Acknowled

The U.S. Department of Energy is acknowledged for its website: http://www.ott.doe.gov/electromagnetic/. This website is reserved for the Advanced Electrodynamics Working Group.

References

1. M. Sachs, 11th chapter in Part 1 of this three-volume set.
2. M. W. Evans, 2nd chapter in this volume (i.e., Part 2 of this compilation).
3. M. W. Evans, *O(3) Electrodynamics*, Kluwer Academic, Dordrecht, 1999.
4. T. W. Barrett and D. M. Grimes, *Advanced Electromagnetism*, World Scientific, Singapore, 1995.
5. A demonstration of this property in O(3) electrodynamics is given by Evans and Jeffers, 1st chapter in Part 3 of this three-volume set; see also Ref(s).—by Vigier in Bibliography at end of that chapter.
6. M. W. Evans, J. P. Vigier, and S. Roy (eds.), *The Enigmatic Photon*, Kluwer Academic, Dordrecht, 1997, Vol. 4.
7. T. E. Bearden, *Energy from the Active Vacuum*, World Scientific, Singapore, in press.
8. T. E. Bearden, 11th and 12th chapters in this volume (i.e., Part 2).
9. M. W. Evans and S. Jeffers, 1st chapter in Part 3 of this three-volume set; see also Ref(s).—by Vigier listed in that chapter.
10. T. W. Barrett, in A. Lakhtakia (Ed.), *Essays on the Formal Aspects of Electromagnetic Theory*, World Scientific, Singapore, 1993.
11. L. H. Ryder, *Quantum Field Theory*, 2nd ed., Cambridge Univ. Press, Cambridge, UK, 1987.
12. T. W. Barrett, *Apeiron* **7**(1), (2000).
13. T. E. Bearden et al., *Phys. Scripta* **61**, 513 (2000).

THE LINK BETWEEN THE TOPOLOGICAL THEORY OF RAÑADA AND TRUEBA, THE SACHS THEORY, AND O(3) ELECTRODYNAMICS

M. W. EVANS

CONTENTS

I. REVIEW OF THE LITERATURE AND GENERAL CONCEPTS

The topological approach of Rañada and Trueba, the general relativistic approach of Sachs, and the O(3) electrodynamics are interlinked and shown to be based on the concept of Faraday's lines of force.

In the review by Rañada and Trueba [1], electric and magnetic lines of force were discussed as real, physical entities, based on the original concepts of Faraday. These authors discussed Kelvin's suggestion of 1868 that atoms are knots of links of vortex lines of the ether, a topological concept, and that Kelvin found the concept of point particle to be extremely unsatisfactory. Point particles are eliminated from consideration in the Sachs [2] theory of electrodynamics, and are replaced by curvature of spacetime. The O(3) electrodynamics of Evans [3] has been demonstrated [4] to be a subtheory of the Sachs theory. Rañada and Trueba discuss the fact that, in contemporary topology, invariant numbers characterize configurations that can deform, distort, or warp. These concepts are similar to the curving of spacetime in general relativity [2], of which O(3) electrodynamics [3] is a subtheory, and also a gauge theory. Topology [1] shows that the variety of chemical elements is due to the way in which curves can be knotted and linked, transmutability of the elements

Modern Nonlinear Optics, Part 2, Second Edition, Advances in Chemical Physics, Volume 119, Edited by Myron W. Evans. Series Editors I. Prigogine and Stuart A. Rice.
ISBN 0-471-38931-5

is due to the breaking and reconnection of lines, and the quantum character of the spectrum is due to the natural topological configurations of the vector field. Rañada and Trueba [1] reinstated lines of force, which they describe as having been relegated in importance in Maxwell's treatise and replaced by the concepts of field and vector potential. Lines of force are integral lines of the magnetic and electric fields, and so exist also in the Sachs [2] and Evans [3] theories.

The Aharonov–Bohm effect requires topological consideration [1], (i.e., a structured vacuum), and there exist conservation laws of topological origin, the simplest one is given by the sine–Gordon equation, which also appears in the discussion of O(3) electrodynamics by Evans and Crowell [5].

All topological theories are nonlinear, a feature of both the Sachs and Evans theories, and the whole of quantum theory can be replaced by topology [1], which reduces in some circumstances to the Yang–Mills theory [1], of which O(3) electrodynamics [3] is an example. O(3) electrodynamics has been developed into an O(3) symmetry quantum field theory by Evans and Crowell [5], and Witten [1] has developed a topological quantum field theory. In the theory of Rañada and Trueba [1], the Maxwell equations are linearized by change of variables of a set of nonlinear equations, and are compatible with topological constants of motion nonlinear in A^μ and $F_{\mu\nu}$. One of these is $\boldsymbol{B}^{(3)}$ [3], whose rigorous form in general relativity is the quaternion-valued equivalent [6] of the Sachs theory. One of these constants of motion is the electromagnetic helicity of a knot, which has been discussed elsewhere [7] in terms of $\boldsymbol{B}^{(3)}$. However, helicity is not conserved in the Sachs theory [2] because the latter contains parity violation as an intrinsic feature. The electromagnetic helicity of a knot is defined by Rañada and Trueba [1] as

$$H = \frac{1}{2}(\boldsymbol{A}\cdot\boldsymbol{B} + \boldsymbol{C}\cdot\boldsymbol{E})d^3\boldsymbol{r} \tag{1}$$

where, in the Maxwell–Heaviside theory, the magnetic and electric fields are defined as

$$\boldsymbol{B} = \nabla\times\boldsymbol{A}; \qquad \boldsymbol{E} = \nabla\times\boldsymbol{C} \tag{2}$$

The helicity H of a knot can, however, be defined as the $\boldsymbol{B}^{(3)}$ field as follows:

$$\begin{aligned} H &= \left|-ig\int\nabla\cdot(\boldsymbol{A}^{(1)}\times\boldsymbol{A}^{(2)})\,dZ\right| = \left|-ig\int\frac{\partial}{\partial Z}\boldsymbol{A}^{(1)}\times\boldsymbol{A}^{(2)}\,dZ\right| \\ &= |-ig\boldsymbol{A}^{(1)}\times\boldsymbol{A}^{(2)}| = |\boldsymbol{B}^{(3)}| \end{aligned} \tag{3}$$

This definition can be rewritten in the form (1) using:

$$\nabla\cdot(\boldsymbol{A}^{(1)}\times\boldsymbol{A}^{(2)}) = \boldsymbol{A}^{(2)}\cdot(\nabla\times\boldsymbol{A}^{(1)}) - \boldsymbol{A}^{(1)}\cdot(\nabla\times\boldsymbol{A}^{(2)}) \tag{4}$$

This definition is related to the difference between left- and right-handed photons because $\boldsymbol{B}^{(3)}$ switches sign between left and right circularly polarized electromagnetic radiation. Therefore, H and $\boldsymbol{B}^{(3)}$ constitute electromagnetic helicities of a knot, and there is also a link between $\boldsymbol{B}^{(3)}$ and the Sachs theory [1], as shown in the review [6] by Evans, linking O(3) electrodynamics and the Sachs theory.

As discussed by Rañada and Trueba [1], Faraday thought of lines of force as real and physical, and these authors represent magnetic lines of force by a complex function $\phi(t, X, Y, Z)$ of time and Cartesian coordinates. Any magnetic field [1] is therefore defined by

$$\boldsymbol{B} = g(\phi, \phi^*)\nabla\phi^*\times\nabla\phi \tag{5}$$

and any electric field by

$$\boldsymbol{E} = -g(\phi, \phi^*)(\partial_0\phi^*\nabla\phi - \partial_0\phi\nabla\phi^*) \tag{6}$$

so that the product

$$\boldsymbol{E}\cdot\boldsymbol{B} = 0 \tag{7}$$

vanishes. Electromagnetism by evolution of magnetic lines is discussed by Rañada and Trueba [1] in terms of level curves of a complex function, and leads to the appearance of a rich topological structure.

The $\boldsymbol{B}^{(3)}$ field [3] of O(3) electrodynamics is defined in terms of the cross-product of plane wave potentials $\boldsymbol{A}^{(1)} = \boldsymbol{A}^{(2)}$

$$\boldsymbol{B}^{(3)} = -ig\boldsymbol{A}^{(1)}\times\boldsymbol{A}^{(2)} \tag{8}$$

where the parameter g is

$$g = \frac{\kappa}{|\boldsymbol{A}^{(1)}|} \equiv \frac{\kappa}{A^{(0)}} \tag{9}$$

and where κ is the wave number. The modulus of the $\boldsymbol{B}^{(3)}$ field is therefore defined as

$$|\boldsymbol{B}^{(3)}| \equiv -ig(A_X^{(1)}A_Y^{(2)} - A_X^{(2)}A_Y^{(1)}) \tag{10}$$

which can be written in terms of lines of force as

$$|\boldsymbol{B}^{(3)}| = g_0\left(\frac{\partial\phi_X^{(1)}}{\partial Z}\frac{\partial\phi_Y^{(2)}}{\partial Z} - \frac{\partial\phi_X^{(2)}}{\partial Z}\frac{\partial\phi_Y^{(1)}}{\partial Z}\right) \tag{11}$$

where the phi functions are

$$\phi_Y^{(1)} = i\exp(-i(\omega t - \kappa Z)); \qquad \phi_Y^{(2)} = -i\exp(i(\omega t - \kappa Z))$$
$$\phi_X^{(1)} = \exp(-i(\omega t - \kappa Z)); \qquad \phi_X^{(2)} = \exp(i(\omega t - \kappa Z)) \tag{12}$$

and where

$$\gamma \equiv \omega t - \kappa Z \tag{13}$$

is the electromagnetic phase. Using the results:

$$\frac{\partial \phi_X^{(2)}}{\partial Z} = -i\kappa\phi_x^{(2)}; \qquad \frac{\partial \phi_Y^{(1)}}{\partial Z} = -\kappa\phi_Y^{(1)} \tag{14}$$

the modulus of the $\boldsymbol{B}^{(3)}$ field becomes

$$|\boldsymbol{B}^{(3)}| = -2ig_0\kappa^2 \tag{15}$$

and the constant g_0

$$g_0 = i\frac{A^{(0)}}{2\kappa^2} \tag{16}$$

is a function of ϕ and ϕ^* as required.

Therefore, the $\boldsymbol{B}^{(3)}$ field can be defined in terms of lines of force, and the topological considerations of Rañada and Trueba [1] can be extended to O(3) electrodynamics, and thence to the Sachs theory.

It is also shown straightforwardly that the electric equivalent of $\boldsymbol{B}^{(3)}$, the putative $\boldsymbol{E}^{(3)}$ field, vanishes, as discussed elsewhere [8]. The demonstration uses the definition:

$$\boldsymbol{E}^{(3)} = -g(\phi^{(1)}, \phi^{(2)})\left(\frac{1}{c}\frac{\partial\phi^{(1)}}{\partial t}\frac{\partial\phi^{(2)}}{\partial Z} - \frac{1}{c}\frac{\partial\phi^{(2)}}{\partial t}\frac{\partial\phi^{(1)}}{\partial Z}\right)\boldsymbol{k} \tag{17}$$

and considers a component such as the X component of the phi field. The result is

$$\boldsymbol{E}^{(3)} = -g(\phi, \phi^*)\left(\frac{1}{c}\frac{\partial\phi_X^{(1)}}{\partial t}\frac{\partial\phi_X^{(2)}}{\partial Z} - \frac{1}{c}\frac{\partial\phi_X^{(2)}}{\partial t}\frac{\partial\phi_X^{(1)}}{\partial Z}\right)\boldsymbol{k} \tag{18}$$

and the same result is obtained by considering the Y component of the phi field.

II. SUMMARY

In this short review, we have extended the topological considerations of Rañada and Trueba [1] to O(3) electrodynamics [3] and therefore also linked these concepts to the Sachs theory reviewed elsewhere in this three-volume compilation [2]. In the same way that topology and knot theory applied to the Maxwell–Heaviside theory produce a rich structure, so does topology applied to the higher-symmetry forms of electrodynamics such as the Sachs theory and O(3) electrodynamics.

Acknowledgments

The U.S. Department of Energy gratefully acknowledged for the website http://www.ott.doe.gov/electromagnetic/, which is reserved for the Advanced Electrodynamics Working Group.

References

1. A. F. Rañada and J. L. Trueba, 2nd chapter in Part 3 of this three-volume set.
2. M. Sachs, 11th chapter in Part 1 of this compilation.
3. M. W. Evans, this volume.
4. M. W. Evans, this volume.
5. M. W. Evans and L. B. Crowell, *Classical and Quantum Electrodynamics and the* $\boldsymbol{B}^{(3)}$ *Field*, World Scientific, Singapore, 2000.
6. M. W. Evans, this volume.
7. M. W. Evans, J. P. Vigier, and S. Roy, *The Enigmatic Photon*, Kluwer, Dordrecht, 1997, Vol. 4.
8. M. W. Evans, J. P. Vigier, S. Roy, and S. Jeffers, *The Enigmatic Photon*, Kluwer, Dordrecht, 1994–1999, in five volumes.