\mu\nu} = \begin{bmatrix} 0 & -D^1 & -D^2 & -D^3 \\ D^1 & 0 & \frac{-H^3}{c} & \frac{H^2}{c} \\ D^2 & \frac{H^3}{c} & 0 & \frac{-H^1}{c} \\ D^3 & \frac{-H^2}{c} & \frac{H^1}{c} & 0 \end{bmatrix} \tag{734}$$

and the 4-current by

$$J^{\nu} = \left(\rho, \frac{\boldsymbol{J}}{c}\right) \tag{735}$$

The equations of O(3) electrodynamics can therefore be written in condensed form as Eqs. (730) and (731) in the vacuum. These equations can be written as a single conservation law under all conditions (vacuum and field–matter interaction):

$$\partial_{\mu}\tilde{G}^{\mu} = \partial_{\mu}H^{\mu} = 0 \tag{736}$$

$$\tilde{G}^{\mu} \equiv \tilde{G}^{\mu\nu}; \qquad H^{\mu} \equiv H^{\mu\nu}\tilde{\varepsilon}_{\nu} \tag{736a}$$

In general, define the unit generators

$$\varepsilon^{\mu} = \left(1, \frac{v}{c}, -\frac{v}{c}, \frac{v}{c}\right) \tag{737a}$$

$$\tilde{\varepsilon}^{\mu} = \left(\frac{v}{c}, 1, 1, 1\right) \tag{737b}$$

where v is linear velocity and c the speed of light. Equation (737a) defines a unit energy-momentum 4-vector orthogonal to the unit energy momentum 4-vector in Eq. (737b). The existence of such generators signals that the electromagnetic field in general has a rotation–translation character, so forward momentum is always accompanied simultaneously by a transverse momentum. Thus $\epsilon_\mu \tilde{\epsilon}^\mu = 0$, that is, ϵ_μ is orthogonal to $\tilde{\epsilon}_\mu$. This feature develops Eq. (736) into two field equations. In the vacuum, $v = c$, and these field equations become Eqs. (730) and (731) with vacuum charge and current defined by Eqs. (732) and (733), respectively. In field–matter interaction, $v < c$ in the charge–current 4-vector of Eq. (735). If $\boldsymbol{B}^{(3)}$ is zero, the vacuum electromagnetic field is lost. Because of its simultaneous rotation and translation, the electromagnetic field has left- and right-handed circular polarization and is chiral. The Pauli–Lubanski construct can be either a pseudovector or vector.

We first consider the conservation law

$$\partial_\mu \tilde{G}^\mu = 0 \tag{738}$$

where ($c = 1$ units)

$$\tilde{G}^\mu = \tilde{G}^{\mu\nu}\varepsilon_\nu = \left(-\frac{v}{c}B^1 + \frac{v}{c}B^2 - \frac{v}{c}B^3, B^1 - \frac{v}{c}E^2, B^2 + \frac{v}{c}E^1, B^3 + \frac{v}{c}E^2 + \frac{v}{c}E^1\right) \tag{739}$$

giving the conservation equation:

$$\frac{v}{c}(-\partial_0 B^1 + \partial_0 B^2 - \partial_0 B^3) + \partial_1\left(B^1 - \frac{v}{c}E^2\right) + \partial_2\left(B^2 + \frac{v}{c}E^1\right) + \partial_3\left(B^3 + \frac{v}{c}E^2 + \frac{v}{c}E^1\right) = 0 \tag{740}$$

In vector form, this becomes (in SI units)

$$\boldsymbol{v}\cdot\left(\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times \boldsymbol{E}\right) = c^2 \nabla \cdot \boldsymbol{B} \tag{741}$$

which is a balance of the Faraday law of induction and the Gauss law for all v, including $v = c$. This result is true for all v, and therefore under all conditions, and is precisely equivalent to the result (730), the condensed form of Eqs. (95)–(100) of O(3) electrodynamics. Apparently, magnetic monopole was never observed and the Faraday law was never violated. This is consistent with O(3) electrodynamics as argued already.

Next, we consider the conservation law:

$$\partial_\mu H^\mu = 0 \tag{742}$$

where ($c = 1$ units)

$$H^\mu = H^{\mu\nu}\tilde{\varepsilon}_\nu = \left(D^1 + D^2 + D^3, \frac{v}{c}D^1 + H^3 - H^2, \frac{v}{c}D^2 - H^3 + H^1, \frac{v}{c}D^3 + H^2 - H^1\right)$$

Using Eq. (742)

$$\partial_0(D^1 + D^2 + D^3) + \partial_1\left(\frac{v}{c}D^1 + H^3 - H^2\right) + \partial_2\left(\frac{v}{c}D^2 - H^3 + H^1\right) + \partial_3\left(\frac{v}{c}D^3 + H^2 - H^1\right) = 0 \tag{744}$$

which in vector form is (in SI units):

$$\nabla \times \mathrm{H} - \frac{\partial \boldsymbol{D}}{\partial t} = v(\nabla \cdot \boldsymbol{D}) \tag{745}$$

and is a combination of the Ampère–Maxwell law:

$$\nabla \times \mathrm{H} - \frac{\partial \boldsymbol{D}}{\partial t} = \boldsymbol{J} = v(\nabla \cdot \boldsymbol{D}) = v\rho \tag{746}$$

and the Coulomb law:

$$\nabla \cdot \boldsymbol{D} = \rho \tag{747}$$

Equation (745) can be written as

$$\nabla \times \boldsymbol{H} = \left(\frac{\partial}{\partial t} + \mathbf{v}\nabla\cdot\right)\boldsymbol{D} = \frac{\partial \boldsymbol{D}}{\partial t} \tag{748}$$

where

$$\frac{\partial}{\partial t} \equiv \frac{\partial}{\partial t} + \boldsymbol{v}\nabla\cdot \tag{749}$$

is the convective derivative. The charge–current 4-vector in general is

$$J^{\mu} \equiv \left(\rho, \frac{\boldsymbol{J}}{c}\right) = \left(\rho, \frac{v}{c}\rho\right) \tag{750}$$

and in the vacuum is

$$J^{\mu}_{\text{vac}} \equiv \left(\rho_{\text{vac}}, \frac{1}{c}J_{\text{vac}}\right); \quad v = c \tag{751}$$

Therefore, charge density and current density in the vacuum and in matter take the same form, [see Eqs. (732) and (733)]. This is a general result of assuming an O(3) vacuum configuration as in Section I. Equations (736) are a form of Noether's theorem and charge/current enters the scene as the result of conservation and topology. Similarly, mass is curvature of the gravitational field.

In the vacuum

$$v = c; \quad \varepsilon^{\mu} = (1, 0, 0, 1) \tag{752}$$

and conservation of the PL pseudovector gives the continuity equation

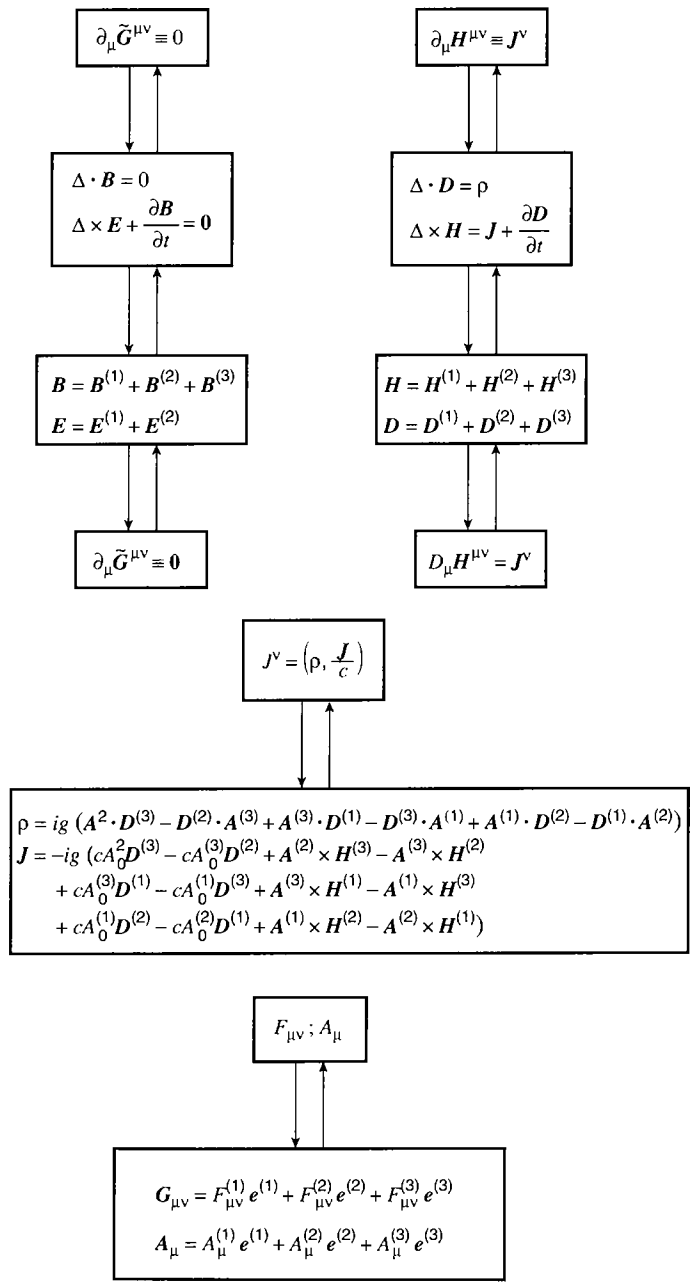
$$\nabla \times \boldsymbol{D}^{(3)} = \nabla \times \boldsymbol{P}^{(3)} = -\frac{\partial \boldsymbol{B}^{(3)}}{\partial t} - c\nabla \cdot \boldsymbol{B}^{(3)}\boldsymbol{e}^{(3)} = \boldsymbol{0} \tag{753}$$

which is a post-Noether-invariant. We have used the vacuum relation

$$\boldsymbol{D}^{(3)} = \varepsilon_0 \boldsymbol{E}^{(3)} + \boldsymbol{P}^{(3)} = \boldsymbol{P}^{(3)} \tag{754}$$

The vacuum polarization component $\boldsymbol{P}^{(3)}$ is equal to the vacuum displacement $\boldsymbol{D}^{(3)}$ and aligned along one axis, so its curl vanishes. If $\boldsymbol{B}^{(3)}$ were zero, then for a light-like $\varepsilon^{\mu}, \tilde{G}^{\mu}$ would be null and the electromagnetic field would vanish a reduction to absurdity proof of the existence of $\boldsymbol{B}^{(3)}$ if we adopt the Poincaré group. The adoption of the latter group leads to the post-Noether-invariant equations (736), which break out into the field equations of O(3) electrodynamics. Since U(1) is an O(3) symmetry with one null axis (the Z axis), U(1) is in a sense a sub symmetry of O(3), and this property leads to the fact that O(3) equations can be expressed in the form of U(1) equations without self-contradiction. The following diagram, which outlines the rules for connecting U(1) and O(3), may help the reader understand how this process occurs.

Rules for Connecting U(1) and O(3)



In the vacuum limit, we also obtain the following equation for the vacuum displacement $\boldsymbol{D}^{(3)}$ and vacuum polarization $\boldsymbol{P}^{(3)}$:

$$\frac{\partial \boldsymbol{D}^{(3)}}{\partial t} + c\frac{\partial \boldsymbol{D}^{(3)}}{\partial Z} = \boldsymbol{0} \tag{755}$$

Now use

$$\nabla \times \mathrm{H} = \left(\frac{\partial \boldsymbol{D}}{\partial t} + v(\nabla \cdot \boldsymbol{D})\right) \tag{756}$$

in the limit $v \rightarrow c$, and take the (3) component to find that:

$$\nabla \times \boldsymbol{H}^{(3)} = \boldsymbol{0} \tag{757}$$

which gives

$$\nabla \times (\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}) = \boldsymbol{0} \tag{758}$$

a result that is consistent with the definition of $\boldsymbol{B}^{(3)}$ in the vacuum, Eq. (38), because the curl of $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ is zero. The 3-component of Eq. (741) is simply

$$\nabla \cdot \boldsymbol{B}^{(3)} = \frac{\partial \boldsymbol{B}^{(3)}}{\partial t} = \boldsymbol{0} \tag{759}$$

because $\boldsymbol{E}^{(3)}$ is zero as proved already. The fact that $\boldsymbol{E}^{(3)}$ is zero is a direct consequence of the Jacobi identities (86) or (578). The same identities imply that there is no magnetic monopole or magnetic current in O(3) electrodynamics under any circumstances. The $\boldsymbol{B}^{(3)}$ component is topological in origin, and does not originate in a magnetic monopole *as a material particle*. These theoretical results are consistent with empirical data [11–20], which imply the presence of $\boldsymbol{B}^{(3)}$ and the absence of a magnetic monopole in nature.

In the Poincaré group, therefore, the fundamental spin of the electromagnetic field is represented ineluctably by the PL vector:

$$\tilde{B}^{\mu} = (B^{(3)}, 0, 0, B^{(3)}) \tag{760}$$

The integral of $\tilde{B}^{\mu}$ over a hypersurface in four-dimensions is always zero, a result of the ordinary Stokes theorem in four dimensions:

$$\oint \tilde{B}_{\mu}\, dx^{\mu} = \frac{1}{2}\int (\partial_{\mu}\tilde{B}_{\nu} - \partial_{\nu}\tilde{B}_{\mu})\partial\sigma^{\mu\nu} = 0 \tag{761}$$

The equivalent result in 3-space dimensions has been given by Evans and Jeffers [102]:

$$\oint \boldsymbol{B}^{(3)} \cdot d\boldsymbol{r} = 0 \tag{762}$$

and is simply a consequence of the fact that $\boldsymbol{B}^{(3)}$ is irrotational by definition. Therefore we obtain from Eq. (761) the results

$$\partial_{\mu}\tilde{B}_{\nu} = \partial_{\nu}\tilde{B}_{\mu} = 0 \tag{763}$$

and

$$\partial_{\mu}H_{\nu} = \partial_{\nu}H_{\mu} = J_{\mu\nu} = J_{\nu\mu} \tag{764}$$

These are alternative forms of the Lehnert or Panofsky–Phillips equations (612), which can be expanded out into the O(3) equations (95)–(106) using the rules in the above flowchart shown above [after text that followss Eq. (754)]. Conservation of helicity therefore requires the charge current tensor to be symmetric. Similarly, conservation of angular momentum requires the energy-momentum tensor to be symmetric in dynamics [6]. Therefore conservation of helicity generates the field equations and new conservation laws based on topology. Charge current itself is the result of topology as discussed by Ryder. [6, p. 93].

The Lie algebra of the PL vector within the Poincaré group is not well known and is given here for convenience. The PL vector is defined by

$$\tilde{W}_{\mu} = \tilde{J}_{\mu\nu}P^{\nu} \tag{765}$$

where

$$\tilde{J}_{\mu\nu} = \begin{bmatrix} 0 & J_1 & J_2 & J_3 \\ -J_1 & 0 & K_3 & -K_2 \\ -J_2 & -K_3 & 0 & K_1 \\ -J_3 & K_2 & -K_1 & 0 \end{bmatrix} \tag{766}$$

is a matrix of Poincaré group generators: the boost (K) and rotation (J) generators [6,11–20]. Here, P^{ν} is the generator of spacetime translation, which is missing from the Lorentz group. Therefore the PL vectors written out in full are

$$\begin{aligned} \tilde{W}_0 &= -J_1P^1 + J_2P^2 + J_3P^3 \\ \tilde{W}_1 &= -J_1P^0 + K_3P^2 - K_2P^3 \\ \tilde{W}_2 &= -J_2P^0 - K_3P^1 + K_1P^3 \\ \tilde{W}_3 &= -J_1P^0 + K_2P^1 - K_1P^2 \end{aligned} \tag{767}$$

These are linear operator relations implying the property

$$[P^0, \tilde{W}^\mu] = 0 \tag{768}$$

showing that the Hamiltonian operator $H = P^0$ [6,11–20] commutes with the complete vector $\tilde{W}^\mu$ under all conditions. Equation (768) implies

$$\partial_\mu \tilde{W}^\mu = 0 \tag{769}$$

as in Eq. (736). Relativistic helicity has no 4-divergence. From Eqs. (767), we obtain the closed Lie algebra

$$\begin{aligned}
[\tilde{W}^1, \tilde{W}^2] &= i(P^0\tilde{W}^3 + P^3\tilde{W}^0) \\
[\tilde{W}^2, \tilde{W}^3] &= i(P^0\tilde{W}^1 + P^1\tilde{W}^0) \\
[\tilde{W}^3, \tilde{W}^1] &= i(P^0\tilde{W}^2 - P^2\tilde{W}^0) \\
[\tilde{W}^0, \tilde{W}^1] &= i(P^3\tilde{W}^2 - P^2\tilde{W}^3) \\
[\tilde{W}^0, \tilde{W}^2] &= i(P^1\tilde{W}^3 - P^3\tilde{W}^1) \\
[\tilde{W}^0, \tilde{W}^3] &= i(P^2\tilde{W}^1 + P^1\tilde{W}^2)
\end{aligned} \tag{770}$$

and Jacobi identities such as

$$[\tilde{W}^1, [\tilde{W}^2, \tilde{W}^3]] + [\tilde{W}^2, [\tilde{W}^3, \tilde{W}^1]] + [\tilde{W}^3, [\tilde{W}^1, \tilde{W}^2]] = 0 \tag{771}$$

checking that $\tilde{W}^\mu$ is a valid generator of the Poincaré group. The Casimir invariants $P_\mu P^\mu$ and $\tilde{W}_\mu \tilde{W}^\mu$ are the two fundamental invariants of the Poincaré group.

In electromagnetic theory, we replace $\tilde{W}^\mu$ by $\tilde{G}^\mu$ the relativistic helicity of the field. Therefore, Eq. (770) forms a fundamental Lie algebra of classical electrodynamics within the Poincaré group. From first principles of the Lie algebra of the Poincaré group, the field $\boldsymbol{B}^{(3)}$ is nonzero.

If a light beam is considered propagating at c in Z, we obtain from Eqs. (770) the Lie algebra of the E(2) Euclidean group [6,11–20], which is a mathematical group with no physical meaning:

$$[\tilde{W}^1, \tilde{W}^2] = 0 \tag{772a}$$

$$[\tilde{W}^2, \tilde{W}^3] = iP^0\tilde{W}^1 \tag{772b}$$

$$[\tilde{W}^3, \tilde{W}^1] = iP^0\tilde{W}^2 \tag{772c}$$

compared with the O(3) Lie algebra

$$[\tilde{W}^1, \tilde{W}^2] = iP^0\tilde{W}^3 \tag{772d}$$

$$[\tilde{W}^2, \tilde{W}^3] = iP^0\tilde{W}^1 \tag{772e}$$

$$[\tilde{W}^3, \tilde{W}^1] = iP^0\tilde{W}^2 \tag{772f}$$

and similarly for $\tilde{G}^\mu$. The E(2) group is the Wigner little group for a particle whose mass is identically zero, and so such a particle does not exist in nature. This proves that the photon and neutrino both have identically nonzero mass. The Wigner little group for a particle with mass is the physical O(3) group. In terms of field components, Eq. (772b) gives (in $c = 1$ units)

$$[B^2 - E^1, B^3] = iB^{(0)}B^1 \tag{773}$$

which is satisfied by

$$\begin{aligned}
[B^2, B^3] &= iB^{(0)}B^1 \\
[B^3, E^1] &= iB^{(0)}E^2
\end{aligned} \tag{774}$$

The first of these equations is an equation of the B cyclic theorem, which therefore emerges from the symmetry of the Poincaré group in free space. Similarly, Eq. (772c) gives:

$$[B^3, B^1 + E^2] = iB^{(0)}(B^2 - E^1) \tag{775}$$

which is satisfied by

$$\begin{aligned}
[B^3, B^1] &= iB^{(0)}B^2 \\
[B^3, E^2] &= -iB^{(0)}E^1
\end{aligned} \tag{776}$$

The first of this pair of equations is another of the B cyclic equations. Finally, Eq. (772a) gives

$$[B^1 + E^2, B^2 - E^1] = 0 \tag{777}$$

which is satisfied by

$$\begin{aligned}
[B^1, B^2] &= [E^1, E^2] \\
[E^2, B^2] &\equiv [E^2, B^2]
\end{aligned} \tag{778}$$

where the first of this pair give the third and final equation of the B cyclic theorem.

The structure of the O(3) equations in condensed form [i.e., Eqs. (612)] emerges from the symmetry of the Poincaré group. Consider, for example, the three equations:

$$\begin{aligned}[P_2, J_3] &= iP_1 \\ [P_3, J_2] &= -iP_1 \\ [P_0, K_1] &= iP_1\end{aligned} \tag{779}$$

By definition, the generator of space-time translation is

$$P \equiv i\partial_\mu \tag{780}$$

so Eq. (779) becomes

$$([\partial_2, J_3] - [\partial_3, J_2] - [\partial_0, K_1])\psi = P_1\psi \tag{781}$$

where ψ is an eigenfunction. Equation (781) can be written as

$$(\partial_2 J_3 - \partial_3 J_2 - \partial_0 K_1 - (J_3\partial_2 - J_2\partial_3 - K_1\partial_0))\psi = 0 \tag{782}$$

which is a relation between operators on ψ. Now use

$$\begin{aligned}J_3\psi &= j_3\psi \\ J_2\psi &= j_2\psi \\ J_1\psi &= j_1\psi\end{aligned} \tag{783}$$

where lowercase letters denote eigenvalues. We have

$$\begin{aligned}\partial_2(j_3\psi) &= (\partial_2 j_3)\psi + j_3(\partial_2\psi) \\ \partial_3(j_2\psi) &= (\partial_3 j_2)\psi + j_2(\partial_3\psi) \\ \partial_0(k_1\psi) &= (\partial_0 k_1)\psi + k_1(\partial_0\psi)\end{aligned} \tag{784}$$

Assume that

$$J_3(\partial_2\psi) + J_2(\partial_3\psi) + K_1(\partial_0\psi) = j_3(\partial_2\psi) + j_2(\partial_3\psi) + k_1(\partial_0\psi) \tag{785}$$

an equation that is compatible with:

$$(\partial_2 + \partial_3 + \partial_0)\psi = \text{constant } \psi \tag{786}$$

Equations (781)–(786) give the eigenvalue relation

$$\partial_2 j_3 - \partial_3 j_2 - \partial_0 k_1 = P_1 \tag{787}$$

which is one component of

$$\nabla \times \boldsymbol{J} - \frac{1}{c}\frac{\partial \boldsymbol{K}}{\partial t} = \boldsymbol{P} \tag{788}$$

If we write

$$\psi \equiv e^{i\phi}\psi_0 \tag{789}$$

where ϕ is a phase factor, then

$$J_3\psi = J_3(e^{i\phi}\psi_0) = j_3^{(0)}e^{i\phi}\psi_0 \equiv j_3\psi \tag{790}$$

and so on. Therefore the eigenvalues appearing in Eq. (788) are phase-dependent in general. It is clear that the structure of Eq. (788) is the same as one of Eqs. (612). The complete set of operator relations leading to this equation is

$$\begin{aligned}([\partial_1, J_2] - [\partial_2, J_1] - [\partial_0, K_3])\psi &= P_3\psi \\ ([\partial_2, J_3] - [\partial_3, J_2] - [\partial_0, K_1])\psi &= P_1\psi \\ ([\partial_3, J_1] - [\partial_1, J_3] - [\partial_0, K_2])\psi &= P_2\psi\end{aligned} \tag{791}$$

Similarly, the Lie algebra

$$([\partial_2, K_3] - [\partial_3, K_2] + [\partial_0, J_3])\psi = 0 \tag{792}$$

and so on leads to the eigenvalue relation

$$\nabla \times \boldsymbol{k} + \frac{1}{c}\frac{\partial \boldsymbol{j}}{\partial t} = \boldsymbol{0} \tag{793}$$

as another of Eqs. (612).

The Lie algebra

$$([\partial_1, J_1] + [\partial_2, J_2] + [\partial_3, J_3])\psi = 0 \tag{794}$$

gives

$$((\partial_1 J_1 - J_1\partial_1) + (\partial_2 J_2 - J_2\partial_2) + (\partial_3 J_3 - J_3\partial_3))\psi = 0 \tag{795}$$

Using

$$\begin{aligned}J_1\psi &= j_1\psi \\ \partial_1(j_1\psi) &= j_1(\partial_1\psi) + (\partial_1 j_1)\psi\end{aligned} \tag{796}$$

and assuming

$$J_1(\partial_1\psi) + J_2(\partial_2\psi) + J_3(\partial_3\psi) = j_1(\partial_1\psi) + j_2(\partial_2\psi) + j_3(\partial_3\psi) \tag{797}$$

leads to

$$\begin{aligned} \partial_1 j_1 + \partial_2 j_2 + \partial_3 j_3 &= 0 \\ \nabla \cdot \boldsymbol{j} &= 0 \end{aligned} \tag{798}$$

Therefore the complete set of equations (612) emerges in the form

$$\begin{aligned} \nabla \cdot \boldsymbol{k} &= 3p_0 \\ \nabla \cdot \boldsymbol{j} &= 0 \\ \nabla \times \boldsymbol{k} + \frac{1}{c}\frac{\partial \boldsymbol{j}}{\partial t} &= \boldsymbol{0} \\ \nabla \times \boldsymbol{j} - \frac{1}{c}\frac{\partial \boldsymbol{k}}{\partial t} &= \boldsymbol{p} \end{aligned} \tag{799}$$

simply by considering the symmetry of the Poincaré group. The vacuum charge–current is therefore intrinsic to the structure of the Poincaré group, but not of the Lorentz group, in which $\boldsymbol{p}$ is undefined. Structure (799) exists under all conditions because the Poincaré group applies under all conditions. Therefore O(3) electrodynamics emerges self-consistently from the symmetry of the Poincaré group, without a magnetic monopole or magnetic current *as material entities*, but with vacuum charge and current. This is a powerful result of symmetry.

Consideration of the symmetry of the Poincaré group also shows that the B cyclic theorem is independent of Lorentz boosts in any direction, and also reveals the physical meaning of the E(2) little group of Wigner. This group is unphysical for a photon without mass, but is physical for a photon with mass. This proves that Poincaré symmetry leads to a photon with identically nonzero mass. The proof is as follows. Consider in the particle interpretation the PL vector

$$W^\mu = -\frac{1}{2}\varepsilon^{\lambda\mu\nu\rho} P_\mu J_{\nu\rho} \tag{800}$$

Barut [102] shows that this PL vector obeys the cyclic conditions:

$$[W^\lambda, W^\mu] = -i\varepsilon^{\mu\nu\sigma\rho} P_\sigma W_\rho \tag{801}$$

For a particle (including the photon) with mass, the spacetime translation operator P^μ in the rest frame is

$$P^\mu = (P^0, 0, 0, 0) \tag{802}$$

and in the light-like condition

$$P^\mu = (P^0, 0, 0, P^0) \tag{803}$$

In the rest frame, Eq. (801) becomes [15]

$$\begin{aligned} [J_1, J_2] &= iJ_3 \\ [J_2, J_3] &= iJ_1 \\ [J_3, J_1] &= iJ_2 \end{aligned} \tag{804}$$

which is the Lie algebra of the rotation generators of the Lorentz group [6]. In the light-like condition, Eq. (801) becomes

$$\begin{aligned} [J_X + K_Y, J_Y - K_X] &= i(J_Z - J_Z) \\ [J_Y - K_X, J_Z] &= i(K_Y + J_X) \\ [K_Y + J_X, J_Z] &= i(K_X - J_Y) \end{aligned} \tag{805}$$

which has the symmetry of the E(2) group. Equation (805) can be written as

$$\begin{aligned} [J_X, J_Y] + [K_X, K_Y] &= iJ_Z - iJ_Z \\ [J_Y, J_Z] + [J_Z, K_X] &= iJ_X + iK_Y \\ -[J_Z, J_X] + [K_Y, J_Z] &= -iJ_Y + iK_X \end{aligned} \tag{806}$$

If we assume that the Lie algebra (804) is independent of Lorentz boosts in any direction, we obtain the Lie algebra:

$$\begin{aligned} [K_X, K_Y] &= -iJ_Z \\ [J_Z, K_X] &= iK_Y \\ [K_Y, J_Z] &= iK_X \end{aligned} \tag{807}$$

This is a Lie algebra of the Poincaré group [15] and of the Lorentz group [6], and is therefore self-consistently independent of spacetime translation. Therefore the meaning of the E(2) little group of Wigner is that it is a combination of the Lie algebra (804), which is independent of Lorentz boosts and spacetime translations; and of the Lie algebra (807), which is independent of spacetime translations. Note that the relation

$$[K_X, K_Y] = -iJ_Z \tag{808}$$

is the Thomas precession [6].

In the field interpretation [11–20], the Lie algebra (804) becomes [15]

$$[\hat{B}^{(1)}, \hat{B}^{(2)}] = -iB^{(0)}\hat{B}^{(3)} \qquad (809)$$

$$\ldots$$

in the basis ((1),(2),(3)), which in vector notation is the B cyclic theorem:

$$\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} = iB^{(0)}\boldsymbol{B}^{(3)*} \qquad (810)$$

$$\ldots$$

The latter is therefore independent of Lorentz boosts of any kind, and independent of spacetime translations of any kind. As demonstrated previously in this chapter, this result can be arrived at independently and self-consistently by considering the following definition:

$$\boldsymbol{B}^{(3)*} \equiv -ig\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} \qquad (811)$$

The B cyclic theorem is therefore a Lie algebra independent of boosts and spacetime translations and is the same in the rest mass and light-like conditions for the photon. This result leads to the Lie algebra (807) for a particle with mass. The E(2) group becomes physical if the photon with mass is boosted to the speed of light, or, more precisely, infinitesimally close to the speed of light.

This symmetry analysis of the generators of the Poincaré group also shows in the field interpretation that the E(2) group contains the $\boldsymbol{B}^{(3)}$ field (corresponding in the particle interpretation to the J_Z generator) but does not contain the $\boldsymbol{E}^{(3)}$ field, corresponding in the particle interpretation to the K_Z generator. The Poincaré group also gives the structure of the O(3) equations of motion, Eqs. (799). In the field interpretation, the P^μ generator of the particle interpretation corresponds to charge–current. Therefore charge is analogous with energy and current with linear momentum. The magnetic field is analogous with the rotation generator, and the electric field is analogous with the boost generator. The Poincaré group Lie algebra produces the O(3) equations (799), and not the Maxwell–Heaviside equations. Our analysis throughout this chapter is therefore shown to be entirely self-consistent on the O(3) level, while there are many self-inconsistencies on the U(1) level. The normalized helicity of the photon with mass is -1, 0, 1, as for any boson with mass. In the rest frame, there is no helicity, because there is no forward momentum for a particle in its own rest frame. In the light-like condition (i.e., infinitesimally near the light-like condition), the three helicities are the space parts of the PL vector in that state:

$$\begin{aligned} W_1 &= J_1P_0 + K_2P_3 \\ W_2 &= J_2P_0 - K_1P_3 \\ W_3 &= J_3P_0 \end{aligned} \qquad (812)$$

The time-like part of the PL vector is

$$W_0 = -J_3P_3 \qquad (813)$$

It can be seen that the PL vector is not proportional to P^μ in the light-like condition, thus removing another paradox [6] of the concept of massless photon.

In the U(1) gauge the vacuum field equations are:

$$\begin{aligned} (\partial^\nu + igA^\nu)\tilde{F}_{\mu\nu} &= 0 \\ (\partial^\nu + igA^\nu)F_{\mu\nu} &= 0 \end{aligned} \qquad (814)$$

and become the Maxwell equations if and only if

$$\begin{aligned} A^\nu\tilde{F}_{\mu\nu} &= 0 \\ A^\nu F_{\mu\nu} &= 0' \end{aligned} \qquad (815)$$

which in vector notation correspond to

$$\begin{aligned} \boldsymbol{A}\cdot\boldsymbol{B} &= 0 \\ \boldsymbol{A}\times\boldsymbol{E} &= \boldsymbol{0} \\ \boldsymbol{A}\cdot\boldsymbol{E} &= 0 \\ \boldsymbol{A}\times\boldsymbol{B} &= \boldsymbol{0} \end{aligned} \qquad (816)$$

Therefore $\boldsymbol{A}\cdot\boldsymbol{B} = 0$ in the U(1) gauge in the vacuum. Unfortunately, the helicity in the U(1) gauge is defined by [103]

$$h \equiv \int \boldsymbol{A}\cdot\boldsymbol{B}\, dV \qquad (817)$$

which is the linking number of field lines. This is zero because $\boldsymbol{A}\cdot\boldsymbol{B} = 0$, and helicity cannot be defined in the vacuum in the U(1) gauge. It is necessary to go to the O(3) level and to define helicity by

$$h_{O(3)} \equiv \int \boldsymbol{A}^{(1)}\cdot\boldsymbol{B}^{(2)}\, dV \qquad (818)$$

It is only on this level that the link between helicity and topological quantization [103] can be understood properly. The O(3) group, like the U(1) group, is multiply connected. The group space of U(1) is a circle [6, p. 105]. As explained earlier in this review, this is not simply connected because a path that goes twice

around a circle cannot be continuously deformed while staying on a circle into one that goes around once. The group space of SU(2) is S^3 [6, p. 411]. Every closed curve S^1 on S^3 may be shrunk to a point. The group O(3) is not simply connected but doubly connected, [6, p. 412]. Therefore the Aharonov–Bohm effect is possible only in O(3), as described in early sections of this review. We have the relation SO(3)=SU(2)/Z2. There are only two types of closed path S^1 in the group space of O(3): homotopic to a point and line [6]; therefore it is doubly connected. The topological theory of classical electromagnetism proposed by Ranada [103] thus can be extended systematically to the O(3) level. On the U(1) level used by Ranada, the electromagnetic knot is locally equivalent to the Maxwell–Heaviside equations. The electromagnetic knot is a field defined by the condition that their force lines are closed curves, and any pair of magnetic or electric lines is a link [103]. The linking lines are two integers that are interpreted as the Hopf indices of two applications from the sphere S^3 to the sphere S^2 at any instant. In the vacuum, the knots are such that $n_m = n_e$. Since $\boldsymbol{A} \cdot \boldsymbol{B}$ is identically zero in the U(1) gauge (Maxwell–Heaviside theory), this elegant theory needs to be upgraded to the O(3) level.

XIV. GAUGE FREEDOM AND THE LAGRANGIAN

We have just seen that the symmetry of the Poincaré group leads to vacuum charge and current as proposed by Panofsky and Phillips [86], Lehnert and Roy [10], and others. We must therefore seek a Lagrangian that gives the structure of the O(3) equations, a structure that, in condensed form, is identical with the Panofsky–Phillips and Lehnert–Roy equations. The Lagrangian leading to the Maxwell–Heaviside equations is deficient. It must also be explained why photon mass can enter gauge theory without making ten Lagrangian gauge not invariant. The problem with the Proca equation is that it removes gauge freedom, but at the expense of rendering the Lagrangian gauge noninvariant [6]. The original Proca equation is not therefore an entirely satisfactory approach to photon mass. The origin of photon mass (m_0) in O(3) electrodynamics is therefore topological, because the origin of charge–current is topological. The topology is expressed through gauge theory and group theory as discussed in Section I. On the U(1) level in the received view, a Lagrangian that does not contain a photon mass term is needed, Euler Lagrange equations have to be constructed, and constraints are needed to reduce the number of field variables so that there are no undetermined multipliers.

This program is not consistent with the Proca equation on the U(1) level. If the Proca equation

$$\partial_\mu H^{\mu\nu} - J^\nu = -\varepsilon_0 \frac{m_0^2 c^2}{\hbar^2} A^\nu \tag{819}$$

is used by ansatz, then it follows, by taking its divergence [6,15], that

$$m_0^2 \partial_\nu A^\nu = 0 \tag{820}$$

and if m_0 is not zero, the Lorenz condition is always obtained

$$\partial_\nu A^\nu = 0 \tag{821}$$

and the d'Alembert equation becomes

$$-\Box A_\mu = \frac{m_0^2 c^4}{\hbar^2} A_\mu \tag{822}$$

A condition is imposed on one of the four components of A_μ so that there are only three free components. However, the Lagrangian leading to the Proca equation is not gauge invariant due to the presence of a mass term [15]

$$\mathscr{L}_{m_0} = \frac{m_0^2}{2} A_\mu A^\mu \tag{823}$$

and the Proca equation always leads to the Lorenz condition, which is arbitrary and self-inconsistent. These disadvantages offset the advantages of the Proca equation; for example, it allows a three dimensional particle interpretation of the photon and it can be quantized without difficulty.

In U(1) gauge theory, the Lagrangian in general [6] contains the mass term (823), but in order to obtain the inhomogeneous Maxwell equations, this is discarded. This procedure is outlined, for example, on pp. 89ff. of Ref. 6. The U(1) Lagrangian in general is, in reduced units

$$\mathscr{L} = D_\mu \phi D_\mu \phi^* - m^2 \phi^* \phi - \frac{1}{4} H^{\mu\nu} H_{\mu\nu} \tag{824}$$

where ϕ is a scalar complex field and $F_{\mu\nu}$ is the electromagnetic field tensor. The Euler–Lagrange equation in the U(1) gauge is

$$\frac{\partial \mathscr{L}}{\partial A_\mu} - \partial_\nu \left(\frac{\partial \mathscr{L}}{\partial (\partial_\nu A_\mu)} \right) = 0 \tag{825}$$

and Eqs. (824) and (825) give

$$\partial_\mu H^{\mu\nu} = J^\nu = -ig(\phi^* D^\nu \phi - \phi D^\nu \phi^*) \tag{826}$$

The photon mass term in the Lagrangian

$$\mathscr{L} = -\frac{1}{4}H_{\mu\nu}H^{\mu\nu} + \frac{1}{2}m_0^2 A_\mu A^\mu \tag{827}$$

leading to the Proca equation in the received view [6] is not invariant under the gauge transformation

$$A^\mu \rightarrow A^\mu + \partial^\mu \chi \tag{828}$$

and is discarded in order to obtain the inhomogeneous Maxwell–Heaviside equation (826). The constant g appears in this theory as a coupling constant; it couples the ϕ and A^μ electromagnetic fields.

Therefore the fact that $\partial^\mu\chi$ is arbitrary in U(1) theory compels that theory to assert that photon mass is zero. This is an unphysical result based on the Lorentz group. When we come to consider the Poincaré group, as in section XIII, we find that the Wigner little group for a particle with identically zero mass is E(2), and this is unphysical. Since $\partial^\mu\chi$ in the U(1) gauge transform is entirely arbitrary, it is also unphysical. On the U(1) level, the Euler–Lagrange equation (825) seems to contain four unknowns, the four components of A^μ, and the field tensor $H^{\mu\nu}$ seems to contain six unknowns. This situation is simply the result of the term $H^{\mu\nu}$ in the initial Lagrangian (824) from which Eq. (826) is obtained. However, the fundamental field tensor is defined by the 4-curl:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{829}$$

and the six components of the field are interrelated automatically by a constraint. The field tensor therefore contains only the four unknowns of A^μ by definition, and this definition is the constraint. The physical nature of the potential has been reviewed by Barrett [3,4].

It is well known that the Proca equation [6], Eq. (809), for a massive photon is not gauge-invariant because the Lagrangian (827) corresponding to it is not gauge-invariant. In SI units, this Lagrangian is

$$\mathscr{L} = -\frac{\varepsilon_0}{4}V_R H_{\mu\nu}H^{\mu\nu} + \frac{\varepsilon_0 V_R}{2}\frac{m_0^2 c^4}{\hbar^2}A_\mu A^\mu \tag{830}$$

where V_R is the radiation volume, ϵ_0 is the permittivity in vacuo, $H^{\mu\nu}$ is the field tensor, and m_0 is the mass of the photon. It is customary to adopt reduced units, so the Lagrangian becomes [6] Eq. (827), with:

$$\varepsilon_0 V = 1; \quad \frac{c^4}{\hbar^2} = 1 \tag{831}$$

The term $m_0^2 A_\mu A^\mu$ is not gauge-invariant under a local U(1) transform of A^μ. This problem can be circumvented by adopting the notion of the vacuum as the ground state of a scalar field ϕ:

$$\frac{\partial V}{\partial \phi} = 0 \tag{832}$$

where V is potential energy. This definition of the vacuum depends on the spontaneous symmetry breaking [6] of the Lagrangian:

$$\begin{aligned}\mathscr{L} &= (\partial_\mu\phi)(\partial^\mu\phi^*) - m^2\phi^*\phi - \lambda(\phi^*\phi)^2 \\ &= (\partial_\mu\phi)(\partial^\mu\phi^*) - V(\phi,\phi^*)\end{aligned} \tag{833}$$

where λ is the self-interaction parameter, and assuming that $\mathscr{L}$ is invariant under the local transformation

$$\phi \rightarrow e^{i\Lambda(x^\mu)}\phi \tag{834}$$

the vacuum is the ground state

$$\frac{\partial V}{\partial \phi} = 0 = m^2\phi^* + 2\lambda\phi^*(\phi^*\phi) \tag{835}$$

and the parameter m is allowed to become negative. This is the basis of the Higgs mechanism of introducing mass. If $m < 0$, there is a minimum at

$$a^2 \equiv |\phi|^2 = -\frac{m^2}{2\lambda}; \quad |\phi| = a \tag{836}$$

from the equation defining the vacuum [Eq. (835)]. In reduced units, spontaneous symmetry breaking of this type leads to the Lagrangian (824) and to the inhomogeneous field equation (826).

The charge–current density

$$J^\mu = -ig(\phi^* D^\mu\phi - \phi D^\mu\phi^*) \tag{837}$$

is a vacuum current because g exists in the vacuum and Eq. (837) is obtained from the definition of the vacuum, Eq. (835), as the ground state of the scalar field ϕ. The fundamental field $F_{\mu\nu}$ is completely defined in terms of the commutator of covariant derivatives:

$$F_{\mu\nu} \equiv \frac{i}{g}[D_\mu, D_\nu] \tag{838}$$

The Lagrangian (824) can be rewritten using Eq. (836) as [6]

$$\mathscr{L} = -\frac{1}{4}H_{\mu\nu}H^{\mu\nu} + \frac{1}{2}g^2a^2A_\mu A^\mu + \frac{1}{2}(\partial_\mu\phi_1)^2 + \frac{1}{2}(\partial_\mu\phi_2)^2 - 2\lambda a^2\phi_1^2 + \sqrt{2}gaA^\mu\partial_\mu\phi_2 + \cdots \tag{839}$$

The two Lagrangians (824) and (839) contain the same physical information, but in the form (839), the mass of the photon appears as the term $\frac{1}{2}g^2a^2A_\mu A^\mu$ in these reduced units. In SI units, the mass of the photon is

$$m_0\frac{c}{\hbar} = ga = g|\phi| \tag{840}$$

and using

$$g = \frac{\kappa}{|\phi|} \tag{841}$$

we recover the de Broglie guidance theorem [15]:

$$m_0c^2 = \hbar\omega \tag{842}$$

from the Higgs mechanism. The Proca equation is recovered in gauge-invariant form from the Lagrangian (839) if it is assumed that ϕ_2 vanishes as the result of spontaneous symmetry breaking. Using the Euler–Lagrange equation

$$\frac{\partial\mathscr{L}}{\partial A_\mu} - \partial_\nu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\nu A_\mu)}\right) = 0 \tag{843}$$

the gauge-invariant Proca equation is as follows, in SI units:

$$\partial_\mu H^{\mu\nu} = J^\nu = -\varepsilon_0 g^2|\phi|^2\frac{c^2}{\hbar^2}A^\nu \tag{844}$$

The Lagrangian (824), which is the same as the Lagrangian (839), gives the inhomogeneous equation (826) using the same Euler–Lagrange equation (843). Therefore the photon mass can be identified with the vacuum charge–current density as follows (in SI units):

$$J^\nu = -\varepsilon_0 g^2|\phi|^2\frac{c^2}{\hbar^2}A^\nu = -ig\varepsilon_0\frac{c^2}{\hbar^2}(\phi^*D^\mu\phi - \phi D^\mu\phi^*) \tag{845}$$

This result, in turn, shows that the O(3) equations in their condensed form, Eq. (612), indicate the existence of photon mass. This is precisely the result

obtained by Lehnert and Roy [10]. Canonical quantization of the gauge-invariant Proca equation proceeds without any problem to give the photon as a boson with helicities -1, 0, 1. This procedure is described in Ref. 6. In summary, it has been shown that the vacuum charge–current density and photon mass are the result of the Higgs mechanism.

Photon mass is shown to be self-consistent with O(3) electrodynamics by considering the O(3) Lagrangian [6] in reduced units:

$$\mathscr{L} = \frac{1}{2}D_\mu\phi_i)(D^\mu\phi_i) - \frac{m^2}{2}\phi_i\phi_i - \lambda(\phi_i\phi_i)^2 - \frac{1}{4}H^i_{\mu\nu}H^{i\mu\nu} \tag{846}$$

where i is the internal gauge index and D^μ is the covariant derivative of O(3) electrodynamics. The latter gives the usual results

$$D_\mu\phi_i = \partial_\mu\phi_i + g\varepsilon_{ijk}A^j_\mu\phi_k$$
$$G^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g\varepsilon^{ijk}A^j_\mu A^k_\nu \tag{847}$$

and the potential V has a minimum [6] at

$$|\phi_0| = a = \left(-\frac{m^2}{4\lambda}\right)^{1/2} \tag{848}$$

where

$$\phi_0 = a\boldsymbol{e}_3 \equiv a\boldsymbol{e}^{(3)} \tag{849}$$

The O(3) Lagrangian becomes

$$\mathscr{L} = \frac{1}{2}((\partial_\mu\phi_1)^2 + (\partial_\mu\phi_2)^2 + (\partial_\mu(\phi_3 - a))^2) + ag((\partial_\mu\phi_1)A^\mu_2 - (\partial_\mu\phi_2)A^\mu_1) + \frac{a^2g^2}{2}((A^1_\mu)^2 + (A^2_\mu)^2) - \frac{1}{4}H^i_{\mu\nu}H^{i\mu\nu} - 4a^2\lambda\chi^2 \tag{850}$$

and contains the photon mass term

$$\mathscr{L}_m = \frac{a^2g^2}{2}((A^1_\mu)^2 + (A^2_\mu)^2) \tag{851}$$

in gauge-invariant form. The photon mass in O(3) electrodynamics is therefore given again by Eq. (840). If it is assumed that

$$g = \frac{\kappa}{|\phi_0|} \tag{852}$$

the de Broglie guidance theorem (842) is again recovered self-consistently.

The Lagrangian (850) shows that O(3) electrodynamics is consistent with the Proca equation. The inhomogeneous field equation (32) of O(3) electrodynamics is a form of the Proca equation where the photon mass is identified with a vacuum charge-current density. To see this, rewrite the Lagrangian (850) in vector form as follows:

$$\mathscr{L} = D_\mu \boldsymbol{\phi} \cdot D^\mu \boldsymbol{\phi} - m^2 \boldsymbol{\phi} \cdot \boldsymbol{\phi} - \frac{1}{4} \boldsymbol{H}_{\mu\nu} \cdot \boldsymbol{H}^{\mu\nu} \tag{853}$$

The inhomogeneous O(3) field equation (32) is obtained through the Euler–Lagrange equation:

$$\frac{\partial \mathscr{L}}{\partial (A^i_\mu)} = \partial_\nu \left(\frac{\partial \mathscr{L}}{\partial (\partial_\nu A^i_\mu)} \right) \tag{854}$$

which gives Eq. (32) with the current term (in SI units):

$$D_\mu \boldsymbol{H}^{\mu\nu} = \boldsymbol{J}^\nu = \frac{\varepsilon_0 c^2}{\hbar^2} g (D^\nu \boldsymbol{\phi}) \times \boldsymbol{\phi} \tag{855}$$

In analogy with Eq. (845), the photon mass is defined in SI units by

$$D_\mu \boldsymbol{H}^{\mu\nu} = \boldsymbol{J}^\nu = -\varepsilon_0 g^2 |\phi_0|^2 \frac{c^2}{\hbar^2} \boldsymbol{A}^\nu \tag{856}$$

The individual terms of the charge current density ($\boldsymbol{J}^\nu$) in the vacuum are Noether currents of the type (101)–(106) and we have the following identifications under all conditions:

$$\begin{aligned}
\rho^{(1)} &= ig(\boldsymbol{A}^{(2)} \cdot \boldsymbol{D}^{(3)} - \boldsymbol{D}^{(2)} \cdot \boldsymbol{A}^{(3)}) \\
\rho^{(2)} &= ig(\boldsymbol{A}^{(3)} \cdot \boldsymbol{D}^{(1)} - \boldsymbol{D}^{(3)} \cdot \boldsymbol{A}^{(1)}) \\
\rho^{(3)} &= ig(\boldsymbol{A}^{(1)} \cdot \boldsymbol{D}^{(2)} - \boldsymbol{D}^{(1)} \cdot \boldsymbol{A}^{(2)}) \\
\boldsymbol{J}^{(1)*} &= -ig(cA_0^{(2)} \boldsymbol{D}^{(3)} - cA_0^{(3)} \boldsymbol{D}^{(2)} + \boldsymbol{A}^{(2)} \times \boldsymbol{H}^{(3)} - \boldsymbol{A}^{(3)} \times \boldsymbol{H}^{(2)}) \\
\boldsymbol{J}^{(2)*} &= -ig(cA_0^{(3)} \boldsymbol{D}^{(1)} - cA_0^{(1)} \boldsymbol{D}^{(3)} + \boldsymbol{A}^{(3)} \times \boldsymbol{H}^{(1)} - \boldsymbol{A}^{(1)} \times \boldsymbol{H}^{(3)}) \\
\boldsymbol{J}^{(3)*} &= -ig(cA_0^{(1)} \boldsymbol{D}^{(2)} - cA_0^{(2)} \boldsymbol{D}^{(1)} + \boldsymbol{A}^{(1)} \times \boldsymbol{H}^{(2)} - \boldsymbol{A}^{(2)} \times \boldsymbol{H}^{(1)})
\end{aligned} \tag{857}$$

The photon with mass has three degrees of freedom, so the O(3) procedure is again self-consistent. The key advantage of the O(3) procedure is that it produces a Proca equation that does not indicate the necessity for the Lorenz condition.

The U(1) Proca equation (819) implies that the Lorenz condition always holds, because Eq. (819) leads to

$$\partial_\nu A^\nu = 0 \tag{858}$$

The O(3) Proca equation (856) does not have this artificial constraint on the potentials, which are regarded as physical in this chapter. This overall conclusion is self-consistent with the inference by Barrett [104] that the Aharonov–Bohm effect is self-consistent only in O(3) electrodynamics, where the potentials are, accordingly, physical.

Having derived the Proca equation in gauge-invariant form on the U(1) and O(3) levels, canonical quantization can be attempted. Defining the photon mass in reduced units as

$$m_0 \equiv g|\phi|, \quad (c = 1, \hbar = 1) \tag{859}$$

canonical quantization of the Proca equation is similar to that of the Klein–Gordon equation discussed in section X. The difference is that the Klein–Gordon equation produces a massless photon. With the definition $m_0 = g|\phi|$, the canonical momentum from the gauge-invariant Lagrangian (827) is

$$\pi^\mu = \frac{\partial \mathscr{L}}{\partial \dot{A}_\mu} = \partial^\mu A^0 - \dot{A}^\mu \tag{860}$$

from which [6] it follows that

$$\pi^i = -\dot{A}^i; \quad \pi^0 = 0 \tag{861}$$

So on this U(1) level, the scalar photon represented by A^0 is set to zero and the Lorenz condition always applies, meaning no gauge freedom. This is self-inconsistent because the original Lagrangian from which Eq. (827) is obtained is a U(1) Lagrangian with gauge freedom. If so, the Lorenz condition cannot always apply. Leaving these problems aside for the sake of argument, the commutation relations fundamental to the method of canonical quantization become [6]

$$[A^i(\boldsymbol{x}, t), \pi_j(\boldsymbol{x}', t)] = i\delta^i_j \delta^3(\boldsymbol{x} - \boldsymbol{x}') \tag{862}$$

$$[\dot{A}_i(\boldsymbol{x}, t), A_j(\boldsymbol{x}', t)] = ig_{ij} \delta^3(\boldsymbol{x} - \boldsymbol{x}') \tag{863}$$

and the field can be expanded in the Fourier series

$$A_\mu(k) = \int \frac{\partial^3 k}{(2\pi)^3 2k_0} \sum_{\lambda=1}^{3} \varepsilon_\mu^{(\lambda)}(k) (a^{(\lambda)}(k) e^{-ikx} + a^{(\lambda)+} e^{ikx}) \tag{864}$$

implying

$$[a^{(\lambda)}(k), a^{(\lambda')+}(k')] = \delta_{\lambda\lambda'}\partial k^0(2\pi)^3\delta^3(\boldsymbol{k}-\boldsymbol{k}') \tag{865}$$

and a Hamiltonian:

$$H = \int \frac{\partial^3 k}{(2\pi)^3 2k_0} k_0 \sum_{\lambda=1}^{3} a^{(\lambda)+}(k)a^{(\lambda)}(k) \tag{866}$$

This gives a straightforward interpretation of the photon with mass as a particle, but this interpretation is self-inconsistent on the U(1) level, as argued.

Self-consistent quantization of the photon with mass can occur using the Higgs mechanism. Symmetry breaking of a U(1) theory gives one massive photon, A_μ; and symmetry breaking on the O(3) level gives one massive photon, A^1_μ, and one massive photon, A^2_μ. On the U(1) level, the time-like component of the photon is canceled by the scalar field, leaving three polarization states for the space-like part of the photon. On the O(3) level, symmetry breaking leads to one massive scalar field and two massive vector fields. The massive scalar field can be interpreted as a physical time-like photon with mass. This massive scalar field appears in the term $-4a^2\lambda\chi^2$ in the Lagrangian (850), where $\chi = \phi_3 - a$. It is also possible to define an effective physical longitudinal photon whose amplitude is the same as that of the physical scalar photon. This should not be confused with the superheavy photon that emerges from electroweak theory with an O(3) electromagnetic sector and observed as described in Section XII. In summary, physical time-like and longitudinal photons are missing from symmetry breaking of a U(1) theory, but are present after symmetry breaking of an O(3) theory. It can be seen from Eq. (826) that electric charge current density is defined by the scalar field ϕ, and the basic requirement for charge to exist from Noether's theorem [6] is that ϕ be complex. It is therefore possible to build up electromagnetic theory from topological considerations, in particular the complex scalar field ϕ, whose ground state is the vacuum.

From the foregoing, it becomes clear that fields and potentials are freely intermingled in the symmetry-broken Lagrangians of the Higgs mechanism. To close this section, we address the question of whether potentials are physical (Faraday and Maxwell) or mathematical (Heaviside) using the non-Abelian Stokes theorem for any gauge symmetry:

$$\oint D_\mu dx^\mu = -\frac{1}{2}\int [D_\mu, D_\nu] d\sigma^{\mu\nu} \tag{867}$$

On the U(1) level, this becomes

$$\oint A_\mu dx^\mu = -\frac{1}{2}\int F_{\mu\nu} d\sigma^{\mu\nu} \tag{868}$$

or in vector notation

$$\oint \boldsymbol{A}\cdot d\boldsymbol{r} = \int \boldsymbol{B}\cdot d\boldsymbol{A}\boldsymbol{r} = \int \nabla\times\boldsymbol{A}\cdot d\boldsymbol{A}\boldsymbol{r} \tag{869}$$

The gauge transformation rule on the U(1) level is

$$\boldsymbol{A} \rightarrow \boldsymbol{A} - \nabla\chi \tag{870}$$

and when applied to Eq. (869), it is found that

$$\oint \nabla\chi\cdot d\boldsymbol{r} = 0 \tag{871}$$

which is self-consistent with

$$\nabla\times(\nabla\chi) = \boldsymbol{0} \tag{872}$$

The Dirac phase factor

$$\exp\left(ig\oint A_\mu dx^\mu\right) = \exp\left(-i\frac{g}{2}\int F_{\mu\nu} d\sigma^{\mu\nu}\right) \tag{873}$$

is therefore gauge-invariant [3,4] and fully describes the electromagnetic phase factor on the U(1) level.

On the O(3) level, a gauge transformation applied to the theorem (867) produces

$$\oint\left(SA_\mu S^{-1} - \frac{i}{g}(\partial_\mu S)S^{-1}\right)dx^\mu = -\frac{1}{2}\int SG_{\mu\nu}S^{-1}d\sigma^{\mu\nu} \tag{874}$$

where

$$S = \exp\left(iM^a\Lambda^a(x^\mu)\right); \quad A^\mu = M^a A^a_\mu \tag{875}$$

Here, M^a are *physical* rotation generators of the O(3) group and Λ^a are *physical* angles [11–20]. The gauge transform produces

$$A^{(i)}_\mu \rightarrow A^{(i)}_\mu + \frac{1}{g}\partial_\mu\Lambda^{(i)}(x^\mu); \qquad i = 1,2,3 \tag{876}$$

so that the potential components of:

$$\boldsymbol{A}_\mu = A^{(1)}_\mu \boldsymbol{e}^{(1)} + A^{(2)}_\mu \boldsymbol{e}^{(2)} + A^{(3)}_\mu \boldsymbol{e}^{(3)} \tag{877}$$

are also *physical*. The gauge transform (874) also produces the result

$$\oint \partial_\mu \Lambda^a(x^\mu)\, dx^\mu = 0 \tag{878}$$

which means that

$$\partial_\nu \partial_\mu \Lambda^a = \partial_\mu \partial_\nu \Lambda^a \tag{879}$$

This result, however, is an identity of Minkowski spacetime itself, namely, $\partial_\nu \partial_\mu$ operating on a function of x^μ produces the same result as $\partial_\nu \partial_\mu$ operating on a function of x^μ. Equation (879) does not mean that Λ^a can take any value. We reach the important conclusion that the vector identity (872) of U(1) is a property of three-dimensional space itself and can always be interpreted as such. Therefore even on the U(1) level, Eq. (872) does not mean that χ can take any value. Even on the U(1) level, therefore, potentials can be interpreted physically, as was the intent of Faraday and Maxwell. On the O(3) level, potentials are always physical.

XV. BELTRAMI ELECTRODYNAMICS AND NONZERO $B^{(3)}$

In this final section, it is shown that the three magnetic field components of electromagnetic radiation in O(3) electrodynamics are Beltrami vector fields, illustrating the fact that conventional Maxwell–Heaviside electrodynamics are incomplete. Therefore Beltrami electrodynamics can be regarded as foundational, structuring the vacuum fields of nature, and extending the point of view of Heaviside, who reduced the original Maxwell equations to their presently accepted textbook form. In this section, transverse plane waves are shown to be solenoidal, complex lamellar, and Beltrami, and to obey the Beltrami equation, of which $\boldsymbol{B}^{(3)}$ is an identically nonzero solution. In the Beltrami electrodynamics, therefore, the existence of the transverse $\boldsymbol{B}^{(1)} = \boldsymbol{B}^{(2)*}$ implies that of $\boldsymbol{B}^{(3)}$, as in O(3) electrodynamics.

As argued by Reed [4], the Beltrami vector field originated in hydrodynamics and is force-free. It is one of the three basic types of field: solenoidal, complex lamellar, and Beltrami. These vector fields originated in hydrodynamics and describe the properties of the velocity field, flux or streamline, $\boldsymbol{v}$, and the vorticity $\nabla \times \boldsymbol{v}$. The Beltrami field is also a Magnus force free fluid flow and is expressed in hydrodynamics as

$$\boldsymbol{v} \times (\nabla \times \boldsymbol{v}) = \boldsymbol{0} \tag{880}$$

The solenoidal vector field is:

$$\nabla \cdot \boldsymbol{v} = 0 \tag{881}$$

and the complex lamellar vector field is

$$\boldsymbol{v}(\nabla \times \boldsymbol{v}) = \boldsymbol{0} \tag{882}$$

The Beltrami condition can also be represented [4] as:

$$\nabla \times \boldsymbol{v} = k\boldsymbol{v} \tag{883}$$

where

$$k = \frac{1}{v^2} \boldsymbol{v} \cdot \nabla \times \boldsymbol{v} \tag{884}$$

for real-valued v.

Beltrami fields have been advanced [4] as theoretical models for astrophysical phenomena such as solar flares and spiral galaxies, plasma vortex filaments arising from plasma focus experiments, and superconductivity. Beltrami electrodynamic fields probably have major potential significance to theoretical and empirical science. In plasma vortex filaments, for example, energy anomalies arise that cannot be described with the Maxwell–Heaviside equations. The three magnetic components of O(3) electrodynamics are Beltrami fields as well as being complex lamellar and solenoidal fields. The component $\boldsymbol{B}^{(3)}$ is identically nonzero in Beltrami electrodynamics if $\boldsymbol{B}^{(1)} = \boldsymbol{B}^{(2)*}$ is so. In the Beltrami electrodynamics, $\boldsymbol{B}^{(3)}$ is a particular solution of the general solution given by Chandrasekhar and Kendall [4] of the Beltrami equation:

$$\nabla \times \boldsymbol{B} = k\boldsymbol{B} \tag{885}$$

This argument shows again that Maxwell–Heaviside electrodynamics is incomplete, because $\boldsymbol{B}^{(3)}$ is zero. General solutions are given in this section of the Beltrami equation, which is an equation of O(3) electrodynamics. Therefore these solutions are also general solutions of O(3) electrodynamics in the vacuum.

The three components of the B cyclic theorem (411) are solenoidal, complex lamellar, and Beltrami. This is a remarkable property of Beltrami electrodynamics when recognized as O(3) electrodynamics for the special case when $\boldsymbol{B}^{(1)} = \boldsymbol{B}^{(2)*}$ are plane waves. Specifically

$$\begin{aligned}
&\nabla \cdot \boldsymbol{B}^{(1)} = 0; \quad \boldsymbol{B}^{(1)} \cdot \nabla \times \boldsymbol{B}^{(1)} = 0; \quad \boldsymbol{B}^{(1)} \times (\nabla \times \boldsymbol{B}^{(1)}) = \boldsymbol{0} \\
&\nabla \cdot \boldsymbol{E}^{(1)} = 0; \quad \boldsymbol{E}^{(1)} \cdot \nabla \times \boldsymbol{E}^{(1)} = 0; \quad \boldsymbol{E}^{(1)} \times (\nabla \times \boldsymbol{E}^{(1)}) = \boldsymbol{0} \\
&\nabla \cdot \boldsymbol{A}^{(1)} = 0; \quad \boldsymbol{A}^{(1)} \cdot \nabla \times \boldsymbol{A}^{(1)} = 0; \quad \boldsymbol{A}^{(1)} \times (\nabla \times \boldsymbol{A}^{(1)}) = \boldsymbol{0}
\end{aligned} \tag{886}$$

and also for indices (2) and (3). Multiplying the Beltrami equation:

$$\nabla \times \boldsymbol{B}^{(1)} = k\boldsymbol{B}^{(1)} \tag{887}$$

on both sides by $\boldsymbol{B}^{(2)}$, it is seen that

$$\boldsymbol{B}^{(2)} \cdot \nabla \times \boldsymbol{B}^{(1)} = k\boldsymbol{B}^{(1)} \cdot \boldsymbol{B}^{(2)} \tag{888}$$

so the constant k is not necessarily zero when dealing with complex fields. To prove that k can be different from zero, consider the complex transverse magnetic plane wave

$$\boldsymbol{A}^{(1)} = \frac{A^{(0)}}{\sqrt{2}}(\boldsymbol{i}\boldsymbol{i} + \boldsymbol{j})e^{i(\omega t - \kappa Z)} \tag{889}$$

which obeys the B cyclic theorem (411). From Eqs. (883) and (884)

$$k = \frac{1}{A^{(0)2}}\boldsymbol{A}^* \cdot \nabla \times \boldsymbol{A} = \frac{\boldsymbol{A}^* \cdot \boldsymbol{B}}{A^{(0)2}} = \kappa \tag{890a}$$

$$\nabla \times \boldsymbol{A}^{(1)} = \kappa \boldsymbol{A}^{(1)} \tag{890b}$$

and all three components—(1), (2) and (3)—are solutions of the same Beltrami equation. Similarly, if we define the complete magnetic field vector by

$$\boldsymbol{B} \equiv \boldsymbol{B}^{(1)} + \boldsymbol{B}^{(2)} + \boldsymbol{B}^{(3)} \tag{891}$$

the complete vector $\boldsymbol{B}$ obeys Eq. (885).

On the U(1) level, if we start with the free-space Maxwell–Heaviside equations

$$\nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{0}; \quad \nabla \times \boldsymbol{B} - \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} = \boldsymbol{0} \tag{892}$$

it follows that

$$\nabla \times \boldsymbol{B} = k\boldsymbol{B} \tag{893a}$$

$$\nabla \times \boldsymbol{E} = k\boldsymbol{E} \tag{893b}$$

$$\nabla \times \boldsymbol{A} = k\boldsymbol{A} \tag{893c}$$

where $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ as usual, and where $k = \pm\kappa$. Here, k is a pseudo scalar that changes sign between left and right circularly polarized radiation. The Beltrami equation for $\boldsymbol{B}^{(3)}$ is

$$\nabla \times \boldsymbol{B}^{(3)} = k\boldsymbol{B}^{(3)} \tag{894}$$

where $k = 0$. It follows that all components of transverse plane waves are described by Beltrami equations in vacuo. For left-handed plane waves

$$\begin{aligned}
\boldsymbol{E}_L^{(1)} &= \frac{E^{(0)}}{\sqrt{2}}(\boldsymbol{i} - i\boldsymbol{j})e^{-i(\omega t - \kappa Z)} = \boldsymbol{E}_L^{(2)*} \\
\boldsymbol{B}_L^{(1)} &= \frac{B^{(0)}}{\sqrt{2}}(i\boldsymbol{i} + \boldsymbol{j})e^{-i(\omega t - \kappa Z)} = \boldsymbol{B}_L^{(2)*} \\
\boldsymbol{A}_L^{(1)} &= \frac{\boldsymbol{A}^{(0)}}{\sqrt{2}}(i\boldsymbol{i} + \boldsymbol{j})e^{-i(\omega t - \kappa Z)} = \boldsymbol{A}_L^{(2)*}
\end{aligned} \tag{895}$$

For right-handed transverse plane waves

$$\begin{aligned}
\boldsymbol{E}_R^{(1)} &= \frac{E^{(0)}}{\sqrt{2}}(\boldsymbol{i} + i\boldsymbol{j})e^{-i(\omega t - \kappa Z)} = \boldsymbol{E}_R^{(2)*} \\
\boldsymbol{B}_R^{(1)} &= \frac{B^{(0)}}{\sqrt{2}}(-i\boldsymbol{i} + \boldsymbol{j})e^{-i(\omega t - \kappa Z)} = \boldsymbol{B}_R^{(2)*} \\
\boldsymbol{A}_R^{(1)} &= \frac{A^{(0)}}{\sqrt{2}}(-i\boldsymbol{i} + \boldsymbol{j})e^{-i(\omega t - \kappa Z)} = \boldsymbol{A}_R^{(2)*}
\end{aligned} \tag{896}$$

and for the longitudinal $\boldsymbol{B}^{(3)}$*field*

$$\boldsymbol{B}_L^{(3)} = -\boldsymbol{B}_R^{(3)} = B^{(0)}\boldsymbol{k} \tag{897}$$

Therefore

$$\begin{aligned}
\nabla \times \boldsymbol{B}_L^{(1)} &= -\kappa \boldsymbol{B}_L^{(1)}; & \nabla \times \boldsymbol{B}_R^{(1)} &= \kappa \boldsymbol{B}_R^{(1)} \\
\nabla \times \boldsymbol{E}_L^{(1)} &= -\kappa \boldsymbol{E}_L^{(1)}; & \nabla \times \boldsymbol{E}_R^{(1)} &= \kappa \boldsymbol{E}_R^{(1)} \\
\nabla \times \boldsymbol{A}_L^{(1)} &= -\kappa \boldsymbol{A}_L^{(1)}; & \nabla \times \boldsymbol{A}_R^{(1)} &= \kappa \boldsymbol{A}_R^{(1)}
\end{aligned} \tag{898}$$

and similarly for index (2). For the longitudinal index (3)

$$\nabla \times \boldsymbol{B}_R^{(3)} = \nabla \times \boldsymbol{B}_L^{(3)} = \boldsymbol{0} \tag{899}$$

and all components are described by Beltrami equations in vacuo. Since $\boldsymbol{E}$ and $\boldsymbol{B}$ are the fundamental fields of electrodynamics, these equations are valid under all conditions. In particular, Eq. (893c) for the potential is not gauge-invariant under the transform:

$$\boldsymbol{A} \rightarrow \boldsymbol{A} - \nabla\chi \tag{900}$$

revealing that in Beltrami electrodynamics, $\boldsymbol{A}$ is physical. This result again supports Maxwell's postulate of a physical vector potential and does not support Heaviside's postulate of an unphysical vector potential. Equation (893c) is self-consistent, however, on the O(3) level, where potentials are physical. The covariant form of Eq. (893c) is

$$F_{\mu\nu} = \kappa A_{\mu\nu} \tag{901}$$

so the field tensor is directly proportional to an axial potential 4-tensor. This suggests that the vector potential can be polar or axial in nature. The solutions of Eq. (901) are also solutions of the d'Alembert equation in vacuo. In this view, the field tensor is directly proportional to the axial potential tensor $A_{\mu\nu}$, and so gauge freedom is lost because, if $F_{\mu\nu}$ is gauge-invariant, so is $A_{\mu\nu}$. This result is another internal inconsistency of the Maxwell–Heaviside point of view.

The Faraday law of induction does not distinguish between left and right circular polarization, that is, the structure of the equation is the same for R and L:

$$\begin{aligned} \nabla \times \boldsymbol{E}_L^{(1)} &= -\frac{\partial \boldsymbol{B}_L^{(1)}}{\partial t} \\ \nabla \times \boldsymbol{E}_R^{(1)} &= -\frac{\partial \boldsymbol{B}_R^{(1)}}{\partial t} \end{aligned} \tag{902}$$

On the other hand, the corresponding Beltrami equations are distinct:

$$\begin{aligned} \nabla \times \boldsymbol{E}_L^{(1)} &= -\kappa \boldsymbol{E}_L^{(1)} \\ \nabla \times \boldsymbol{E}_R^{(1)} &= \kappa \boldsymbol{E}_R^{(1)} \end{aligned} \tag{903}$$

The handedness, or chirality, inherent in foundational electrodynamics at the U(1) level manifests itself clearly in the Beltrami form (903). The chiral nature of the field is inherent in left- and right-handed circular polarization, and the distinction between axial and polar vector is lost. This result is seen in Eq. (901), where $A_{\mu\nu}$ is a tensor form that contains axial and polar components of the potential. This is precisely analogous with the fact that the field tensor $F_{\mu\nu}$ contains polar (electric) and axial (magnetic) components intermixed. Therefore, in propagating electromagnetic radiation, there is no distinction between polar and axial. In the received view, however, it is almost always asserted that $\boldsymbol{E}$ and $\boldsymbol{A}$ are polar vectors and that $\boldsymbol{B}$ is an axial vector.

The $\boldsymbol{B}^{(3)}$ component [which is nonzero only on the O(3) level] is a solution of the Beltrami equation (885) with $k = 0$. Therefore, in Beltrami electrodynamics, $\boldsymbol{B}^{(3)}$ is a solenoidal, irrotational, complex lamellar and Beltrami field in the vacuum, and is also a propagating field. The $\boldsymbol{B}^{(3)}$ component in Beltrami

electrodynamics is part of the general solution of the solenoidal Beltrami equation given in Ref. 4, and is identically nonzero in the vacuum. This statement is equivalent to saying that electrodynamics is an O(3) Yang–Mills theory in the vacuum. The general solution in cylindrical components of Eq. (885) is

$$\boldsymbol{B} = \sum_{m,n} B_{mn} \boldsymbol{b}^{mn}(r, \theta, z) \tag{904}$$

where m is a nonnegative integer and where $\boldsymbol{b}^{mn}$ depends on ϕ and Z through $\phi = m\theta + nZ$. The expressions for the modes depend on linear combinations of Bessel and Neumann functions, J_m and N_m, similar to the solutions of the Helmholtz equation [5]. When the domain of solution involves the axis $r = 0$, and solutions are restricted to axisymmetric wave equations, then

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) = -k^2\psi \tag{905}$$

The solution of this equation is [4]

$$\psi = C J_0(kr) \tag{906}$$

where C is any constant, and the solution specializes to:

$$\boldsymbol{B} = B_0(0, J_1(kr), J_0(kr)) \tag{907}$$

for the mode $m = n = 0; a = (0, 0, 1)$. Therefore the unit vector $a = (0, 0, 1)$ designates the Z axis. The solution for the $\boldsymbol{B}^{(3)}$ component is

$$\boldsymbol{B}^{(3)} = B_0(0, J_1(0), J_0(0)) \tag{908}$$

and depends on the Bessel functions $J_1(0)$ and $J_0(0)$. Therefore

$$\begin{aligned} \boldsymbol{B}^{(3)} &= \boldsymbol{B}(k = 0, m = 0, n = 0) \\ &= B_0(0, 0, 1) = B^{(0)}\boldsymbol{k} \end{aligned} \tag{909}$$

and $\boldsymbol{B}^{(3)}$ is an identically nonzero, phaseless function directed in the Z axis. This result is self-consistent with that of O(3) electrodynamics.

In conducting media, the wave number κ becomes complex [5], and by separating real and imaginary parts, we can obtain the Beltrami equations:

$$\nabla \times \boldsymbol{A} = k\boldsymbol{A}; \quad k \equiv i\kappa'' \tag{910}$$

$$\nabla \times B = kB; \qquad k \equiv i\kappa'' \tag{911}$$

Taking the curl of Eq. (910) gives

$$\nabla \times (\nabla \times \boldsymbol{A}) = k\nabla \times \boldsymbol{A} = k^2\boldsymbol{A} \tag{912}$$

which can be rewritten as

$$\nabla^2\boldsymbol{A} = \kappa''^2\boldsymbol{A} \tag{913}$$

using the vector identity

$$\nabla \times (\nabla \times \boldsymbol{A}) = \nabla \times (\nabla \cdot \boldsymbol{A}) - \nabla^2\boldsymbol{A} \tag{914}$$

The covariant form of Eq. (914) is

$$\Box A^\mu = -\kappa''^2 A^\mu \tag{915}$$

If we assume

$$\kappa'' = \frac{m_0 c}{\hbar} \tag{916}$$

Eq. (915) becomes the Proca equation, and Eq. (916), the de Broglie guidance theorem.

Similarly, Eq. (911) becomes the equation of the Meissner effect in superconductivity:

$$\nabla^2\boldsymbol{B} = \kappa''^2\boldsymbol{B} \tag{917}$$

Finally, using

$$\nabla \times \boldsymbol{B} = k\boldsymbol{B} = k\nabla \times \boldsymbol{A} = k^2\boldsymbol{A} \tag{918}$$

we obtain the London equation:

$$\boldsymbol{J} = \nabla \times \boldsymbol{B} = -\kappa''^2\boldsymbol{A} \tag{919}$$

It is seen that the acquisition of mass by the photon is the result of an equation of superconductivity, and this is, of course, the basis of spontaneous symmetry breaking and the Higgs mechanism (Section XIV). Beltrami equations account for all these phenomena, and are foundational in nature. Note that the London equation (919) is not gauge-invariant on the U(1) level because a physical gauge-invariant current is proportional to the vector potential, which, in the received view, is gauge-noninvariant. This is another flaw of U(1) electrodynamics in the

received opinion. The electric field from the London equation is zero because the current $\boldsymbol{J}$ is time-independent:

$$\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} = \boldsymbol{0} \tag{920}$$

By Ohm's law, the resistance of the conducting medium vanishes, and the medium becomes a superconductor. The Higgs mechanism and spontaneous symmetry breaking were derived using the properties of superconductors.

TECHNICAL APPENDIX A: THE NON-ABELIAN STOKES THEOREM

The non-Abelian Stokes theorem is a relation between covariant derivatives for any gauge group symmetry:

$$\oint D_\mu dx^\mu = -\frac{1}{2}\int [D_\mu, D_\nu] d\sigma^{\mu\nu} \tag{A.1}$$

This expression can be expanded as

$$\oint (\partial_\mu - igA_\mu) dx^\mu = -\frac{1}{2}\int [\partial_\mu - igA_\mu,\ \partial_\nu - igA_\nu] d\sigma^{\mu\nu} \tag{A.2}$$

The terms

$$\oint \partial_\mu dx^\mu = [\partial_\mu,\ \partial_\nu] = 0 \tag{A.3}$$

are zero because by symmetry

$$\partial_\nu \partial_\mu = \partial_\mu \partial v \tag{A.4}$$

so

$$\oint \partial_\mu dx^\mu = -\frac{1}{2}\int [\partial_\mu,\ \partial v] d\sigma^{\mu\nu} = 0 \tag{A.5}$$

The half-commutators are evaluated as follows

$$[A_\mu, \partial_\nu] = -\partial_\nu A_\mu; \qquad [\partial_\mu, A_\nu] = \partial_\mu A_\nu \tag{A.6}$$

giving the non-Abelian Stokes theorem

$$\oint A_\mu dx^\mu = -\frac{1}{2}\int G_{\mu\nu} d\sigma^{\mu\nu} \tag{A.7}$$

where the field tensor for any gauge group is

$$G_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \tag{A.8}$$

On the U(1) level, the 4-potential is

$$A_\mu = (\phi, cA) \tag{A.9}$$

and the field tensor is

$$F_{\mu\nu} = \begin{bmatrix} 0 & \frac{E_1}{c} & \frac{E_2}{c} & \frac{E_3}{c} \\ \frac{-E_1}{c} & 0 & B_3 & -B_2 \\ \frac{-E_2}{c} & -B_3 & 0 & B_1 \\ \frac{-E_3}{c} & B_2 & -B_1 & 0 \end{bmatrix} \tag{A.10}$$

Summing over repeated indices gives the time-like relation

$$\oint \phi \, dt = \frac{1}{2c^2}\left(\int E_X d\sigma^{01} + \int E_Y d\sigma^{02}\right) \tag{A.11}$$

where the SI units on either side are those of electric field strength multiplied by area. Summing over space indices gives

$$\oint A_1 dx^1 + A_2 dx^2 + A_3 dx^3 = -\frac{1}{2}\int F_{ij} d\sigma^{ij} \tag{A.12}$$

which can be rewritten as

$$\begin{aligned} \oint A_1 dx^1 &= -\frac{1}{2}\int F_{23} d\sigma^{23} + F_{32} d\sigma^{32} = -\int B_1 d\sigma^{23} \\ \oint A_2 dx^2 &= -\frac{1}{2}\int F_{31} d\sigma^{31} + F_{13} d\sigma^{13} = -\int B_2 d\sigma^{31} \\ \oint A_3 dx^3 &= -\frac{1}{2}\int F_{12} d\sigma^{12} + F_{21} d\sigma^{21} = -\int B_3 d\sigma^{12} \end{aligned} \tag{A.13}$$

In Cartesian coordinates, this is

$$\begin{aligned} \oint A_X dX &= \int B_x d\sigma^{YZ} \\ \oint A_Y dY &= \int B_Y d\sigma^{ZX} \\ \oint A_Z dZ &= \int B_Z d\sigma^{XY} \end{aligned} \tag{A.14}$$

or in condensed notation

$$\oint \boldsymbol{A} \cdot d\boldsymbol{r} = \int \boldsymbol{B} \cdot d\boldsymbol{A}\boldsymbol{r} \tag{A.15}$$

This is the Stokes theorem as usually found in textbooks. For plane waves, $\boldsymbol{A}$ is always perpendicular to the path, so in free space

$$\oint \boldsymbol{A} \cdot d\boldsymbol{r} = 0 \equiv \oint A_Z dZ \Rightarrow \nabla \times \boldsymbol{A}_Z = \boldsymbol{0} \tag{A.16}$$

On the O(3) level, there is a nonzero commutator and an additional term

$$\oint A_3^{(3)} dx^3 = -i\frac{g}{2}\left(\int [A_1^{(1)}, A_2^{(2)}] d\sigma^{12} + \int [A_2^{(1)}, A_1^{(2)}] d\sigma^{21}\right) \tag{A.17}$$

in the basis ((1),(2),(3)) defined by

$$\begin{aligned} \boldsymbol{e}^{(1)} \times \boldsymbol{e}^{(2)} &= i\boldsymbol{e}^{(3)*} \\ &\cdots \end{aligned} \tag{A.18}$$

In Cartesian form, Eq. (A.17) becomes

$$\oint A_Z^{(3)} dZ = -ig \int [A_X^{(1)}, A_Y^{(2)}] dAr = \int B_Z^{(3)} dAr \tag{A.19}$$

and explains the Sagnac effect as in the text. There are time-like relations such as

$$\oint A_0 dx^0 = -\frac{1}{2}\int \partial_0 A_\nu - \partial_\nu A_0 - ig[A_0, A_\nu] d\sigma^{0\nu} \tag{A.20}$$

which define the scalar potential in O(3) electrodynamics to be nonzero and structured.

TECHNICAL APPENDIX B: 4-VECTOR MAXWELL–HEAVISIDE EQUATIONS

In this second technical appendix, it is shown that the Maxwell–Heaviside equations can be written in terms of a field 4-vector $G^\mu = (0, c\boldsymbol{B} + i\boldsymbol{E})$ rather than as a tensor. Under Lorentz transformation, G^μ transforms as a 4-vector. This shows that the field in electromagnetic theory is not uniquely defined as a 4-tensor. The Maxwell–Heaviside equations can be written in terms of the 4-vectors:

$$G^\mu = (0, c\boldsymbol{B} + i\boldsymbol{E}) \tag{B.1}$$

and

$$H^{(\mu)} = (0, \boldsymbol{H} + ic\boldsymbol{D}) \tag{B.2}$$

as

$$\begin{aligned} &\partial_\mu G^\mu = 0 \\ &\quad [\partial_i, G_j] + i[\partial_0, G_k] = 0 \\ &\partial_\mu H^\mu = i\rho c \\ &\quad [\partial_i, H_j] + i[\partial_0, H_k] = J_k \end{aligned} \tag{B.3}$$

Under Lorentz transformation:

$$\begin{aligned} G_\mu G^\mu &= G'_\mu G^{\mu'} \\ H_\mu H^\mu &= H'_\mu H^{\mu'} \end{aligned} \tag{B.4}$$

Using the fact that ρ and $\boldsymbol{J}$ themselves form the components of a 4-vector, the Maxwell–Heaviside equations for field matter interaction can be combined into one relation between 4-vectors:

$$(-i\partial_\mu H^\mu, ([\partial_i, H_j] + i[\partial_0, H_k])) = c\left(\rho, \frac{1}{c} J_k\right) \tag{B.5}$$

The free-space equivalent is

$$(\partial_\mu G^\mu, [\partial_i, G_i] + i[\partial_0, G_k]) = 0 \tag{B.6}$$

A Lorentz boost in the Z direction of the vector G^μ produces

$$\begin{aligned} cB'_X + iE'_X &= cB_X + iE_X \\ cB'_Y + iE'_Y &= cB_Y + iE_Y \\ cB'_Z + iE'_Z &= \gamma(cB'_Z + iE'_Z) \\ cB'_0 + iE'_0 &= -\gamma\beta(cB'_Z + iE'_Z) \end{aligned} \tag{B.7}$$

but a Lorentz transform in the Z direction applied to $F^{\mu\nu}$ produces

$$\begin{aligned} cB'_X &= \gamma(cB_X + \beta E_Y) \\ cB'_Y &= \gamma(cB_Y - \beta E_X) \\ cB'_Z &= cB_Z \\ B'_0 &= 0 \end{aligned} \tag{B8}$$

The results (B.7) and (B.8) are different, even though both describe a boost of the same vector equations, the Maxwell–Heaviside equations:

$$\begin{aligned} \nabla \cdot \boldsymbol{B} &= 0 \\ \nabla \times E + \frac{\partial \boldsymbol{B}}{\partial t} &= \boldsymbol{0} \end{aligned} \tag{B.9}$$

The only common factor is that the charge–current 4-tensor transforms in the same way. The vector representation develops a time-like component under Lorentz transformation, while the tensor representation does not. However, the underlying equations in both cases are the Maxwell–Heaviside equations, which transform covariantly in both cases and obviously in the same way for both vector and tensor representations.

If we define the vectors

$$\begin{aligned} \boldsymbol{a} &\equiv \frac{1}{2}(c\boldsymbol{B} + i\boldsymbol{E}) \\ \boldsymbol{b} &\equiv \frac{1}{2}(c\boldsymbol{B} - i\boldsymbol{E}) \end{aligned} \tag{B.10}$$

then

$$\begin{aligned} [a_X, a_Y] &= ia_Z \cdots \\ [b_X, b_Y] &= ib_Z \cdots \\ [a_i, a_i] &= 0 \qquad (i,j = X, Y, Z) \end{aligned} \tag{B.11}$$

and $\boldsymbol{a}$ and $\boldsymbol{b}$ both generate a group SU(2). The Lorentz group is then SU(2) $\otimes$ SU(2) and transforms in a well-defined way labeled by two angular momenta $(\boldsymbol{j}, \boldsymbol{j}')$, the first corresponding to $\boldsymbol{a}$ and the second to $\boldsymbol{b}$. Thus $\boldsymbol{a}$ and $\boldsymbol{b}$ are generators of the Lorentz group. The vector G^μ also transforms as a rest frame Pauli–Lubanski vector, suggesting that the vector representation is suitable for intrinsic photon spin, and the tensor representation for orbital angular momentum. This is also suggested by O(3) electrodynamics where the fundamental intrinsic spin of the field is $\boldsymbol{B}^{(3)}$.

TECHNICAL APPENDIX C: ON THE ABSENCE OF MAGNETIC MONOPOLES AND CURRENTS IN O(3) ELECTRODYNAMICS

The non-Abelian Stokes theorem

$$\oint D_\mu dx^\mu + \frac{1}{2}\int [D_\mu, D_\nu] d\sigma^{\mu\nu} = 0 \tag{C.1}$$

is the integral form of the Jacobi identity

$$\sum_{\sigma,\mu,\nu} [D_\sigma, [D_\mu, D_\nu]] = 0 \tag{C.2}$$

which is an identity between spacetime translation generators of the Poincaré group. Since

$$D_\mu = \partial_\mu - igA_\mu \tag{C.3}$$

for any gauge group symmetry, it follows that the identity (C.2) holds for the different components of D_μ. In an O(3) gauge, group symmetry identity (C.2) can be written as the field equation (31) of the text, so it follows that

$$\partial_\mu \tilde{\boldsymbol{G}}^{\mu\nu} \equiv \boldsymbol{0} \tag{C.4}$$

$$A_\mu \times \tilde{\boldsymbol{G}}^{\mu\nu} \equiv \boldsymbol{0} \tag{C.5}$$

Equation (C.5) means that there are no magnetic charge or current densities in O(3) electrodynamics.

It follows that

$$\boldsymbol{A}^{(2)} \cdot \boldsymbol{B}^{(3)} - \boldsymbol{B}^{(2)} \cdot \boldsymbol{A}^{(3)} = \boldsymbol{0} \tag{C.6}$$

$$\boldsymbol{A}^{(3)} \cdot \boldsymbol{B}^{(1)} - \boldsymbol{B}^{(3)} \cdot \boldsymbol{A}^{(1)} = \boldsymbol{0} \tag{C.7}$$

$$\boldsymbol{A}^{(1)} \cdot \boldsymbol{B}^{(2)} - \boldsymbol{B}^{(1)} \cdot \boldsymbol{A}^{(2)} = \boldsymbol{0} \tag{C.8}$$

The third equation is always true if

$$\boldsymbol{B}^{(1)} = \nabla \times \boldsymbol{A}^{(1)}; \qquad B^{(2)} = \nabla \times A^{(2)} \tag{C.9}$$

because of the vector identity

$$\nabla \cdot (\boldsymbol{F} \times \boldsymbol{G}) = \boldsymbol{G} \cdot (\nabla \times \boldsymbol{F}) - \boldsymbol{F} \cdot (\nabla \times \boldsymbol{G}) \tag{C.10}$$

and the first two equations are always true because (3) is always orthogonal to (1) and (2).

It also follows that

$$cA_0^{(3)}\boldsymbol{B}^{(2)} - cA_0^{(2)}\boldsymbol{B}^{(3)} + \boldsymbol{A}^{(2)} \times \boldsymbol{E}^{(3)} - \boldsymbol{A}^{(3)} \times \boldsymbol{E}^{(2)} = \boldsymbol{0} \tag{C.11}$$

$$cA_0^{(1)}\boldsymbol{B}^{(3)} - cA_0^{(3)}\boldsymbol{B}^{(1)} + \boldsymbol{A}^{(3)} \times \boldsymbol{E}^{(1)} - \boldsymbol{A}^{(1)} \times \boldsymbol{E}^{(3)} = \boldsymbol{0} \tag{C.12}$$

$$cA_0^{(2)}\boldsymbol{B}^{(1)} - cA_0^{(1)}\boldsymbol{B}^{(2)} + \boldsymbol{A}^{(1)} \times \boldsymbol{E}^{(2)} - \boldsymbol{A}^{(2)} \times \boldsymbol{E}^{(1)} = \boldsymbol{0} \tag{C.13}$$

and using

$$A_0^{(2)} = A_0^{(1)} \equiv 0; \qquad |\boldsymbol{A}^{(3)}| = A_0^{(3)} \tag{C.14}$$

Eqs. (C.11) and (C.12) give

$$c\boldsymbol{B}^{(1)} = \boldsymbol{k} \times \boldsymbol{E}^{(1)} - \boldsymbol{A}^{(1)} \times \frac{\boldsymbol{E}^{(3)}}{cA_0^{(3)}} \tag{C.15}$$

$$c\boldsymbol{B}^{(2)} = \boldsymbol{k} \times \boldsymbol{E}^{(2)} - \boldsymbol{A}^{(2)} \times \frac{\boldsymbol{E}^{(3)}}{cA_0^{(3)}} \tag{C.16}$$

However, we know from Eq. (C.4) that

$$\nabla \times E^{(1)} + \frac{\partial \boldsymbol{B}^{(1)}}{\partial t} = \boldsymbol{0} \tag{C.17}$$

$$\nabla \times E^{(2)} + \frac{\partial \boldsymbol{B}^{(2)}}{\partial t} = \boldsymbol{0} \tag{C.18}$$

so

$$c\boldsymbol{B}^{(1)} = \boldsymbol{k} \times \boldsymbol{E}^{(1)} \tag{C.19}$$

$$c\boldsymbol{B}^{(2)} = \boldsymbol{k} \times \boldsymbol{E}^{(2)} \tag{C.20}$$

and $\boldsymbol{E}^{(3)}$ is identically zero because $A_0^{(3)}$, $\boldsymbol{A}^{(1)}$, and $\boldsymbol{A}^{(2)}$are nonzero. It follows that

$$\frac{\partial \boldsymbol{B}^{(3)}}{\partial t} = \boldsymbol{0} \tag{C.21}$$

and there is no Faraday induction due to $\boldsymbol{B}^{(3)}$. Equation (C.13) gives

$$\boldsymbol{A}^{(1)} \times \boldsymbol{E}^{(2)} = \boldsymbol{A}^{(2)} \times \boldsymbol{E}^{(1)} \tag{C.22}$$

which is self-consistent with Eqs. (C.9), (C.17), and (C.18).

The B cyclic theorem follows from

$$c\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} = c\boldsymbol{B}^{(1)} \times (\boldsymbol{k} \times \boldsymbol{E}^{(2)}) \tag{C.23}$$

which becomes

$$\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} = iB^{(0)}\boldsymbol{B}^{(3)*} \tag{C.24}$$

using the vector identity

$$\boldsymbol{F} \times (\boldsymbol{G} \times \boldsymbol{H}) = \boldsymbol{G}(\boldsymbol{F} \cdot \boldsymbol{H}) - \boldsymbol{H}(\boldsymbol{F} \cdot \boldsymbol{G}) \tag{C.25}$$

Similarly

$$c\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(3)} = (\boldsymbol{k} \times \boldsymbol{E}^{(1)}) \times \boldsymbol{B}^{(3)} \tag{C.26}$$

becomes

$$\boldsymbol{B}^{(3)} \times \boldsymbol{B}^{(1)} = iB^{(0)}\boldsymbol{B}^{(2)*} \tag{C.27}$$

using

$$\boldsymbol{E}^{(1)} = -ic\boldsymbol{B}^{(1)} \tag{C.28}$$

and we obtain the Poincaré invariant B cyclic theorem because $\boldsymbol{E}^{(3)}$ is zero, and because there are no magnetic charge and current desities:

$$\begin{aligned} \boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} &= iB^{(0)}\boldsymbol{B}^{(3)*} \\ \boldsymbol{B}^{(2)} \times \boldsymbol{B}^{(3)} &= iB^{(0)}\boldsymbol{B}^{(1)*} \\ \boldsymbol{B}^{(3)} \times \boldsymbol{B}^{(1)} &= iB^{(0)}\boldsymbol{B}^{(2)*} \end{aligned} \tag{C.29}$$

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