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Chapter 1. The Conjugate Product in Dirac's Electron Theory

In this third volume some of the major practical advantages of the Evans-Vigier field, $\boldsymbol{B}^{(3)}$ [1—15] are described through the concept of fermion resonance, familiar in contemporary NMR and ESR. The key to much of our development is the empirical evidence for the existence of the optical conjugate product, $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$, [16—21], where $\boldsymbol{A}^{(1)} = \boldsymbol{A}^{(2)*}$ is the complex vector potential in vacuo of electromagnetic radiation. It may be shown using Dirac's electron theory [22—25] that the classical electromagnetic field *always* interacts with the half integral fermion spin to generate the energy

$$En_{int} = -\frac{e}{m}\left(\frac{\hbar}{2}\boldsymbol{\sigma}^3\right)\cdot \boldsymbol{B}^{(3)*}, \tag{1}$$

where

$$\boldsymbol{B}^{(3)*} = \boldsymbol{B}^{(3)} = -i\frac{e}{\hbar}\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} \tag{2}$$

is the Evans-Vigier field developed in Vol. 1 and 2 [1,2] using the circular basis (1), (2), (3)) of those volumes. This result is a new general theorem of field-fermion interaction when the field is classical, and is equally valid in quantum electrodynamics, where there are only electrons and photons. In Eq. (1) and (2), which use S.I. units, e is the charge of the fermion, m its mass, $\hbar$ is the Dirac constant, $\boldsymbol{\sigma}^{(3)}$ the third Pauli spinor,

$$\boldsymbol{\sigma}^{(3)} = \boldsymbol{\sigma}_Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3}$$

and $\boldsymbol{B}^{(3)} = \boldsymbol{B}^{(3)*}$ is real and physical, a magnetic flux density [1,2] which is free of electromagnetic phase but which, nevertheless, propagates in vacuo at the speed of light,

c. Clearly, the $\boldsymbol{B}^{(3)}$ field can always be detected experimentally through its interaction with half integral fermion spin, and one of the most sensitive detection methods is resonance, akin to NMR and ESR. The $\boldsymbol{B}^{(3)}$ field is directly proportional in Eq. (2) to the optical conjugate product $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$, and is gauge independent and real, because the conjugate product itself is pure imaginary [1,2]. Equations (1) and (2) describe the transfer of energy and angular momentum from the electromagnetic field, considered classically, to the quantized fermion, for example an electron, proton or neutron. Since $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ clearly has magnetic symmetry from Eq. (2), this is a process closely akin to the original description by Dirac [26] of the motion of the electron in a classical, static magnetic field, leading to his famous explanation of the anomalous Zeeman effect and the Stern-Gerlach experiment. It is well known that this, the original relativistic quantum field theory, leads to NMR and ESR, which are essentially absorption spectroscopies [27—31] in the radio frequency and microwave regions of the spectrum.

Therefore in this first chapter we describe the origin of the conjugate product $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ in the first principles of relativistic quantum field theory, using the minimal prescription [1,2,22—25] to describe the conservation of energy and linear momentum between fermion and field. It is well known that the minimal prescription is a consequence of special relativity through local gauge invariance and charge-current conservation. In this picture, the elementary charge e is a conserved quantity which scales (or determines the magnitude and sign of) the interaction of a gauge field, the electromagnetic field, with matter. Linear momentum is generated through the product $e\boldsymbol{A}$, and potential energy through the product ecA_0, where A_0 is the scalar part of a four-vector of which $\boldsymbol{A}$ is the space component [1,2]. Conservation laws in field fermion interaction are therefore built up from a consideration not of the electric and magnetic field components of the electromagnetic wave, but from its potential four-vector A_μ. The Aharonov-Bohm effects [32—37] confirm that A_μ is physically meaningful in contemporary orthodoxy, which replaces the original nineteenth century perception of A_μ as being a mathematical intermediary leading to the d'Alembert wave equation [22—25]. It has been known since the work of Weyl [38] and others [39] that A_μ is physically meaningful in quantum mechanics and therefore in relativistic quantum field theory. From this twentieth century perspective, electric ($\boldsymbol{E}$) and magnetic ($\boldsymbol{B}$) fields are derivatives of A_μ, and are related to it in such a way as to produce force equations such as that of Lorentz [22—25]. It is well known that these results can be obtained from the principle of least action applied to field-matter interaction. Therefore, Eq. (1), although new to physics, is an entirely natural outcome of the role of the vector potential in conserving energy and momentum in field-fermion interaction. In the last analysis, it is itself a statement of conservation, based on the Noether Theorem [1,2,22—25].

In terms of beam intensity (I, W m^{-2}), otherwise known as power density, and beam angular frequency (ω, rad s^{-1}), the $\boldsymbol{B}^{(3)}$ field from Eq. (1) is [14,15]

$$\boldsymbol{B}^{(3)} = \frac{e\,\mu_0 c}{\hbar}\,\frac{I}{\omega^2}\,\boldsymbol{e}^{(3)} = 5.723 \times 10^{17}\frac{I}{\omega^2}\,\boldsymbol{e}^{(3)}, \quad (4)$$

where μ_0 is the permeability in vacuo in S.I. units $\left(\mu_0 = 4\pi \times 10^{-7}\ \mathrm{Js^2C^{-2}m^{-1}}\right)$ and where $\boldsymbol{e}^{(3)}$ is a unit vector in the (3) axis of frame ((1), (2), (3)) [1,2]. Equation (4) is derived in Chap. 2 using the well-known classical relation

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}, \quad (5)$$

between $\boldsymbol{A}$ and $\boldsymbol{B}$, a relation which originates in the principle of least action [22—25]. Thus if $\boldsymbol{A}$ is a plane wave in vacuo, so are $\boldsymbol{B}$ and $\boldsymbol{E}$, the latter being interrelated through Maxwell's equations. The plane wave $\boldsymbol{A}$ is in general complex because as such it is a solution of d'Alembert's equation in vacuo [1,2,22—25]. If $\boldsymbol{A}$ were pure real the conjugate product $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ would vanish, and this in itself is enough to show that $\boldsymbol{B}^{(3)}$ cannot be a *static* magnetic field, because a static magnetic field is defined as the curl of a pure real $\boldsymbol{A}$, as used by Dirac in his original theory of the electron [26].

For a given intensity, I, therefore, $\boldsymbol{B}^{(3)}$ is inversely proportional to the square of the electromagnetic beam's frequency, $f = \omega/(2\pi)$ (hertz or cycles s^{-1}). This inverse square frequency dependence of $\boldsymbol{B}^{(3)}$ means that it is orders of magnitude more intense (in tesla) for a given I at radio frequencies (MHz) than at visible frequencies (100 THz region). For I, for example, of 10 watts per square cm (10^5 W m^{-2}) the $\boldsymbol{B}^{(3)}$ field reaches an order of magnitude of nanotesla at 5,000 cm^{-1} (inverse centimeters or wavenumbers [40]) in the visible. In the visible under these conditions $\boldsymbol{B}^{(3)}$ is therefore one hundred thousand times weaker than the earth's mean magnetic field, but for a 10.0 MHz radio frequency field it becomes 14.5 *mega*tesla, causing proton resonance from Eq. (1) in the *infra-red* at about two thousand wavenumbers [14,15]. This result is derived in this opening chapter using the original Dirac electron theory [26] but with a complex $\boldsymbol{A}$ instead of a real $\boldsymbol{A}$. This is the only difference between our theory and Dirac's well known original [26].

Proton resonance in the infra-red is entirely out of reach in contemporary NMR practice, which is based [27—31] on permanent magnets. The highest contemporary resonance frequency in conventional NMR is typically about 0.5 GHz (500 MHz). Using Eq. (1), it is clear that NMR can be practiced at will in the much more accessible infra-red-visible frequency range, resulting in an enormous resolution enhancement achieved without the use of expensive superconducting magnets and with an ordinary Fourier transform absorption spectrometer. These advantages result directly from the appearance in Eq. (1) of the Pauli spinor $\boldsymbol{\sigma}^{(3)}$. Whenever a classically described electromagnetic field interacts with a fermion such as a proton, the $\boldsymbol{B}^{(3)}$ field produces *two* observable energy states whose existence can be traced to the topological properties of $\boldsymbol{\sigma}^{(3)}$ [2,22—25]. Proton resonance occurs when a photon, $\hbar\omega_{res}$, of a separate probe field is absorbed to induce a change between the lower and upper energy states defined by the mathematical properties of the spinor $\boldsymbol{\sigma}^{(3)}$ of Eq. (1). Electron resonance occurs similarly when the probe photon is tuned to the right frequency. These resonance processes parallel exactly those occurring in everyday NMR or ESR, but with the permanent magnetic field replaced by $\boldsymbol{B}^{(3)}$ as defined in Eq. (1). Therefore the permanent magnet of a contemporary NMR or ESR spectrometer is replaced by a circularly polarized electromagnetic field from an ordinary radio frequency or microwave generator. Not only is this process of vast potential utility, but it is also an excellent method of proving the existence of $\boldsymbol{B}^{(3)}$ experimentally. Experiments such as these would, if positive, also prove the physical nature of $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ and underline the contemporary view that the potential four-vector is physical in nature. There are compelling reasons, therefore, to pursue this type of investigation, not least of which is the empirical, i.e., experimental, evidence for the existence of $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ already available from magneto-optics [16—21].

Dirac's original methods [26] are used in Sec. 1.1 of this chapter to derive Eq. (1), whose key feature is the interaction of $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ with the spinor $\boldsymbol{\sigma}^{(3)*}$. Using Eq. (2), Eq. (1) is identical in structure with the equation first derived by Dirac [26] of the anomalous Zeeman effect. This equation gives rise to NMR and ESR, in which the fermion is respectively a nucleon (e.g. a proton), and an electron. The factor two which gives rise to the everyday term "half integral spin", is the result of a non-relativistic approximation, to be demonstrated in Sec. 1.1 below, and the fundamental reason for the existence of NMR and ESR is the topology of space itself, which allows a global distinction between SU(2) and O(3) [1,2,22—25]. The same topology allows the existence of equation (1), which gives rise to an entirely new absorption spectroscopy based on the Dirac equation itself, and allows nucleon resonance to be observed to great advantage over contemporary NMR and ESR. It is advantageous always to bear in mind that Dirac derived his equation purely from the general principles of relativity and quantum theory. These considerations force the use of four by four anti-commuting Dirac matrices [26] of which the Pauli matrices are two by two components. Therefore the fermion intrinsic spin has a deeper meaning than angular momentum, and it is well known that the fermion spin cannot be pictured classically (e.g. as a spinning object in space) because it is essentially a consequence of the topology of space itself.

There is no reason, therefore, to assume that NMR and/or ESR must always be practiced with static magnetic fields, or that a Pauli spinor must always interact with a static magnetic field, and Sec. 1.1 will show that the conjugate product $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ can be derived from first principles. This is an advance on the original phenomenological inference of Pershan [41], which first indicated that there might be an inverse Faraday effect (phase free magnetization by light [16—21]), and also an advance on approximate, semi-classical theories [42] involving the related product $\boldsymbol{E}^{(1)} \times \boldsymbol{E}^{(2)}$. It is well established experimentally [16—21] that the optical conjugate product

$$\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} = \frac{c^2}{\omega^2}\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} = \frac{1}{\omega^2}\boldsymbol{E}^{(1)} \times \boldsymbol{E}^{(2)}, \tag{6}$$

produces observable effects, prominent among which is the inverse Faraday effect. Therefore Sec. 1.1 confirms that relativistic quantum field theory produces the inverse Faraday effect. This is a reassuring result both for fundamental field theory and experimental magneto-optics.

The theory in Sec. 1.1 is based on a classical view of the electromagnetic field, whereas more rigorously, there are radiative corrections due to quantum electrodynamics (QED), on which there is an extensive literature [22—25]. In respect of electron resonance, QED leads to a 1% correction in the factor 2 appearing in Eq. (1), and so is not of central importance in resonance due to $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ or the $\boldsymbol{B}^{(3)}$ field, being a minor correction of the classical view. In the delicate understanding of photon-electron interaction however, QED is essential, and it is necessary to show that the conjugate product has a precise meaning in QED. This is left to later chapters of this volume. The experimental effects, now known to great accuracy [22—25], of the anomalous magnetic moment of the electron, first discussed by Schwinger [43], should become observable also with $\boldsymbol{B}^{(3)}$, because $\boldsymbol{B}^{(3)}$ is a physical magnetic field. The original experimental measurements, ably described by Dirac [26,44], should be repeated with a radio frequency, circularly polarized, electromagnetic field rather than a static magnetic field.

This line of reasoning can be extended to all magnetic effects in which a free fermion spin is present in a sample, and there are many of these effects now known. In each case the static magnetic field of the measuring apparatus is replaced by $\boldsymbol{B}^{(3)}$ of an ordinary radio frequency generator, provided that the radiation is circularly polarized. Finally, in Sec. 1.2 of this chapter, the original Dirac theory of Sec. 1.1 is updated and reworked consistently in contemporary notation and standard representation [2,22—25].

1.1 $\boldsymbol{B}^{(3)}$ FROM THE ORIGINAL ELECTRON THEORY OF DIRAC

Dirac's famous theory of the electron [26] is recovered exactly if $\boldsymbol{A}$ is pure real in what follows. In other words, Dirac assumes that $\boldsymbol{A}$ is real in order to recover a theory of the fermion in a static magnetic field, and to explain the elegant experiment of Gerlach and Stern [45] and the anomalous Zeeman effects [46] then known. In this section Dirac's clear and powerful description [26] is deliberately and closely followed to show the existence of $\boldsymbol{B}^{(3)}$ from the first principles of this, the original relativistic quantum field theory. Dirac's development is based [26] on the relativistic quantum wave equation

$$\left(p_0 + eA_0 - \rho_1(\boldsymbol{\sigma}\cdot(\boldsymbol{p} + e\boldsymbol{A})) - \rho_3 mc\right)\psi = 0, \tag{7}$$

for a fermion (in his case an electron) in a classical electromagnetic field. Equation (7) is written in S.I. units whereas Dirac uses Gaussian units, and this is the only difference in notation. The momentum four-vector used by Dirac is, in S.I. units

$$p_\mu := (p_0, \boldsymbol{p}), \tag{8}$$

and the potential four-vector in S.I. units is

$$A_\mu := (A_0, \boldsymbol{A}). \tag{9}$$

The matrices ρ_1 and ρ_3 are [1]

$$\rho_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \rho_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \tag{10}$$

and ψ is a column vector, described in contemporary terms [2,22—25] as the Dirac four-spinor. In his original account [26], Dirac does not use parity to interconvert components of the four-spinor, as is the contemporary practice [2,22—25].

The Hamiltonian for a classical electron in a classical electromagnetic field is now used by Dirac as a guideline to the properties of Eq. (7). The wave equation expected from analogy [26] with the classical theory is

$$\left((p_0 + eA_0)^2 - (\boldsymbol{p} + e\boldsymbol{A})^2 - m^2c^2\right)\psi = 0, \tag{11}$$

and was written from the outset [26] for a *real* $\boldsymbol{A}$. For complex $\boldsymbol{A}$ Eq. (11) becomes

$$\left((p_0 + eA_0)\left(p_0 + eA_0^*\right) - (\boldsymbol{p} + e\boldsymbol{A})\cdot(\boldsymbol{p} + e\boldsymbol{A}^*) - m^2c^2\right)\psi = 0, \tag{12}$$

because mathematically real, and therefore *physical*, quantities may always be formed from the product of a mathematically complex quantity with its complex conjugate. In order to make his theory of the electron resemble Eq. (11) as closely as possible, Dirac multiplies Eq. (7) by the factor

$$p_0 + eA_0 + \rho_1\left(\boldsymbol{\sigma}\cdot(\boldsymbol{p} + e\boldsymbol{A})\right) + \rho_3 mc, \tag{13}$$

which for a complex potential four-vector becomes

$$p_0 + eA_0^* + \rho_1\left(\boldsymbol{\sigma}\cdot(\boldsymbol{p} + e\boldsymbol{A}^*)\right) + \rho_3 mc, \tag{14}$$

giving the product

$$\begin{aligned}&\left(\left(p_0 + eA_0^*\right)\left(p_0 + eA_0\right) - \left(\boldsymbol{\sigma}\cdot(\boldsymbol{p} + e\boldsymbol{A}^*)\right)\left(\boldsymbol{\sigma}\cdot(\boldsymbol{p} + e\boldsymbol{A})\right) - m^2c^2\right. \\ &\left. - \rho_1\left(\left(p_0 + eA_0^*\right)(\boldsymbol{\sigma}\cdot(\boldsymbol{p} + e\boldsymbol{A})) - \left(\boldsymbol{\sigma}\cdot(\boldsymbol{p} + e\boldsymbol{A}^*)\right)(p_0 + eA_0)\right)\right)\psi = 0.\end{aligned} \tag{15}$$

This, in our theory, replaces Eq. (31), Chap. 11, of Ref. 26. The product (15) contains several terms which are developed as follows.

Firstly, the conjugate product term leading to Eq. (1) originates in

$$e^2(\boldsymbol{\sigma}\cdot\boldsymbol{A}^*)(\boldsymbol{\sigma}\cdot\boldsymbol{A})\psi. \tag{16}$$

As shown by Dirac [26], if $\boldsymbol{B}$ and $\boldsymbol{C}$ are any two three-dimensional vectors that commute with what is now known as the Pauli spinor $\boldsymbol{\sigma}$, then

$$(\boldsymbol{\sigma}\cdot\boldsymbol{B})(\boldsymbol{\sigma}\cdot\boldsymbol{C}) = \boldsymbol{B}\cdot\boldsymbol{C} + i(\boldsymbol{\sigma}\cdot\boldsymbol{B}\times\boldsymbol{C}). \tag{17}$$

For a pure real $\boldsymbol{A}$ there is only one term on the right hand side of Eq. (17), but for complex $\boldsymbol{A}$, there enters into the Dirac theory of the electron an all-important new term, which describes the interaction of the conjugate product $\boldsymbol{A}^{(1)}\times\boldsymbol{A}^{(2)}$ with the Pauli spinor component $\boldsymbol{\sigma}^{(3)}$. This term is inherently electromagnetic in origin, and does not exist in a static magnetic field.

Following the development by Dirac, but allowing now for complex $\boldsymbol{A}$, we set

$$\boldsymbol{p} \rightarrow -i\hbar\nabla, \tag{18}$$

and obtain the terms

1) $(\boldsymbol{p}\times\boldsymbol{p})\psi = (i\hbar)^2\nabla\times\nabla\psi = 0,$ (19)

2) $(e\boldsymbol{A} \times \boldsymbol{p})\psi = \boldsymbol{0} = -i\hbar e\boldsymbol{A} \times \nabla\psi = 0,$ (20)

3) $(e\boldsymbol{p} \times \boldsymbol{A}^*)\psi = -i\hbar e\nabla \times \boldsymbol{A}^*\psi - i\hbar e\nabla\psi \times \boldsymbol{A}^* = -i\hbar e\boldsymbol{B}^*\psi.$ (21)

The last of these uses the quantum prescription (18) to describe the interaction of the intrinsic spin of the fermion with the ordinary magnetic component of the electromagnetic field, usually described as a plane wave,

$$\boldsymbol{B}^* = \nabla \times \boldsymbol{A}^*. \tag{22}$$

For a static magnetic field, term (3) leads to the famous result [26] that the intrinsic angular momentum of a fermion is half integral in the non-relativistic (but non-classical) limit [2,22—25]. In our development, however, $\boldsymbol{B}^*$ is a plane *wave*, and averages to zero over many cycles of the field. Term three is therefore of no further interest in the derivation of Eq. (1) from first principles. We follow Dirac in discarding term one, the cross product $\boldsymbol{p} \times \boldsymbol{p}$. The term $e\boldsymbol{A} \times \boldsymbol{p}$ is also discarded, as usual [2,22—26], because it is a classical quantity multiplying a del operator on ψ. (In contrast, the term $e\boldsymbol{p} \times \boldsymbol{A}^*$ becomes $-ie\hbar\nabla \times \boldsymbol{A}^*$ which is a del operator on $\boldsymbol{A}^*$, and this is non-zero.)

Continuing in this way, we find that Dirac's original Eq. (34) of Chap. 11 of Ref. 26 is replaced for complex $\boldsymbol{A}$ by

$$\Big((p_0 + eA_0^*)(p_0 + eA_0) - (\boldsymbol{p} + e\boldsymbol{A}^*)\cdot(\boldsymbol{p} + e\boldsymbol{A}) - ie^2\boldsymbol{\sigma}\cdot\boldsymbol{A}^* \times \boldsymbol{A} - m^2c^2 - e\hbar\boldsymbol{\sigma}\cdot\boldsymbol{B}^* + i\rho_1 e\hbar\boldsymbol{\sigma}\cdot\boldsymbol{E}\Big)\psi = 0, \tag{23}$$

in which there appears the conjugate cross product term, and in which products involving $\boldsymbol{A}$ are conjugate products. The term in $\boldsymbol{\sigma}\cdot\boldsymbol{E}$ in our Eq. (23) can contain a real part if $\boldsymbol{E}$ is a complex plane wave, but the symmetry of this term is $\hat{T}$ and $\hat{P}$ negative [1,2] and does not appear to be physical. (This is the same kind of reasoning that is used to explain why there is no electric analogue of the Faraday or inverse Faraday effects.) For this reason we take no further interest here in this term. It is notable that Dirac [26] also discards this term as unphysical but for different reasons.

So the final result of our calculation is

$$\Big((p_0 + eA_0^*)(p_0 + eA_0) - (\boldsymbol{p} + e\boldsymbol{A}^*)\cdot(\boldsymbol{p} + e\boldsymbol{A}) - m^2c^2 - ie^2\boldsymbol{\sigma}\cdot\boldsymbol{A}^* \times \boldsymbol{A}\Big)\psi = 0, \tag{24}$$

in which there appears the gauge independent term $-ie^2\boldsymbol{\sigma}\cdot\boldsymbol{A}^* \times \boldsymbol{A}$ which allows

resonance to occur between the two physically distinct energy states generated by the topological properties of the spinor $\boldsymbol{\sigma}$. The conjugate cross product $\boldsymbol{A}^* \times \boldsymbol{A}$ plays the role of an entirely novel type of magnetic field, the Evans-Vigier field $\boldsymbol{B}^{(3)}$ of Eq. (1). This term vanishes if $\boldsymbol{A}$ is real, and the original Dirac theory is recovered because $\boldsymbol{B}^*$ becomes a conventional static magnetic field.

The non-relativistic, non-classical, limit of Eq. (24) can be obtained using the standard approximations introduced by Dirac [26],

$$En \sim mc^2, \quad \boldsymbol{p} \sim \boldsymbol{0}, \tag{25}$$

in which case the eigenvalue of the interaction energy between field and electron is

$$W := En - mc^2 \sim \frac{e^2c^2(\boldsymbol{\sigma}\cdot\boldsymbol{A})(\boldsymbol{\sigma}\cdot\boldsymbol{A}^*)}{En + mc^2 + ecA_0} - ecA_0. \tag{26}$$

In the radiation gauge, A_0 is zero, and in the Coulomb gauge A_0 is a constant which can be zero [22—25], and if A_0 is zero,

$$W \sim \frac{e^2}{2m}(\boldsymbol{A}\cdot\boldsymbol{A}^* + i\boldsymbol{\sigma}\cdot\boldsymbol{A} \times \boldsymbol{A}^*). \tag{27}$$

The resonance term is the second term on the right hand side of this equation, and using Eq. (2), which was derived in Vol. 2 [2], we recover Eq. (1) from this resonance term.

This development from original first principles shows that the $\boldsymbol{B}^{(3)}$ field *always* exists in field-fermion interaction provided that the field is an electromagnetic wave. In other words, we reach a new theorem which states that the electromagnetic wave interacts with intrinsic fermion spin as if the wave were a phase free magnetic flux density, the Evans-Vigier field, $\boldsymbol{B}^{(3)}$.

1.2 CONTEMPORARY DEVELOPMENT

1.2.1 MINKOWSKI NOTATION

In contemporary notation [2,22—25] it is convenient to rework Sec. 1.1 from the Dirac equation,

$$\gamma_\mu(p_\mu + eA_\mu)\psi(\boldsymbol{p}) = -mc\psi(\boldsymbol{p}). \tag{28}$$

Here ψ is Dirac's four-spinor, γ_μ is Dirac's matrix in standard representation, and $p_\mu + eA_\mu$ is the energy-momentum of the fermion in the electromagnetic field. We develop the theory in this section with Minkowski notation, as for example in the classic text by Jackson [47]. We continue to use S.I., standard, contemporary units. In vector notation, Eq. (28) becomes [2,22—25]

$$\left(\gamma_0\left(En + ecA_0\right) - c\boldsymbol{\gamma}\cdot(\boldsymbol{p} + e\boldsymbol{A})\right)\psi = mc^2\psi, \tag{29}$$

where

$$\psi = \begin{pmatrix} u \\ v \end{pmatrix}, \quad \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}, \quad \gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{30}$$

Here u and v are two-spinors (column vectors with two components) interconvertible by parity inversion, $\hat{P}$. Therefore Eq. (29) splits into two equations,

$$\begin{aligned} \left(En + ecA_0\right)u - c\boldsymbol{\sigma}\cdot(\boldsymbol{p} + e\boldsymbol{A})v &= mc^2 u, \\ -\left(En + ecA_0\right)v + c\boldsymbol{\sigma}\cdot(\boldsymbol{p} + e\boldsymbol{A})u &= mc^2 v, \end{aligned} \tag{31}$$

which describe the dynamics of the fermion in the field.

It is convenient to work in the circular basis [1,2],

$$\boldsymbol{e}^{(1)} = \boldsymbol{e}^{(2)*} = \frac{1}{\sqrt{2}}(\boldsymbol{i} - i\boldsymbol{j}), \qquad \boldsymbol{e}^{(3)} = \boldsymbol{k}, \tag{32}$$

where $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ are real, Cartesian, unit vectors respectively in X, Y, and Z of the laboratory frame. The basis (32) [1,2], has O(3) symmetry

$$\boldsymbol{e}^{(1)} \times \boldsymbol{e}^{(2)} = i\boldsymbol{e}^{(3)*}, \text{ et cyclicum}, \tag{33}$$

and is therefore a representation of the rotation group in three space dimensions. In this basis, the vector potential of the electromagnetic wave can be written as the plane wave

$$\boldsymbol{A}^{(1)} = \boldsymbol{A}^{(2)*} = \frac{A^{(0)}}{\sqrt{2}}(i\boldsymbol{i} + \boldsymbol{j})e^{i\phi}, \tag{34}$$

and so, from Eq. (5), the magnetic field can be written as

$$\boldsymbol{B}^{(1)} = \boldsymbol{B}^{(2)*} = \frac{B^{(0)}}{\sqrt{2}}(i\boldsymbol{i} + \boldsymbol{j})e^{i\phi}. \tag{35}$$

Here ϕ is the electromagnetic phase [1,2]

$$\phi = \omega t - \boldsymbol{\kappa}\cdot\boldsymbol{r}, \tag{36}$$

where ω is its angular frequency at instant t, and $\boldsymbol{\kappa}$ its wave vector at point $\boldsymbol{r}$.

Using Dirac's original approximations (25), i.e., working in the non-relativistic limit with a very slow fermion, [2,22—25], Eqs. (31) become

$$\begin{aligned} \left(En + ecA_0 - mc^2\right)u &= ec\boldsymbol{\sigma}\cdot\boldsymbol{A}v, \\ \left(En + ecA_0 + mc^2\right)v &= ec\boldsymbol{\sigma}\cdot\boldsymbol{A}u, \end{aligned} \tag{37}$$

and their Hermitian transposed counterparts become

$$\begin{aligned} u^+\left(En + ecA_0 - mc^2\right) &= ecv^+(\boldsymbol{\sigma}\cdot\boldsymbol{A})^+, \\ v^+\left(En + ecA_0 + mc^2\right) &= ecu^+(\boldsymbol{\sigma}\cdot\boldsymbol{A})^+. \end{aligned} \tag{38}$$

In these equations the spinors are row vectors and are operated upon to the left [2]. The superscript "+" denotes *Hermitian transpose* [2], which is transposition with simultaneous complex conjugation of elements. It is convenient to develop (Appendix A) the Pauli spinors in the same circular basis ((1), (2), (3)):

$$\boldsymbol{\sigma}^{(1)} = \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}^{(2)} = \boldsymbol{\sigma}^{(1)+} = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}^{(3)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{39}$$

giving the SU(2) commutative algebra

$$\left[\frac{\boldsymbol{\sigma}^{(1)}}{2}, \frac{\boldsymbol{\sigma}^{(2)}}{2}\right] = -\frac{\boldsymbol{\sigma}^{(3)+}}{2}, \text{ et cyclicum}. \tag{40}$$

The product algebra of

Appendix B, leading to

$$(\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{e}^{(2)})(\boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{e}^{(1)}) = \boldsymbol{e}^{(1)} \cdot \boldsymbol{e}^{(2)} + i\boldsymbol{\sigma}^{(3)} \cdot \boldsymbol{e}^{(1)} \times \boldsymbol{e}^{(2)}, \tag{41a}$$

$$(\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{e}^{(2)}) = (\boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{e}^{(1)})^{+}. \tag{41b}$$

We can therefore write the Dirac equations (37) and (38) as

$$\begin{aligned} (En + ecA_0 - mc^2)u &= ec\boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{A}^{(1)}v, \\ (En + ecA_0 + mc^2)v &= ec\boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{A}^{(1)}u, \end{aligned} \tag{42a}$$

$$\begin{aligned} u^{+}(En + ecA_0 - mc^2) &= ecv^{+}\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{A}^{(2)}, \\ v^{+}(En + ecA_0 + mc^2) &= ecu^{+}\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{A}^{(2)}, \end{aligned} \tag{42b}$$

in which

$$\boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{A}^{(1)} = (\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{A}^{(2)})^{+}, \qquad \boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{A}^{(2)} = (\boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{A}^{(1)})^{+}. \tag{43}$$

It is convenient to solve these equations using firstly the transverse gauge, in which $A_0 = 0$, and secondly the Dirac condition [48,49], in which $A_0 = |\boldsymbol{A}|$. In the transverse (or radiation) gauge, we obtain, with $En \sim mc^2$,

$$Wu = \frac{e^2}{2m}(\boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{A}^{(1)})^2 u, \tag{44a}$$

$$u^{+}W = \frac{e^2}{2m}u^{+}(\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{A}^{(2)})^2. \tag{44b}$$

Multiplying Eq. (44a) by Eq. (44b),

$$u^{+}W^2 u = u^{+}\left(\frac{e^2}{2m}\right)^2 (\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{A}^{(2)}\boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{A}^{(1)})^2 u. \tag{45}$$

The eigenvalue of the square of the energy of interaction of fermion and field is therefore

$$W^2 = \left(\frac{e^2}{2m}\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{A}^{(2)}\boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{A}^{(1)}\right)^2, \tag{46}$$

giving the possibility of positive and negative energy expectation values [2,22—25],

$$W = \pm\frac{e^2}{2m}\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{A}^{(2)}\boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{A}^{(1)}. \tag{47}$$

This leads to the prediction of anti-fermions as is well known and well verified experimentally [2,22—26]. For our purposes the positive expectation value of energy can be developed with the spinor algebra

$$\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{A}^{(2)}\boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{A}^{(1)} = \boldsymbol{A}^{(1)} \cdot \boldsymbol{A}^{(2)} + i\boldsymbol{\sigma}^{(3)} \cdot \boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}, \tag{48}$$

and the second term on the right hand side leads to equation (1) as we set out to show. The original and contemporary approaches to the Dirac theory produce the same, potentially very useful, result, that fermions interact with the electromagnetic wave through the $\boldsymbol{B}^{(3)}$ field.

In the radiation gauge, A_μ is not completely covariant, because A_0 is zero in one special Lorentz frame. This unsatisfactory feature of the radiation gauge has been discussed in the recent literature [1,2]. Roy and Evans [50] have recently suggested a novel gauge theory which reconciles non-zero photon mass with type two gauge invariance

$$A_0 \underset{F.A.P.P.}{\dashrightarrow} |\boldsymbol{A}|, \tag{49}$$

so that

$$A_\mu A_\mu \underset{F.A.P.P.}{\dashrightarrow} 0. \tag{50}$$

In this notation, *FAPP* denotes *for all practical purposes*, and Eq. (50) is a limiting form of, and an excellent approximation to, the condition introduced by Dirac [48,49],

$$A_\mu A_\mu = \text{constant}. \tag{51}$$

We refer to Eq. (49) as the Dirac condition. It has the advantage of rendering A_μ completely covariant and of being compatible with the Lorentz gauge [1,2,22]. It also allows the Proca equation [1,2] to be derived from a suitable Lagrangian, and so has useful properties.

The theory of electromagnetism developed in Vol. 1 and 2 augments the vacuum Maxwell equations with a system of which Eq. (2) is an example. Maxwell's original derivatives are replaced [2] by gauge invariant counterparts, and these are the same as those of the Dirac electron theory being developed in this chapter. Non-Maxwellian terms are developed within O(3) gauge theory [2] and are quadratic in A_μ, as illustrated in Eq. (2). Terms such as this are derivable in Evans-Vigier field theory [1,2] from O(3) gauge theory and are quadratic in the six field $F_{\mu\nu}$. These terms imply the use of complex fields and potentials as in de Broglie's theory of the photon [51]. The Dirac condition (49), which we are about to develop further, is an intrinsic part of the Evans-Vigier theory, some parts of which [1,2] border on the theory of finite photon mass.

The self-consistency and physical good sense of the Dirac condition can be illustrated as follows by incorporating into it Eq. (1), which becomes

$$E_{int} = \frac{e^2c^2}{En + mc^2 + ecA_0}\sigma^{(1)} \cdot A^{(2)}\sigma^{(2)} \cdot A^{(1)} - ecA_0. \tag{52}$$

This equation defines the non-relativistic limit of the interaction energy of a fermion in a circularly polarized electromagnetic field using a completely covariant A_μ defined by Eqs. (49) and (50), and which is light-like in vacuo for all practical purposes. It can be shown as follows that Eq. (52) makes sense by considering the limit

$$ecA_0 \gg (En + mc^2), \tag{53}$$

which represents a very intense, relatively low frequency, electromagnetic wave interacting with a fermion. In the approximation $En \sim mc^2$, Eq. (53) becomes [1,2]

$$\omega \ll \frac{e}{2m}B^{(0)}. \tag{54}$$

In this limit, Eq. (52) becomes

$$E_{int} \rightarrow \frac{ec}{A_0}\left(A^{(1)} \cdot A^{(2)} + i\sigma^{(3)} \cdot A^{(1)} \times A^{(2)}\right) - ecA_0, \tag{55}$$

and using

$$A^{(1)} \cdot A^{(2)} = A_0^2, \qquad A^{(1)} \times A^{(2)} = iA_0^2e^{(3)}, \tag{56}$$

we obtain

$$E_{int1} \rightarrow \mp ecA_0. \tag{57}$$

Finally, using the minimal prescription for the free photon [2],

$$eA_0 = \hbar\kappa = \hbar\frac{\omega}{c}, \tag{58}$$

Eq. (57) becomes

$$E_{int} \rightarrow \mp\hbar\omega, \tag{59}$$

which is consistent with energy conservation. The rotational kinetic energy of one photon, $\hbar\omega$, has been transferred to the fermion in an elastic collision. The $\mp$ sign in Eq. (59) is due to the topological properties of the Pauli spinor, and has no classical meaning. That is to say, the fermion can be in one of two energy states, because its intrinsic spin has eigenvalues $\pm\hbar/2$.

In the radiation gauge on the other hand, Eq. (59) can not be obtained, because the limiting condition (53) is never valid. The radiation gauge is therefore inconsistent with the physical result (59) and is inconsistent in this context with the law of conservation of energy. The origin of the inconsistency is that in the radiation gauge the potential four-vector is not consistently covariant. This leads to other well-known difficulties [1,2] in the canonical quantization of the electromagnetic gauge field. The failure of the radiation gauge in Dirac's theory of the electron becomes apparent, furthermore, only in the limit (53). In the opposite limit, which is almost always used in the development of the non-relativistic limit,

$$ecA_0 \ll (En + mc^2), \tag{60}$$

there is no practical difference between the Coulomb gauge and the Dirac condition, because $A_0 \sim 0$ is equivalent to Eq. (60). In the rest frame approximation $En \sim mc^2$, Eq. (60) becomes

$$\omega \gg \frac{e}{2m} B^{(0)}, \tag{61}$$

in which Eq. (1) is valid in both gauges. In the opposite limit, Eq. (54), Eq. (1) is no longer gauge independent in this way, meaning that the radiation gauge can no longer be used because it does not lead to energy conservation as just shown. Equation (55) is an experimentally verifiable test of the validity of Dirac's condition (49). As such, it is also important for the theory of finite photon mass [1,2], and is a prediction of the Evans-Vigier theory [1,2].

1.2.2 FEYNMAN SLASH NOTATION

In contemporary Dirac theory, field-fermion interaction is described with Feynman slash notation [22—25], which allows a compact description of Dirac algebra. In this section we demonstrate Eq. (1) using these contemporary methods, in which reduced units $\hbar = c = 1$ are used. The field free Dirac equation in this notation is

$$(i\partial\!\!\!/ - m)\psi = 0, \tag{62}$$

where $i\partial\!\!\!/ = p\!\!\!/ := \gamma^\mu p_\mu$, the reduced energy momentum four-vector. The Hermitian transpose of Eq. (62) is, in contemporary notation [22—25],

$$\bar{\psi}(i\overleftarrow{\partial\!\!\!/} + m) = 0. \tag{63}$$

The product of Eqs. (62) and (63) produces

$$\bar{\psi}(p\!\!\!/ p\!\!\!/ - m^2)\psi = 0 \tag{64}$$

in which the quantity between the spinors $\bar{\psi}$ and ψ is a scalar. We first show that Eq. (64) produces the Einstein equation of special relativity through the equivalence principle. It is convenient to proceed by expanding the product $p\!\!\!/ p\!\!\!/$,

$$p\!\!\!/ p\!\!\!/ = p^\mu p_\mu - i\sigma_{\mu\nu}p^\mu p^\nu, \tag{65}$$

where, as usual, $\sigma_{\mu\nu}$ is the commutator of Dirac matrices [22—25],

$$\sigma^{\mu\nu} := \frac{i}{2}[\gamma^\mu, \gamma^\nu]. \tag{66}$$

If it is assumed that there is no spin contribution to $p\!\!\!/ p\!\!\!/$, then,

$$p\!\!\!/ p\!\!\!/ = p^\mu p_\mu = En^2 - p^2, \tag{67}$$

where En is energy and $\boldsymbol{p}$ is linear momentum in reduced units of $\hbar = c = 1$. Therefore Eq. (64) becomes

$$\bar{\psi}(En^2 - p^2 - m^2)\psi = 0, \tag{68}$$

which in S.I. units is

$$\bar{\psi}(En^2 - p^2c^2 - m^2c^4)\psi = 0. \tag{69}$$

This is the quantum mechanical version (Klein Gordon equation) of the Einstein equation of classical relativistic dynamics,

$$En^2 = p^2c^2 + m^2c^4, \tag{70}$$

and Eq. (69) means that the expectation value of the quantity between the spinors is zero.

In the del representation, Eq. (69) is

$$\bar{\psi}(\Box + m^2)\psi = 0, \tag{71}$$

where $\Box$ is the d'Alembertian operator [1,2]. If interpreted as

$$\Box\psi_i = -m^2\psi_i, \quad i = 1, ..., 4, \tag{72}$$

Eq. (72) becomes another form of the Dirac equation with Klein-Gordon components [2] as required. If we interpret the spinor as the column vector,

$$\psi = \begin{pmatrix} \boldsymbol{E} \\ i\boldsymbol{B} \end{pmatrix}, \tag{73}$$

where $\boldsymbol{E}$ is electric field strength and $\boldsymbol{B}$ is magnetic flux density, and if we assert that the mass, m, is zero, we recover the equations

$$\Box \boldsymbol{E} = \Box \boldsymbol{B} = \boldsymbol{0}, \tag{74}$$

which become the d'Alembert equation in vacuo [52] if $|\boldsymbol{A}|$ is proportional to $|\boldsymbol{B}|$ (and thus to $|\boldsymbol{E}|$) in vacuo. This proportionality is satisfied by Eq. (5) which gives,

$$|\boldsymbol{A}| = \frac{c}{\omega}|\boldsymbol{B}| = \frac{1}{\omega}|\boldsymbol{E}|, \tag{75}$$

for plane waves in S.I. units. The representation (73) allows the Maxwell equations to be expressed in the form of a Dirac equation [22—25].

The interaction of fermion and classical field is described in contemporary slash notation using the minimal prescription in the form

$$i\partial\!\!\!/ \rightarrow i\partial\!\!\!/ - eA\!\!\!/; \qquad -i\partial\!\!\!/ \rightarrow -i\partial\!\!\!/ - eA\!\!\!/^*. \tag{76}$$

Since $p\!\!\!/ = i\partial\!\!\!/$, and $\partial\!\!\!/$ is real, the complex conjugate of $p\!\!\!/ - eA\!\!\!/$ is given in general by

$$(p\!\!\!/ - eA\!\!\!/)^* = -p\!\!\!/ - eA\!\!\!/^*. \tag{77}$$

The Hermitian transpose [2,22—25] of the Dirac equation of a fermion in a classical electromagnetic field is therefore obtained from the original by (Appendix C)

$$(p\!\!\!/ - eA\!\!\!/ - m)\psi = 0, \tag{78a}$$

$$\Downarrow$$

$$\bar{\psi}(-p\!\!\!/ - eA\!\!\!/^* - m) = 0. \tag{78b}$$

Multiplying Eq. (78a) and (78b) gives

$$\bar{\psi}(p\!\!\!/ + eA\!\!\!/^* + m)(p\!\!\!/ - eA\!\!\!/ - m)\psi = 0. \tag{79}$$

The expectation value of the scalar quantity which appears between the spinors $\bar{\psi}$ and ψ is zero. The equivalence principle shows that the classical relativistic version of Eq. (79) is

$$En^2 = p^2 + m^2 + e^2A\!\!\!/^*A\!\!\!/ + em(A\!\!\!/^* + A\!\!\!/) + e(p\!\!\!/A\!\!\!/ - A\!\!\!/^*p\!\!\!/), \tag{80}$$

and it is seen that the electromagnetic field has added several terms to the Einstein equation (70).

The resonance phenomenon which we want to demonstrate arises from the conjugate product term $e^2A\!\!\!/^*A\!\!\!/$, which is phase free [1—15] and non-zero after averaging over many cycles of the electromagnetic field at any frequency. It is not, however, a *static* property, because it propagates in vacuo at the speed of light. Terms such as $emA\!\!\!/^*$ average to zero over many field cycles and we take no further interest in them here, as in the previous section. We note that the conjugate product term has appeared from the principle of local gauge invariance, meaning that the standard minimal prescription leads to it as a consequence of current conservation and Noether's Theorem. Not surprisingly, therefore, the conjugate product term is gauge invariant, and therefore physical [1—15]. It has been assumed only that A_μ is complex, as discussed for example by Barut [52], and on countless other occasions. The term $e^2A\!\!\!/^*A\!\!\!/$ is therefore a direct outcome of the Dirac equation and for this reason should produce observable experimental effects.

In reduced units we obtain [22—25]

$$A\!\!\!/^*A\!\!\!/ = -i\boldsymbol{\sigma}^{(3)} \cdot \boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} + A_0^2 - \boldsymbol{A}^{(1)} \cdot \boldsymbol{A}^{(2)}, \tag{81}$$

where A_0 is the scalar component of the four-vector A_μ. For a non-zero A_0 in general we obtain

$$En^2 = m^2 + p^2 + e^2A_0^2 - e^2(i\boldsymbol{\sigma}^{(3)} \cdot \boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} + \boldsymbol{A}^{(1)} \cdot \boldsymbol{A}^{(2)}). \tag{82}$$

In order to proceed we use the approximation

$$(m + eA_0)^2 \sim m^2 + e^2A_0^2, \tag{83}$$

which is valid if: 1) $m \gg eA_0$, 2) $eA_0 \gg m$. The first limit is the non-relativistic one, discussed in several standard texts [22—25]. Using this approximation, and the related approximation $En \sim m$, $p \sim 0$ produces, in reduced units,

$$W := En - m \sim -i\frac{e^2}{2m}\boldsymbol{\sigma}^{(3)} \cdot \boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} + eA_0 - \frac{e^2}{2m}\boldsymbol{A}^{(1)} \cdot \boldsymbol{A}^{(2)}. \tag{84}$$

In S.I. units, this equation becomes

$$W := En - mc^2 \sim -i\frac{e^2}{2m}\boldsymbol{\sigma}^{(3)} \cdot \boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} + ecA_0 - \frac{e^2}{2m}\boldsymbol{A}^{(1)} \cdot \boldsymbol{A}^{(2)}, \tag{85}$$

for the extra energy added to the Einstein equation of the fermion by the electromagnetic field. The expected resonance term is $-i(e^2/2m)\boldsymbol{\sigma}^{(3)} \cdot \boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ and is real, physical, and gauge independent as expected.

In Chap. 2, it is described how this result leads to a new resonance technique which if developed, has the capability of making unnecessary the use of superconducting magnets in NMR and ESR. This is the result of the exact solutions to the Dirac equation that we have been discussing in this chapter.

Chapter 2. Limits and Resonance Conditions

The resonance method of detecting features outlined in Chap. 1 is potentially of great practical utility, because, as explained in this chapter, it can lead theoretically to ultra-high resolution NMR and ESR without magnets. In contemporary terms this is a seemingly impossible aim, the best available superconducting magnets produce about 20 tesla at most, resulting in resonance at about 0.5 GHz [20]. The basic reason for using high intensity magnetic fields in NMR is the need for high resolution. It will be shown that it is possible in theory to remove the magnets and at the same time achieve a much higher resolution than hitherto attainable. Furthermore, this aim can be achieved in principle with mature contemporary radio frequency technology.

2.1 LIMITS FROM THE DIRAC EQUATION

Before developing the theory of resonance which will possibly form the basis of NMR and ESR in the future, it is instructive to infer the physical meaning represented by mathematical limits of the Dirac theory of Chap. 1. In particular, it is necessary to explain how the equation self-consistently deals with seemingly disparate definitions of $\boldsymbol{B}^{(3)}$, namely,

$$1)\ \boldsymbol{B}^{(3)*} = B^{(0)}\boldsymbol{e}^{(3)} = -\frac{i}{B^{(0)}}\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)}, \tag{86}$$

$$2)\ \boldsymbol{B}^{(3)*} = -i\frac{e}{\hbar}\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}. \tag{87}$$

Definition (1) [1,2] simply puts $\boldsymbol{B}^{(3)}$ proportional to the amplitude of the magnetic component of vacuum electromagnetism, meaning that $\boldsymbol{B}^{(3)}$ is proportional to the square root of intensity (I). Definition (2) [2], which was derived rigorously in Chap. 1, has $\boldsymbol{B}^{(3)}$ proportional to the conjugate product $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ and thus to intensity itself. How can definitions (1) and (2) both apply; which should be used to describe the field-fermion interaction; and how can one definition be equivalent to the other? The answers are obtained straightforwardly as follows from a simple inspection of limits.

Definition (2) represents the usual experimental limit, under which all magneto-optic results known to us have been obtained to date. These results appear always to show that the magneto-optic phenomenon is proportional to the intensity (I) of the pump laser or electromagnetic field. Some of the key references to the available experimental data are collected in [16-21]. The lowest frequency circularly polarized pump field used to date was at three GHz, by Deschamps *et al.* [17]. In this experiment the inverse Faraday effect [16] was detected experimentally using pulsed, circularly polarized, microwaves, and the effect found to be proportional to I. The definition (2), also has $\boldsymbol{B}^{(3)}$ proportional to I, through a universal constant $e/\hbar$, and inversely proportional to the square of ω. It is now realized that this result can be shown from the first principles of relativistic quantum theory, as demonstrated in Chap. 1. In Vol. 2 it was inferred from O(3) (non-Abelian) gauge theory.

It corresponds to the limit

$$\hbar\omega \gg ecA_0, \tag{88}$$

where A_0 is the time-like component of A_μ, and where ω is the angular frequency of the field in which the fermion is immersed. In the radiation, or transverse, gauge [47], it is asserted that $A_0 = 0$, so that in this gauge the limit (88) is an exact result for finite ω. In this limit, Eq. (27) shows that the energy transferred to the fermion is second order in the vector potential, i.e., first order in the intensity, I. This result is quantum relativistic in origin, although obtained in a non-relativistic approximation as described in Chap. 1, and although the electromagnetic field is still treated classically. Classical physics cannot account for the existence of the spinor term (e.g. in Eq. (27)), nor can it account for the definition (2) of $\boldsymbol{B}^{(3)}$, a definition which includes $\hbar$, the Dirac constant of quantum mechanics [22-25]. Conservation of energy-momentum is represented in the transverse gauge by an equation such as (27), but the gauge is not manifestly covariant [22-25] and produces well known difficulties in canonical quantization of the field. As discussed in Chap. 1, conservation of energy-momentum in the transverse gauge is also not entirely satisfactory. These caveats apply, however, to a well accepted gauge assumption [47] rather than to the method being used, and it can be shown (next section) that the classical limit of an equation such as (1) is precisely that obtained by Talin *et al.* [54] for the inverse Faraday effect and is precisely that predicted by the classical, but relativistic, Hamilton-Jacobi equation of one classical electron or proton in a field [1]. This is discussed further in Sec. 2.2, where the Dirac equation is transformed into a Pauli equation, and finally into the Hamilton-Jacobi equation [1]. The inverse Faraday effect in the limit (88) is proportional to I as observed [16-21] to date and can be understood through $\boldsymbol{B}^{(3)}$ in definition (2). This high field frequency definition should almost always be used in experimental applications.

The definition (1) of $\boldsymbol{B}^{(3)}$ was the first to be inferred heuristically [1,2], however, and from the work of Chap. 1 it is clear that it applies in the limit,

$$ecA^{(0)} \to \hbar\omega, \qquad A^{(0)} = (\boldsymbol{A}\cdot\boldsymbol{A}^*)^{1/2} \to A_0. \tag{89}$$

In this limit, nearly all the energy-momentum of the photon (anticipating the quantized interpretation of the field) is transferred to the fermion, which is thereby accelerated very close to the speed of light. As discussed in Chap. 1, and in Vol. 1 and 2 [1,2], this is a low frequency, high intensity limit,

$$\omega \ll \frac{e}{2m}B^{(0)}, \tag{90}$$

and since A_0, the time-like component of A_μ, cannot be zero in this view, this limit cannot be attained in the transverse or radiation gauge. Using it in Eq. (1) produces, with definition (1), the result,

$$En_{int} \xrightarrow[ecA_0 \to \hbar\omega]{} -\frac{eB^{(0)}}{m}\frac{\hbar}{2}\boldsymbol{\sigma}^{(3)}\cdot\boldsymbol{e}^{(3)*}, \tag{91}$$

which shows that the energy formed between field and fermion in the limit (89) is proportional to the square root of I and ultimately becomes independent of ω. This is experimentally distinguishable from Eq. (1), which was obtained in the opposite limit of the Dirac equation, although no experimental investigations have been made to our knowledge under the required conditions of very low pump frequency, very high pump intensity. These investigations would show experimentally whether or not the Dirac condition is more representative of free space electromagnetism than the transverse gauge.

It has been demonstrated theoretically, therefore, that limiting forms of the Dirac equation produce both the heuristic definition of $\boldsymbol{B}^{(3)}$ (definition (1)), and the more recent [15] definition (2) appearing in Eq. (1) from fundamental first principles of relativistic quantum mechanics. Clearly, definition (1) of $\boldsymbol{B}^{(3)}$ can only be used under the right conditions, represented by Eq. (89); and $\boldsymbol{B}^{(3)}$ cannot be defined at all under these conditions in the transverse gauge. Self consistently, therefore [1,2], the definition (1) of $\boldsymbol{B}^{(3)}$ requires a finite time-like component of A_μ ($= (A_0, \boldsymbol{A})$). Definition (2) can however be arrived at if the time-like component is zero. In both cases it is possible to retain the useful condition

$$A_\mu A_\mu \xrightarrow[FAPP]{} 0, \tag{92}$$

recently suggested by Roy and Evans [55] as a means of reconciling the theory of type two gauge invariance (which we support) and the theory of finite photon mass [1,2]. Under the Dirac condition (of which Eq. (92) is a limit [55]),

$$\boldsymbol{A}^{(1)}\cdot\boldsymbol{A}^{(1)*} + \boldsymbol{A}^{(2)}\cdot\boldsymbol{A}^{(2)*} + \boldsymbol{A}^{(3)}\cdot\boldsymbol{A}^{(3)*} = A_0^2, \tag{93}$$

can always be obtained if the magnitude of the space-like and time-like components of A_μ are equal. For all practical purposes A_μ is a fully covariant, *light-like*, four-vector in this view. In the transverse gauge, the Roy-Evans condition (92) applies if

$$A^{(0)2}\left(e^{(1)}\cdot e^{(2)*}+e^{(2)}\cdot e^{(2)*}\right)+A^{(3)2}\left(e^{(3)}\cdot e^{(3)*}\right)=0, \qquad A^{(3)}=\sqrt{2}\,iA^{(0)}, \tag{94}$$

i.e., if a pure imaginary and phase free third component is introduced for the space part of A_μ, there being by definition no time-like component [47]. The longitudinal component of $\boldsymbol{A}$ in this view is unphysical, and so the usual assertion that the radiation gauge [47] is pure transverse is retained in the usual physical applications. Therefore we arrive at the rather unexpected conclusion that a *massless* particle may be described mathematically with three states of polarization, provided that only two of these are physical. In the usual Wigner theory [56], there can be only two physical polarization states, and these must be transverse. The Wigner theory does not appear to prohibit a mathematically imaginary longitudinal state of polarization, even for a *massless* particle.

The very concept of a *massless* particle, however, is fraught with great difficulty, because mass is a Lorentz invariant scalar [1,2]. A good text such as that by Marion and Thornton [57] explains clearly that, for this reason, mass does not change upon Lorentz frame transformation any more than it changes from one Cartesian frame to another. The term *rest mass* is misleading therefore because mass in the rest frame is the same as mass in any other Lorentz frame [57]. It is well known that mass is one of the two *invariants* of the Poincaré group [23], the other being loosely referred to as *spin*. The commonly held notion that mass must go to zero for a particle travelling at c is therefore a gross misconception. It is truer to assert that for a particle mathematically translating at c there is no rest frame, and so the meaning of transformation from one frame to another is lost. In other words, there is only one frame possible, and the Lorentz transformation is not defined from a frame in which a particle is moving at less than c to a frame in which it is moving at c, because if the mass is invariant, the transformation produces a zero in the denominator of some transformed quantities. A dynamical quantity can become mathematically infinite if Lorentz transformed into a frame moving at c and such a transformation cannot be physically meaningful. This is easily checked by identifying the relative frame velocity, v, appearing in the transformation [1,2] with c. Once the concept of massless particle is abandoned, derivative deductions such as a massless particle having two degrees of polarization also become physically meaningless. Therefore if a photon is indeed a particle (which is open to serious doubt [6]) it is massive and has three degrees of polarization in three dimensional space.

The artificial nature of the radiation gauge [47] is also discussed in Ref. 1 and 2 and Chap. 1. Its use in canonical quantization leads to serious difficulties because it is not fully covariant. The gauge is an assertion [47] made for convenience. In Dirac's

original development of his electron theory [26] it is not used, because the time-like component of A_μ is finite. The Roy-Evans condition (92) is always valid in the Dirac electron theory if

$$A^{(0)2}\left(e^{(1)}\cdot e^{(1)*}+e^{(2)}\cdot e^{(2)*}\right)\rightarrow A_0^2-A^{(3)2}\left(e^{(3)}\cdot e^{(3)*}\right), \tag{95}$$

and this is always possible, not surprisingly, because the Roy-Evans condition (92) is a limiting form of the Dirac condition itself [48,49],

$$A_\mu A_\mu = \text{constant}. \tag{96}$$

The condition

$$A_\mu A_\mu = 0, \tag{97}$$

should not therefore be referred to as *light-like* in general because it can apply to *any* four-vector. This mathematical result shows that the concept of zero mass for a particle travelling at the speed of light is meaningless.

Our discussion of limits is summarized in Table 1, which is also a summary of the properties of fermion spin in a classical, circularly polarized, electromagnetic field. The Table shows that the intrinsic spin of the fermion always interacts with the $\boldsymbol{B}^{(3)}$ field in one form or another. The most general result is (I), and (II) and (III) are well defined limits of the same fundamental equation. Limit (II) is reached when the fermion energy is much greater than the energy ecA_0. In the radiation gauge, A_0 is zero by definition [47], in which case (I) and (II) become identical and (III) can never be reached because in relativity mc^2 is never zero. That $\boldsymbol{B}^{(3)}$ is much more than a mathematical convenience is revealed by a careful consideration of limit (III), in which the energy ecA_0 is much greater than the total fermion energy, including its rest energy, mc^2. In this limit,

$$ecA_0 \gg \left(En + mc^2\right) \sim 2mc^2. \tag{98}$$

TABLE 1

Limits from the Dirac Equation

	Limit	Description	$\boldsymbol{B}^{(3)}$ Field
I	$(En + mc^2) \sim ecA_0$, $\hbar\kappa = eA^{(0)}$, eA_0	Energy ecA_0 and fermion energy about the same.	$\boldsymbol{B}^{(3)*} \sim -ix\frac{e}{\hbar}\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$
II	$(En + mc^2) \gg ecA_0$ $\hbar\kappa = eA^{(0)} \gg eA_0$	Initial fermion energy much greater than ecA_0.	$\boldsymbol{B}^{(3)*} \to -i\frac{e}{\hbar}\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$
III	$ecA_0 \gg (En + mc^2)$, $\hbar\kappa = eA^{(0)} \to eA_0$	Energy ecA_0 much greater than initial fermion energy.	$\boldsymbol{B}^{(3)*} \to -\frac{i}{B^{(0)}}\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)}$

$$x = 2mc^2/(En + mc^2 + ecA_0)$$

We have seen in Chap. 1 that in the limit (III), and in the quantized interpretation of the field, the photon energy $\hbar\omega$ is transferred wholly to the fermion for all practical purposes (F.A.P.P.),

$$\hbar\omega \to ecA_0, \tag{99}$$

and if

$$A_0 \to A^{(0)} := (\boldsymbol{A}\cdot\boldsymbol{A}^*)^{1/2}, \tag{100}$$

then

$$\boldsymbol{B}^{(3)*} \to -\frac{i}{B^{(0)}}\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)}, \tag{101}$$

where

$$B^{(0)} = \frac{\omega}{c}A^{(0)}. \tag{102}$$

It is important to note that Eq. (101) *is a result of the Dirac equation*, provided that Eq. (98) and (99) are used as limits. They imply,

$$(\hbar\omega)_{\text{photon}} \gg (2mc^2)_{\text{fermion}}, \tag{103}$$

which can be interpreted physically to mean that the energy acquired by the fermion in a F.A.P.P. elastic collision with a photon is much greater than twice the fermion's initial rest energy. If we prefer to retain the classical interpretation of the field, Eq. (103) is

$$ecA_0 \gg 2mc^2. \tag{104}$$

Equation (101) has also been deduced heuristically [1], and implies that $\boldsymbol{B}^{(3)}$ is a property of the free photon — the photomagneton operator [1,2] of the quantized field. Table 1 shows that $\boldsymbol{B}^{(3)}$ in this limit can be isolated experimentally only under the appropriate conditions, also discussed in Vols. 1 and 2 [1,2]. These conditions must be such that the electromagnetic field is of very great intensity, so that its ecA_0 is much greater than the initial fermion energy. In order for this to be possible at all, we must follow Dirac [26] and use a *non-zero* A_0 in the theory. Furthermore, we must use $A^{(0)} \to A_0$ as in Eq. (100), and this is precisely the Roy-Evans condition [55], a limiting form of the Dirac condition [48,49] itself.

In classical electrodynamics [47] it is well known that the classical electron, when translating infinitesimally near c, radiates plane waves indistinguishable from transverse electromagnetic waves under this condition. This conclusion can be reached using special relativity [47]. Therefore, in quantum mechanics, if a fermion (electron) were to acquire $\hbar\omega$ as in limit (III), it would *become a photon*. In this state *it would retain the* $\boldsymbol{B}^{(3)}$ *field* defined in limit (III), a definition in which the electron's charge, e, has been incorporated into the scalar amplitude $B^{(0)}$. Conservation of C symmetry in this exchange has been discussed in Ref. 1; a symmetry conservation which requires, consistently, both e and $B^{(0)}$ to be C negative. Conservation of angular momentum [1-15] in classical physics also requires the $\boldsymbol{B}^{(3)}$ of the incoming electromagnetic field to be transferred to the classical electron, and in an elastic exchange the angular momentum is transferred entirely to the electron along with the $\boldsymbol{B}^{(3)}$ field. Any other conclusion for a finite ecA_0 would be in violation of the Dirac equation, because the latter conserves energy and angular momentum.

It can always be asserted that A_0 is zero, as is the accepted practice [47] in the transverse, or radiation, gauge, and this assertion can be used to force us to the flawed conventional wisdom that $\boldsymbol{B}^{(3)}$ *does not exist*. As discussed in Chap. 1, this is in violation of the principle of conservation of energy if it is assumed, reasonably, that a photon's incoming energy is $\hbar\omega$, and that under well defined experimental conditions this can be transferred, F.A.P.P. in its entirety, to a fermion in an elastic collision. However, this same assertion, that A_0 is zero, has caused fundamental difficulties [1,2,22—25] ever since the first attempts were made to apply the principles of canonical quantization [22] to the classical electromagnetic field. Therefore, an attempt to assert that $\boldsymbol{B}^{(3)}$ in limit (III) is zero on the grounds that A_0 must be zero by assertion will run into exactly the same problems. Such an assertion also spoils the symmetry of Table 1, in that it forces limit (I) to become the same as limit (II), and makes limit (III) impossible if it is accepted that rest energy, mc^2, can never be zero. It also means that A_μ, which is now accepted to have physical meaning in *classical* electrodynamics [58], must always have a time-like component which is zero. This cannot be true for a four-vector in general, because as soon as Lorentz transformation is applied, an initially zero time-like component can become non-zero in another Lorentz frame. It can only be true if the mass associated with A_μ is rigorously zero, so that Lorentz transformation *loses meaning* because there is no rest frame [1,57]. (Alternatively, we can think of a zero mass particle as implying the existence of only one Lorentz frame, so that Lorentz transformation is by definition prohibited.) However, we have already argued that the notion of a zero mass particle is physically *meaningless*, and the photon as particle should be interpreted as the originators of special relativity probably intended, as an aid to understanding. It has been pointed out repeatedly down the years [59,60] that the photon is an enigma. Much of the received wisdom about the photon as particle is flawed, as pointed out, for example, by Einstein [61]. Table 1, based on an elementary development of the Dirac equation of a fermion in the classical field, shows that the assertion that $\boldsymbol{B}^{(3)}$ is zero [62—65] is incorrect.

The conservation of C symmetry in Table 1 leads to an interesting line of thought that will be pursued later in this volume. In agreement with Noether's Theorem [22—25] and the much older Ampère Hypothesis [66] C conservation shows that the symmetry of e and $\boldsymbol{B}^{(0)}$ are both negative. The photon as an idea in natural philosophy must allow for elementary charge to be present in its theoretical framework. This is simply an expression of charge-current conservation and the contemporary orthodoxy that elementary charge is a conserved quantity in nature [22—25]. The notion that the photon is an uncharged particle [67] must therefore be based on the idea that the photon shows no *net* charge, because if charge is elementary and conserved in nature, it cannot be annihilated. In the received view of electron-positron annihilation [67] *two* photons are produced,

$$e^- + e^+ \rightarrow 2\gamma . \tag{105}$$

The charge on the electron is elementary and negative, that on the positron is elementary and positive, so it is asserted usually that each of the two photons produced on the right hand side of equation (105) must be uncharged. This view means that the charge and mass present on the left hand side of Eq. (105) have disappeared from the right hand side, because conventionally, photons are also claimed to be massless particles [22—25] travelling at exactly the speed of light in vacuo, c.

If mass and charge are elementary and conserved in nature, however, they cannot be annihilated, which leads us to the view that the electron and positron combine to form a single entity, a new type of photon, which translates infinitesimally close to c, and which is made up of a combination of plus and minus e. This new photon structure therefore rests on equal and opposite elementary charge and allows for the smallest possible unit of elementary mass to be present, the mass of the photon [68]. In this view rest energy is mc^2, as usual, but m can never be zero, because if so, rest energy would vanish entirely, in contradiction to the principles of special relativity. In the conventional view, mass *can* be zero by assertion, and in a *massless particle*, there is zero rest energy and no rest frame. In the conventional view, therefore, there is no notion of elementary mass (an indivisible, constant, universal and scalar unit of mass), because mass is allowed under certain circumstances to disappear. If so, there cannot be an indivisible and constant elementary unit, and in our view, this conclusion is unphysical. Mac Gregor [69] has recently discussed the history of the new type of photon which we wish to develop, a photon which can be shown to be consistent with relativity. For example, French [70] has shown that equal and opposite elementary charges, co-propagating exactly at c, can never attract, so that this photon can never collapse in on itself. In a photon with our very small elementary mass travelling infinitesimally near c, the attraction is very weak and only very gradually will the photon collapse, as it slows below c. This process becomes noticeable [68] only after the photon has travelled millions of light years. It is well known [68] that Tolman described this eventuality many years ago, and coined the term *tired light*. More contemporary texts such as that of Itzykson and Zuber also discuss photons with finite mass [23]. It will be shown later in this volume that this photon implies the existence of the $\boldsymbol{B}^{(3)}$ field if one charge is situated at the origin and the other at the tip of the rotating electric or magnetic field vector of the classical electromagnetic field.

2.2 THE PAULI EQUATION AND OTHER LIMITS OF THE DIRAC EQUATION

By reducing the Dirac equation to the Pauli equation [22], it can be shown that the effect of $\boldsymbol{A} \times \boldsymbol{A}^*$ on a fermion is indistinguishable in general from that of a magnetic field, provided that the latter is interpreted properly as discussed in Sec. 2.1. The Pauli equation is obtained from the Dirac equation (23) as follows, in a non-relativistic approximation. Various limits of the Pauli equation are then discussed in this section.

We consider as starting point equation (23), with

$$En = p_0 c , \tag{106}$$

and consider that $A_0 = A_0^*$ is the real, scalar, amplitude of the time-like component of the four-vector A_μ. In the transverse gauge this amplitude is zero by assertion, but more generally it is non-zero, as used by Dirac [26]. Therefore Eq. (23) becomes

$$\begin{aligned}\big((En - mc^2 + ecA_0)(En + mc^2 + ecA_0) - c^2(\boldsymbol{p} + e\boldsymbol{A}^*)\cdot(\boldsymbol{p} + e\boldsymbol{A})\\ - ec^2\hbar\boldsymbol{\sigma}\cdot\boldsymbol{B}^* - ie^2c^2\boldsymbol{\sigma}\cdot\boldsymbol{A}^*\times\boldsymbol{A}\big)\psi = 0,\end{aligned} \tag{107}$$

in which $\boldsymbol{B}^*$ is the oscillating magnetic component of the classical electromagnetic field, and in which $\boldsymbol{A}\times\boldsymbol{A}^*$ is phase free. This is a Dirac wave equation in which the eigenfunction is a four-spinor, ψ. It reduces to a Pauli equation in the non-relativistic limit in which $En \sim mc^2$, with the additional condition,

$$(En + mc^2) \gg ecA_0, \tag{108}$$

As we have seen, this condition is exact if we choose to work in the transverse gauge, but then we introduce difficulties in canonical quantization and energy conservation as discussed already.

In the limit (108) we obtain the Pauli wave equation in the form

$$\begin{aligned}(En - mc^2)\psi \sim \bigg(\frac{1}{2m}(\boldsymbol{p} + e\boldsymbol{A}^*)\cdot(\boldsymbol{p} + e\boldsymbol{A}) - ecA_0\\ + \frac{e}{2m}(\hbar\boldsymbol{\sigma}\cdot\boldsymbol{B}^* + ie\boldsymbol{\sigma}\cdot\boldsymbol{A}^*\times\boldsymbol{A})\bigg)\psi.\end{aligned} \tag{109}$$

This is a wave equation in a four-spinor ψ, whose two two-spinor components are interchangeable by parity inversion [2,22—25]. It is similar in structure to the Schrödinger wave equation and its four components are Klein-Gordon equations [22—25]. The equation (109) differs from the standard Pauli equation only in one respect, that $\boldsymbol{A}$ is complex. Since $\boldsymbol{A}$ is accepted as complex in the literature [47], our new term $ie\boldsymbol{\sigma}\cdot\boldsymbol{A}^*\times\boldsymbol{A}$ will lead in theory to novel resonance effects described later in this chapter.

Equation (109) is a Pauli equation for one fermion in a circularly polarized electromagnetic field, and as such, it is able to describe both the intrinsic spin ($2\boldsymbol{S}$) and orbital ($\boldsymbol{L}$) angular momentum of the electron through the well known factor $\boldsymbol{L} + 2\boldsymbol{S}$. There are several limits of the Pauli equation which are well discussed in the standard texts [22—25], but which are considered here in order to explore the effect of the new magnetic term in $\boldsymbol{\sigma}\cdot\boldsymbol{A}\times\boldsymbol{A}^*$. The positive energy solutions of the terms

$$(En - mc^2)\psi = \left(\frac{1}{2m}(\boldsymbol{p} + e\boldsymbol{A}^*)\cdot(\boldsymbol{p} + e\boldsymbol{A}) - ecA_0\right)\psi, \tag{110}$$

give the classical Hamiltonian as an expectation value of this wave equation

$$H_{\text{class}} = \frac{1}{2m}(\boldsymbol{p} + e\boldsymbol{A}^*)\cdot(\boldsymbol{p} + e\boldsymbol{A}) - ecA_0. \tag{111}$$

The classical result (111) identifies ecA_0 as a kinetic energy and shows that the vector potentials $\boldsymbol{A}$ and $\boldsymbol{A}^*$ also contribute to the kinetic energy. The rest energy mc^2 has been incorporated into the factor two of the classical Eq. (111) in our non-relativistic limit. The first kinetic energy term in Eq. (111) can be expanded as

$$H_{\text{class}} = \frac{1}{2m}(\boldsymbol{p}\cdot\boldsymbol{p} + e^2\boldsymbol{A}^*\cdot\boldsymbol{A} + e(\boldsymbol{p}\cdot\boldsymbol{A} + \boldsymbol{A}^*\cdot\boldsymbol{p})), \tag{112}$$

where $\boldsymbol{p}$ is the initial, non-relativistic, fermion linear momentum. Averaging over many cycles of the field leaves

$$\langle H_{\text{class}}\rangle = \frac{1}{2m}(\boldsymbol{p}\cdot\boldsymbol{p} + e^2\boldsymbol{A}^*\cdot\boldsymbol{A}), \tag{113}$$

an equation which has the same dynamical content as the equation first derived by Volkow in 1935, using a different route [22], of the Dirac electron in a linearly polarized plane wave. In Volkow's notation [22],

$$\overline{A^2} = \boldsymbol{A}^*\cdot\boldsymbol{A} = \boldsymbol{A}^{(2)}\cdot\boldsymbol{A}^{(1)}. \tag{114}$$

This checks that our own derivation of the Pauli equation and its above classical limit is consistent with literature derivations. The latter, however, miss the key new term in $\boldsymbol{A}\times\boldsymbol{A}^*$, which in our Pauli equation (109) multiplies the Pauli spinor and therefore gives rise to *two* energy states between which resonance can be made to occur. This is a perfectly general result that applies to all types of fermions, including nucleons, so that $\boldsymbol{A}\times\boldsymbol{A}^*$ can cause nuclear resonance as well as electron resonance.

If we reinstate this term into Eq. (110),

$$(En - mc^2)\psi = \frac{1}{2m}\left(\boldsymbol{p}\cdot\boldsymbol{p} + e^2(\boldsymbol{A}\cdot\boldsymbol{A}^* - i\boldsymbol{\sigma}\cdot\boldsymbol{A}\times\boldsymbol{A}^*) - ecA_0\right)\psi. \quad (115)$$

The term in $\boldsymbol{B}^*$ has been left out because it vanishes after averaging over many cycles of the field [47], $\boldsymbol{B}^*$ being a classical plane wave component, and not a static magnetic field as in the standard derivations of the anomalous Zeeman effect [22—25]. The term in $\boldsymbol{A}\times\boldsymbol{A}^*$, however, is non-zero after averaging over many field cycles, simply because it is free of the electromagnetic phase [1,2]. Equation (2) of Chap. 1 shows that it is directly proportional to a new type of magnetic field, $\boldsymbol{B}^{(3)}$. Using Eq. (2) reduces Eq. (115) to a new type of Pauli equation [22],

$$(En - mc^2)\psi = \left(\frac{1}{2m}\boldsymbol{p}\cdot\boldsymbol{p} + \frac{e}{2m}\hbar\boldsymbol{\sigma}^{(3)}\cdot\boldsymbol{B}^{(3)*} + \frac{e^2}{2m}\boldsymbol{A}\cdot\boldsymbol{A}^* - ecA_0\right)\psi, \quad (116)$$

in which the Evans-Vigier field, $\boldsymbol{B}^{(3)}$, is the magnetic field.

Obviously, $\boldsymbol{B}^{(3)}$ in this equation must be interpreted as in Eq. (2), and Table 1 defines the condition under which this interpretation is applicable. The Pauli equation (116) has been obtained from the Dirac equation (23), in which the standard property was used that $\boldsymbol{A}$ is complex [47] in general. The standard Pauli equation can be reduced [22] to the form,

$$(En - mc^2)\psi = \left(\frac{1}{2m}\boldsymbol{p}\cdot\boldsymbol{p} - \frac{e}{2m}(\boldsymbol{L} + 2\boldsymbol{S})\cdot\boldsymbol{B} - ecA_0\right)\psi, \quad (117)$$

in which $\boldsymbol{B}$ is the standard static magnetic field, where $\boldsymbol{L}$ denotes the electron's orbital angular momentum, a classical concept, and in which $2\boldsymbol{S}$ denotes its spin angular momentum with $\boldsymbol{S}$ defined as $\boldsymbol{\sigma}/2$. It is well known that $2\boldsymbol{S}$ does not appear in classical physics. The reduction of Eq. (116) to this general form can be achieved with the Hamilton-Jacobi formalism of Chap. 12 of Vol. 1, a formalism which allows the calculation of the orbital angular momentum of the fermion through the classical

$$L_Z = Xp_Y - Yp_X. \quad (118)$$

Here $\boldsymbol{p}$ is the linear momentum and $\boldsymbol{r}$ $(:= X\boldsymbol{i} + Y\boldsymbol{j} + Z\boldsymbol{k})$ the position of the fermion in three dimensional space. The momentum components are given by [1],

$$p_X = eA^{(0)}\cos\omega t, \qquad p_Y = -eA^{(0)}\sin\omega t, \quad (119)$$

and the position components by

$$X = -\frac{ecA^{(0)}}{\gamma\omega}\sin\omega t, \qquad Y = -\frac{ec}{\gamma\omega}A^{(0)}\cos\omega t, \quad (120)$$

where

$$\gamma^2 := m^2c^2 + e^2A_0^2, \quad (121)$$

is the relativistic factor. In the non-relativistic limit [1,54],

$$\gamma \rightarrow mc, \quad (122)$$

and the angular momentum of the fermion in the field is

$$L_Z = \frac{e^2A^{(0)2}}{m\omega}. \quad (123)$$

Incorporating this into the Pauli equation (116) gives in the limit Eq. (122),

$$W\psi = \left(\frac{1}{2m}\boldsymbol{p}\cdot\boldsymbol{p} + \frac{1}{2}(1 + \boldsymbol{\sigma}^{(3)*}\cdot\boldsymbol{e}^{(3)})\,\omega L_Z\right)\psi. \quad (124)$$

The physical meaning of this equation is found through comparison with the work of Talin *et al.* [54], whereupon it becomes clear that the term in $\omega L_Z/2$ describes the *inverse Faraday effect* [1,2] for one fermion. Talin *et al.* [54] left out of consideration the spinor term in $(1/2)\boldsymbol{\sigma}^{(3)*}\cdot\boldsymbol{e}^{(3)}\omega L_Z$ because the conjugate product $\boldsymbol{A}^{(1)}\times\boldsymbol{A}^{(2)}$ was unconsidered in their classical theory. The quantum relativistic theory developed here for the first time reveals the existence of the spinor term.

If we use the definition (2) for the field $\boldsymbol{B}^{(3)}$ we obtain

$$L_Z = \frac{\hbar}{\omega}\frac{e}{m}B_Z, \qquad \boldsymbol{B}^{(3)} := B_Z\boldsymbol{e}^{(3)}, \quad (125)$$

i.e., the angular momentum of the fermion is proportional directly to $\boldsymbol{B}^{(3)}$. The final form of the Pauli equation is therefore

$$W\psi = \left(\frac{1}{2m}\boldsymbol{p}\cdot\boldsymbol{p} + \hbar\frac{e}{2m}(1 + \boldsymbol{\sigma}^{(3)*}\cdot\boldsymbol{e}^{(3)})B_Z\right)\psi, \tag{126}$$

which shows that the Evans-Vigier field is responsible for the inverse Faraday effect, and for a novel, as yet undetected effect due to the fermion's intrinsic spin. The following sections develop resonance theory based on the existence of this new term. We shall see that in theory, this term makes conventional NMR and ESR obsolete, because the $\boldsymbol{A}^{(1)}\times\boldsymbol{A}^{(2)}$ property of a radio frequency field can be substituted for the static magnetic field of a permanent magnet, one of the most costly parts of a contemporary NMR spectrometer.

It should be noted that despite the presence of the Dirac constant in Eq. (126), the $\boldsymbol{B}^{(3)}$ field has been defined as proportional to the inverse of $\hbar$, so the all important new term in $\boldsymbol{\sigma}^{(3)*}$ is, overall, independent of $\hbar$. It is not classical, however, because spinors do not appear in the classical understanding [54] of the inverse Faraday effect. This term is new to physics, and has many interesting properties, theoretical and practical.

2.3 RESONANCE EQUATIONS IN NOVEL NMR AND ESR SPECTROSCOPIES

Equation (115) shows that the intrinsic spin of a fermion forms an interaction energy with the optical property $\boldsymbol{A}\times\boldsymbol{A}^*$, and that the classical angular momentum of the fermion also contributes to this fundamental process of field-fermion interaction through $\boldsymbol{A}\cdot\boldsymbol{A}^*$. The latter process was first described by Talin *et al.* [54] when developing the classical theory of the inverse Faraday effect. The conjugate cross product $\boldsymbol{A}\times\boldsymbol{A}^*$ is inversely proportional to the square of the field angular frequency ω. These properties make it ideal for the development of powerful new resonance spectroscopies, and in this section the fundamental equations are given of the optical equivalent of NMR and ESR.

In S.I. units the fundamental equation linking $\boldsymbol{A}$ to a magnetic field $\boldsymbol{B}$ is, in classical electrodynamics [47],

$$\boldsymbol{B} = \nabla\times\boldsymbol{A}. \tag{127}$$

So if $\boldsymbol{A}$ is a plane wave in vacuo then so is $\boldsymbol{B}$ (and its electric counterpart $\boldsymbol{E}$). If the plane wave $\boldsymbol{A}$ is a solution of the vacuum d'Alembert equation then it may be written

as

$$\boldsymbol{A}^{(1)} = \boldsymbol{A}^{(2)*} = \frac{A^{(0)}}{\sqrt{2}}(i\boldsymbol{i} + \boldsymbol{j})e^{i\phi}. \tag{128}$$

From Eq. (127), the plane wave $\boldsymbol{B}$ is

$$\boldsymbol{B}^{(1)} = \boldsymbol{B}^{(2)*} = \frac{\omega}{c}\boldsymbol{A}^{(1)} = \frac{B^{(0)}}{\sqrt{2}}(i\boldsymbol{i} + \boldsymbol{j})e^{i\phi}, \tag{129}$$

and using the classical vacuum Maxwell equation, the plane wave $\boldsymbol{E}$ is

$$\boldsymbol{E}^{(1)} = \boldsymbol{E}^{(2)*} = \frac{E^{(0)}}{\sqrt{2}}(\boldsymbol{i} - i\boldsymbol{j})e^{i\phi}. \tag{130}$$

Here ϕ is the electromagnetic phase [1,2], $A^{(0)}$, $B^{(0)}$ and $E^{(0)}$ are scalar amplitudes, and $\boldsymbol{i}$ and $\boldsymbol{j}$ are unit Cartesian vectors in X and Y, perpendicular to the propagation direction Z of the wave. The following key relations then follow using elementary algebra,

$$\boldsymbol{A}^{(1)}\times\boldsymbol{A}^{(2)} = \frac{c^2}{\omega^2}\boldsymbol{B}^{(1)}\times\boldsymbol{B}^{(2)} = \frac{1}{\omega^2}\boldsymbol{E}^{(1)}\times\boldsymbol{E}^{(2)}, \tag{131}$$

and show that the product $\boldsymbol{A}^{(1)}\times\boldsymbol{A}^{(2)}$ is proportional to $\boldsymbol{B}^{(1)}\times\boldsymbol{B}^{(2)}$ divided by the square of the angular frequency. Expressing $\boldsymbol{B}^{(1)}\times\boldsymbol{B}^{(2)}$ in terms of beam intensity or power density (I in W m^{-2})

$$\boldsymbol{B}^{(1)}\times\boldsymbol{B}^{(2)} = i\frac{\mu_0}{c}I\boldsymbol{e}^{(3)*}, \tag{132}$$

where μ_0 is the vacuum permeability in S.I. (Chap. 1).

The basis of the resonance phenomenon to be developed here is that a probe photon $\hbar\omega_{res}$ at a resonance angular frequency ω_{res} can be absorbed under the *resonance condition* [71],

$$\hbar\omega_{res} = \frac{e^2c^2B^{(0)2}}{2m\omega^2}(1-(-1)), \tag{133}$$

defined by a transition from the negative to positive states of the spinor $\sigma^{(3)}$. This process is precisely analogous to ordinary optical absorption, in which terms NMR is a radio frequency spectroscopy, and ESR a microwave spectroscopy. The probe resonance frequency (f_{res}) in this theory is therefore

$$f_{res} = \frac{\omega_{res}}{2\pi} = \left(\frac{e^2\mu_0 c}{2\pi\hbar m}\right)\frac{I}{\omega^2}, \tag{134}$$

and is inversely proportional to the square of the angular frequency ω of the circularly polarized, pump electromagnetic field which takes the place of the superconducting magnet of conventional NMR or ESR [71,27—31].

For ^1H (proton) resonance the result (134) is adjusted empirically in our development for the experimentally different Landé factors of the proton and electron [71], respectively 5.5857 and 2.002. A more complete theory must rest on the internal structure of the proton, and similarly for other fermions whose Landé factors differ from that of the electron. In principle the theory developed here gives rise to a means of investigating nuclear properties using readily available radio frequency generators instead of magnets.

For proton resonance our final equation is

$$\omega_{res} = \left(\frac{5.5857\, e^2\mu_0 c}{2.002\,\hbar m}\right)\frac{I}{\omega^2} = 1.532 \times 10^{25}\frac{I}{\omega^2}, \tag{135}$$

and some data from this equation are given in Table 2, which shows that it becomes possible in theory to practice NMR at much greater resolution than at present using much higher resonance frequencies. The latter can easily be chosen at will by changing the power density of the radiation being used to generate $A^{(1)} \times A^{(2)}$. The relative separation between chemically shifted resonance lines [27—31,71] becomes much greater at, say, visible frequencies than the radio frequencies accessible in contemporary apparatus [27—31], and so the instrumental resolution is increased dramatically. This is the major advantage of our technique in chemical physics, in which contemporary NMR is already used extensively. In ESR the same overall advantage obtains, the equivalent of equation (135) becomes

$$\omega_{res} = 1.007 \times 10^{28}\frac{I}{\omega^2}, \tag{136}$$

TABLE 2

Resonance Frequencies from Equation (135)

Pump Frequency	$B^{(3)}$	Resonance Frequency
5,000 cm^{-1} (visible)	64.5 nT	0.28 Hz
500 cm^{-1} (infra-red)	6.45 μT	28.0 Hz
1.8 GHz (microwave)	448 T	1.8 GHz*
1.0 GHz (microwave)	1.45 kT	6.18 GHz
0.1 GHz (r.f./micr.)	145 kT	20.6 cm^{-1} a
10.0 MHz (r.f.)	14.5 MT	2,060 cm^{-1} b
1.0 MHz (r.f.)	1.45 GT	206,000 cm^{-1} c

*. auto-resonance, at which the resonance frequency is the same as the applied field frequency
a. far infra-red
b. infra-red
c. ultra-violet

and it is possible, in principle, to practice ESR without magnets at any convenient frequency, say in the visible. In this design the probe radiation can be broad band radiation from the source of a contemporary Fourier transform infra-red spectrometer [40]. The spectrum picked up by the interferometer is then the ESR (or NMR) spectrum, giving a great resolution advantage over contemporary NMR spectra laboriously gathered in the radio frequency range. Fine details of NMR and ESR spectra, currently obscured by lack of resolution, become easily observable in theory using the interaction energy between $A^{(1)} \times A^{(2)}$ and the third Pauli spinor $\sigma^{(3)*}$. This novel term is the key that nature provides through fundamental topology [1,2,22].

The magnetic field in Table 2 is calculated directly from Eq. (2) expressed in terms of field intensity and frequency,

$$B^{(3)} = \frac{e\mu_0 c}{\hbar}\frac{I}{\omega^2}e^{(3)} = 5.723 \times 10^{17}\frac{I}{\omega^2}. \tag{137}$$

The flux density in tesla from this equation is tabulated for various frequencies for an illustrative power density I of ten watts per square centimeter (10^5 W m^{-2}). At visible frequencies it is in the nanotesla range, ten thousand times weaker than the Earth's magnetic field. This produces NMR resonance in the hertz range (Table 2), roughly the same as some experimental data obtained by Warren *et al.* [20], data which support the general validity of our simple one fermion theory.

For a 1.0 GHz *pump field* (by which term we mean the field that produces $A^{(1)} \times A^{(2)}$) the magnetic field defined in Eq. (2) is in the kilotesla range. A 10 MHz pump field of intensity 10 watts per square centimeter produces NMR resonance in the infra red, in which range it can be picked up in principle by the probe field of an ordinary Fourier transform infra red (FT IR) spectrometer. If this advantage can be realized in practice it would make contemporary NMR obsolete in the sense that expensive superconducting magnets would no longer be needed, and it would be possible to pick up the nucleon resonance with ordinary absorption spectroscopy. As we have seen, these major developments in practice are the result of the Dirac equation, i.e., of rigorous relativistic quantum theory based on the fundamental principles of relativity and quantum mechanics. These principles have been known and used for over sixty years, our contribution has been to use a complex $\boldsymbol{A}$ instead of a real $\boldsymbol{A}$, and to realize that the spinor interacts with $\boldsymbol{A} \times \boldsymbol{A}^*$.

There are other major advantages of the new resonance technique which can be sketched using simple Maxwell-Boltzmann statistics as follows. If the relevant energy of interaction of field and fermion is taken to be, from Eq. (115)

$$\Delta En(m) = En(m+1) - En(m) = \frac{e^2 \mu_0 c}{m} \frac{I}{\omega^2}, \tag{138}$$

then the relevant Maxwell-Boltzmann exponential is

$$\exp\left(-\frac{\Delta En}{kT}\right) \sim 1 - \frac{\Delta En}{kT} + \ldots, \tag{139}$$

and the population ratio [75] of the up and down states of the spinor is

$$\frac{N(m)}{N(m+1)} \sim 1 - \frac{e^2 \mu_0 c I}{mkT}. \tag{140}$$

For a radio frequency-microwave range field of intensity 100 watts per square centimeter at 100 MHz at 293 K, this ratio is 0.87, indicating a 13% difference in spin population between the two fermion states. This compares with about seven parts per million for a contemporary NMR instrument at 293 K, 1.0 tesla magnet. An elementary population analysis reveals another advantage of our new technique, which for want of a more evocative description we refer to as radio frequency NMR (RF-NMR). The advantage is that the applied radio frequency field does not have to be manufactured to any great degree of homogeneity because we can tolerate fluctuations in the 10% range.

In summary therefore, the Dirac equation of one fermion (a proton or an electron for example) in a classical electromagnetic field produces a powerful new spectroscopy which if developed makes obsolete our contemporary NMR and ESR technology. It is convenient to understand the magnetic nature of the new phenomenon through the Evans-Vigier field defined in equation (2) of Chap 1. It is equally straightforward to understand it in terms of the product $\boldsymbol{A} \times \boldsymbol{A}^*$, which forms an interaction energy with

the intrinsic fermion spin, a famous concept in twentieth century physics. In the next chapter we discuss the impact of this term on the Aharonov-Bohm effects, which typify the advances being made in contemporary fundamental field theory.

Chapter 3. Optical Aharonov-Bohm Effects of $\boldsymbol{A} \times \boldsymbol{A}^*$

It has been argued that $\boldsymbol{A} \times \boldsymbol{A}^*$ is a physically observable optical property, which forms an interaction energy with a Pauli spinor, an energy which appears as a term in the Dirac equation. The existence of $\boldsymbol{A} \times \boldsymbol{A}^*$ is a direct consequence of the first principles of relativity and quantum mechanics, because these form the basis of Dirac's original work [26]. Therefore the physical cross product $\boldsymbol{A} \times \boldsymbol{A}^*$ must be gauge invariant [1,2], and its proportionality to the physical $\boldsymbol{B}^{(3)}$ ensures this. In Vol. 2, it was shown [2] that Eq. (2) is, indeed, a consequence of gauge theory provided that this is developed within the O(3) group. This non-Abelian theory of vacuum electrodynamics produces Eq. (2) in what seems to us to be a very natural way, and this development finds a clear echo in the Dirac equation, as we have seen. The heuristic arguments of Vol. 1 and 2 [1,2] were used to propose the existence of an optical equivalent of the Aharonov-Bohm effects [32—37], in which $\boldsymbol{B}^{(3)}$ is the magnetic field. In this chapter we apply relativistic quantum theory to develop the optical Aharonov-Bohm effect (OAB) from the Dirac equation.

The Aharonov-Bohm effects [32—37] are central to contemporary field theory, because they show that the vector potential is physically meaningful in classical electrodynamics. Excellent discussions of the effects are found in texts such as that of Ryder [23], who develops the ideas of non-trivial vacuum topology from the original paper of Aharonov and Bohm [37]. Bohm *et al.* have provided a concise summary [72] quoted as follows:

> "If the price of avoiding non-locality is to make an intuitive explanation impossible, one has to ask whether the cost is not too great."

This statement means that the experimentally verified existence of effects due to the vector potential implies action at a distance in classical electromagnetism. For example, the gauge transformation into the vacuum of the vector potential, $\boldsymbol{A}$ through whose curl we define $\boldsymbol{B}$, produces another vector potential which is able to affect the trajectory of an electron matter wave. The trajectory is changed physically even though the original field, $\boldsymbol{B}$, is not present. Therefore action at a distance has occurred and has been observed experimentally. The topology of the vacuum itself [23] must therefore be such as to be able to sustain this physical phenomenon due to gauge *transformation*.

In this chapter we examine whether such an effect exists due to the generation of $\boldsymbol{A} \times \boldsymbol{A}^*$ by the Dirac equation for a quantized fermion matter wave interacting with a classical electromagnetic wave, and find that in theory it does, being proportional to I and inversely proportional to the *cube* of the angular frequency, ω, of the electromagnetic field used to generate $\boldsymbol{A} \times \boldsymbol{A}^*$. We continue to maintain a classical description of the field, but there are undoubtedly effects of this kind due to quantum field theory. Perhaps there are non-classical effects akin to light squeezing which require careful statistical development [73].

3.1 VACUUM TOPOLOGY NEEDED FOR AN OAB EFFECT

The $\boldsymbol{B}^{(3)}$ field manifests itself in interaction with matter as described by the Dirac equation and contemporary developments thereof [22—25]. The equation is fairly simple in structure and can be solved exactly (Chaps. 1 and 2) when matter is represented by a single fermion, more accurately the de Broglie matter wave of a single fermion. The existence of an OAB effect due to $\boldsymbol{B}^{(3)}$ would mean that the group of vacuum electromagnetism is more naturally O(3), because $\boldsymbol{B}^{(3)}$ is longitudinal, i.e., is an axial vector in the axis of propagation. This makes the transverse gauge seem to be an unnatural assertion of classical electrodynamic theory [47] rather than a property of nature itself. It is clearer to think of electromagnetic waves propagating in vacuo as producing a quantized flux vortex [23], which is identifiable with $\boldsymbol{B}^{(3)}$ [1,2]; a vortex which is stabilized by the doubly connected vacuum topology of the group space of O(3) [1,2,23]. The interaction of $\boldsymbol{B}^{(3)}$ with one fermion then takes place as in Table 1, which defines limits which again emerge from the Dirac equation itself. Under experimentally accessible conditions the analysis summarized in Table 1 shows that $\boldsymbol{B}^{(3)}$ must be interpreted through Eq. (2), i.e., through the conjugate cross product $\boldsymbol{A} \times \boldsymbol{A}^*$. The latter therefore generates an optical Aharonov-Bohm effect (OAB) if $\boldsymbol{B}^{(3)}$ is a physical magnetic field.

This is an understanding based on relativistic quantum theory and is an advance on the heuristic theory of $\boldsymbol{B}^{(3)}$ pursued in Vol. 1 and 2. That theory must be interpreted in appropriate limits of the Dirac equation and is the high field limit, the applied electromagnetic field is of such very high intensity as to accelerate the fermion infinitesimally close to the speed of light. In the opposite case it has now become clear that Eq. (2) follows from the Dirac equation in the low field limit, and the Dirac equation must be solved under the circumstances appropriate to the sample being used. For example, if the sample has no free electron spin, Eq. (2) is not directly applicable. In such circumstances a non-relativistic, semi-classical theory such as that of Woźniak, Evans and Wagnière [42] must be developed with the Dirac equation, a completely non-trivial task, even for the fermions of the hydrogen atom. The most appropriate methods to use are based on RF-NMR as developed in the previous chapter. Experiments such as those of Rikken [74], carried out under inappropriate circumstances, will, unsurprisingly, produce a negative result. To interpret such a result as implying the non-existence of $\boldsymbol{B}^{(3)}$ implies a rejection of the Dirac equation itself.

Vacuum topology is now thought to be highly non-trivial, and to be responsible for the existence of the various Aharonov-Bohm effects now known [32—37]. They can exist only in certain types of group space, such as the Abelian U(1) or non-Abelian O(3). They cannot exist in the SU(2) group space because the latter is singly connected. The topological arguments to date have been applied to the original Aharonov-Bohm effect, and have assumed that the group space of vacuum electromagnetism is the Abelian U(1) = O(2). In Vol. 2 [2] however, we have argued that electromagnetism in vacuo exists in the non-Abelian O(3) group space, because of the existence in vacuo of the new field $\boldsymbol{B}^{(3)}$. Equation (2) was derived [2] using O(3) gauge theory, and e interpreted in vacuo as a constant of this non-Abelian gauge theory, meaning that the momentum magnitude of a photon in the vacuum is both $\hbar\kappa$ and $eA^{(0)}$. The vacuum Maxwell equations are written down [47] with source terms missing, the charge and current being located infinitely far away from the field at a given point in phase space. Despite this, the origin of the fields (an oscillating charge at infinity) betrays itself in the $\hat{C}$ negative symmetry [1] of the amplitudes $B^{(0)}$ and $E^{(0)}$, so charge is, after all, present in the vacuum Maxwell equations. This is a consequence of the conservation of elementary charge, e, i.e., of Noether's theorem and Ampère's hypothesis.

Conservation of charge is the *origin* of the minimal prescription [22—25] through type two gauge invariance, and the minimal prescription defines the additional energy-momentum, eA_μ, acquired by a charged particle in a classical field. If eA_μ is such as to accelerate the fermion infinitesimally close to c, the fermion becomes essentially indistinguishable from a photon, and F.A.P.P., complete transfer of energy-momentum has occurred from field to fermion. If so, the momentum acquired by the fermion must be both eA and $\hbar\boldsymbol{\kappa}$. This allows the $\boldsymbol{B}^{(3)}$ field in vacuo to be defined in two ways

$$\boldsymbol{B}^{(3)*} = -i\frac{e}{\hbar}\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} = -\frac{i}{B^{(0)}}\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)}, \tag{141}$$

and this definition is the high field limit. It is easily checked that the identity in Eq. (141) produces

$$eA^{(0)} = \hbar\kappa, \tag{142}$$

which was derived independently in Vol. 2 [2] as the charge quantization condition. It is a high field limit because the field intensity needed to accelerate a fermion infinitesimally close to the speed of light is enormous. It demonstrates clearly however that e can be a field property, because it is simply the charge on the fermion accelerated infinitesimally close to c. Under these conditions the transverse gauge does not apply, as argued in Chaps. 1 and 2, because the highly accelerated fermion must have a time-like component in its energy-momentum four-vector, i.e., must have energy as well as momentum, and part of this energy must be its original rest energy, an invariant of the Poincaré group for a given fermion mass. In other words eA_μ must be a physical energy-momentum, as indicated by the Aharonov-Bohm effects [32—37].

In the opposite limit (II) of Table 1, when the electromagnetic field's intensity is such that the initial fermion energy $En + mc^2$ is much greater than ecA_0, the Dirac equation shows that $\boldsymbol{B}^{(3)}$ must be interpreted as Eq. (2), because the fermion energy-momentum does not begin to resemble that of a photon. In this limit, the identity in Eq. (141) *does not apply*, and the transverse gauge can be used as an aid to calculation, but not, in our opinion, as anything that is physically meaningful. If eA_μ is physical in nature, then in general its time-like component is not zero, as asserted in the transverse gauge. In general however, when a photon collides with a fermion as in the Compton effect [45] a *fraction* of $\hbar\boldsymbol{\kappa}$ is transferred in an inelastic collision, and this can always be put equal to $e\boldsymbol{A}$. There then occurs the well known Compton frequency shift, indicating a loss of momentum by the photon. This is a gain of $e\boldsymbol{A}$ by the fermion according to the minimal prescription. In this sense, the charge quantization condition *always* applies because it is a consequence of Noether's theorem and the quantum hypothesis. Therefore $\boldsymbol{B}^{(3)}$ as defined by Eq. (2) in this situation is a fraction of the free space $\boldsymbol{B}^{(3)}$.

The simplest theory of electron-photon collision [45] can be used to illustrate the above discussion with equations. Consider the traditionally massless photon colliding with an initially stationary electron of rest energy mc^2. The electron acquires momentum (p) and its final energy is [45] $(p^2c^2 + m^2c^4)^{\frac{1}{2}}$ in the observer's frame of reference. The change in energy of the electron is, according to the minimal prescription [1,2,22—25], ecA_0. Therefore

$$ecA_0 = (p^2c^2 + m^2c^4)^{\frac{1}{2}} - mc^2 \tag{143}$$

is the energy gained by the electron, where

$$\boldsymbol{p} = e\boldsymbol{A}, \qquad p^2 = e^2\boldsymbol{A}\cdot\boldsymbol{A}^*. \tag{144}$$

The energy lost by the photon is the Compton frequency shift in its simplest form, $\hbar(\omega_i - \omega_f)$. So,

$$ecA_0 = \hbar\Delta\omega = \hbar(\omega_i - \omega_f), \tag{145}$$

where ω_i and ω_f are the initial and final angular frequencies. Conservation of energy therefore leads to

$$\hbar\Delta\omega = (e^2c^2\boldsymbol{A}\cdot\boldsymbol{A}^* + m^2c^4)^{\frac{1}{2}} - mc^2, \tag{146}$$

which, for the electron, is a classical but relativistic equation with no spinors. The left hand side is a quantum mechanical representation of the change in electromagnetic energy, whereas the right hand side contains a classical representation of the same thing through the minimal prescription. Equation (146) is therefore an expression of the equivalence principal, as well as the result of conservation of energy.

The initial electron momentum is zero and its final momentum is $\boldsymbol{p} = e\boldsymbol{A}$. The momentum gained by the electron is the same as that lost by the photon, $\hbar(\boldsymbol{\kappa}_i - \boldsymbol{\kappa}_f)$. Conservation of momentum leads to

$$\hbar\Delta\boldsymbol{\kappa} := \hbar(\boldsymbol{\kappa}_i - \boldsymbol{\kappa}_f) = e\boldsymbol{A}. \tag{147}$$

Physically, this equation means that the (classical) electron is put into a helical trajectory [75] by a travelling, circularly polarized, plane wave. The Lorentz force on the electron is

$$\boldsymbol{F} = e(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}), \tag{148}$$

where

$$\boldsymbol{E} = -\frac{\partial\boldsymbol{A}}{\partial t} - c\nabla A_0, \tag{149}$$

is the rotating electric field component of the plane wave and where

$$\boldsymbol{B} = \nabla\times\boldsymbol{A}, \tag{150}$$

is the rotating magnetic field component. Here $\boldsymbol{v}$ is the velocity of the electron in the field. The transverse momentum, $\hbar\boldsymbol{\kappa}$, of the photon is used on the left hand side of Eq. (147) because $e\boldsymbol{A}$ is the classical transverse momentum of the electron in its helical trajectory. Again, the intrinsic spin of the electron has not been considered in Eq. (147).

Equation (142) is recovered from Eq. (147) if $\boldsymbol{\kappa}_f$ is zero, and where the amplitude of $\boldsymbol{\kappa}_i$ is the scalar $\kappa = \omega/c$ of the free photon. The scalar magnitude $A^{(0)}$ in Eq. (142)

is

$$A^{(0)} = (\boldsymbol{A}\cdot\boldsymbol{A}^*)^{\frac{1}{2}}. \tag{151}$$

In these calculations, $\boldsymbol{A}$, $\boldsymbol{E}$ and $\boldsymbol{B}$ are complex plane waves of the circularly polarized classical field in vacuo, given by Eqs. (128) to (130) respectively. Equation (142) is therefore the limit in which all the photon's transverse momentum is given to the electron. If so, the free photon has *also* lost all its energy, the angular frequency of the incoming field has been changed from ω_i to 0, and the photon (the field's quantum of energy) has been lost entirely, i.e., *annihilated*. In this limit Eq. (146) becomes

$$\hbar\omega := \hbar\omega_i = (e^2c^2\boldsymbol{A}\cdot\boldsymbol{A}^* + m^2c^4)^{\frac{1}{2}} - mc^2, \tag{152}$$

and this can be consistent with Eq. (142) if and only if

$$A_0 \rightarrow A^{(0)} = (\boldsymbol{A}\cdot\boldsymbol{A}^*)^{1/2} \gg \frac{mc}{e}. \tag{153}$$

This is limit III of Table 1 when we make the theory more precise through use of the Dirac equation of an electronic matter wave interacting with a classical electromagnetic wave. Obviously, Eq. (141) applies only in limit (153), but nevertheless shows that $\boldsymbol{B}^{(3)}$ emerges from the Dirac equation in this limit in the form

$$\boldsymbol{B}^{(3)*} \rightarrow -\frac{i}{B^{(0)}}\boldsymbol{B}^{(1)}\times\boldsymbol{B}^{(2)}. \tag{154}$$

The $\boldsymbol{B}^{(3)}$ field is therefore fundamental in nature [1,2]. The electromagnetic A_μ of the plane wave always interacts with one fermion as if the electromagnetic plane wave were a magnetic field $\boldsymbol{B}^{(3)}$. Since this is a fundamental interaction between field and fundamental particle, it occurs in matter such as atoms and molecules.

Using $A^{(0)} = cB^{(0)}/\omega$, Eq. (153) becomes

$$B^{(0)} \gg \frac{m}{e}\omega, \tag{155}$$

which is Eq. (411) of Vol. 1 [1], obtained from an independent analysis based on the classical but relativistic Hamilton-Jacobi equation of the classical electron in the field. Equation (411) of Vol. 1 becomes recognizable in its quantized form as limit III of

Table 1. Using $I = cB^{(0)2}/\mu_0$ shows that Eq. (155) is equivalent to

$$I \gg \left(\frac{cm^2}{\mu_0 e^2}\right)\omega^2 = 7.72\times10^{-9}\omega^2, \tag{156}$$

for the electron. For the proton

$$I \gg 0.026\,\omega^2. \tag{157}$$

We refer to these as equations of *the strong field limit*. For a visible frequency laser as used by Rikken [74], with $\omega = 1.77\times10^{16}\ rad\ s^{-1}$ (10,640 cm^{-1}), Eq. (156) gives

$$I \gg 2.4\times10^{22}\ W\,m^{-2}. \tag{158}$$

The peak intensity used in the experiment [74] was only $5.5\times10^{12}\ W\,m^{-2}$ and the sample used was benzene, with no free electrons. Under these conditions, Eq. (141) does not apply because: 1) the field intensity is at least ten orders of magnitude too low; 2) there are no free electrons.

In the opposite limit, limit II of Table 1, or Eq. (414) of Vol. 1 [1], the field energy ecA_0 is much smaller than the rest energy mc^2 of the fermion. In the transverse gauge it is asserted [47] that A_0 is zero, in which case Eq. (145) shows that there can be no Compton effect, contrary to experiment [71]. In the limit $A_0 \rightarrow 0$, however, $\omega_f \rightarrow \omega_i$, and there can be a small but non-zero Compton shift. From Eq. (146) in this limit,

$$mc^2 \gg e^2\boldsymbol{A}\cdot\boldsymbol{A}^*, \tag{159}$$

which in terms of intensity becomes

$$I \ll \left(\frac{cm^2}{\mu_0 e^2}\right)\omega^2. \tag{160}$$

We refer to this, purely for convenience, as *the weak field limit*. A glance at Eq. (160) shows that most, if not all, experiments in magneto-optics to date have been carried out well within the weak field limit, under which $\boldsymbol{B}^{(3)}$ is defined through Eq. (2) as a result of the Dirac equation. Even in this limit, however, the experimental results by Rikken [74] cannot be compared with Eq. (1), because the sample used, benzene, has no free electron spin. There *is* however, an optical (*B term*) Faraday effect in benzene due to perturbation of the polarizability by $\boldsymbol{A} \times \boldsymbol{A}^*$ [76—78]. The experiment failed to detect this effect, but it had been detected already by Flytzannis *et al.* [79] under more appropriate conditions, and was first predicted by Kielich [80]. Our demonstration of the origins of the fundamentally *non*-zero $\boldsymbol{A} \times \boldsymbol{A}^*$ in relativity (Chaps. 1 and 2) shows the incorrectness of any claim that $\boldsymbol{B}^{(3)}$, as defined by Eq. (2), is zero. The only way to sustain such a view is to assert, arbitrarily, that the left and right hand sides of Eq. (2) cannot be identified. There is no known symmetry argument or dimensionality argument that supports this assertion, and no known experimental data.

We have therefore illustrated the emergence of the condition $eA^{(0)} = \hbar\kappa$ in the strong field limit. This relation was obtained independently in Vol. 2 [2] from O(3) gauge theory. The primary theme of this section is to show that vacuum topology [23] allows the existence of the OAB in the O(3) group space as well as that of U(1). In order for any Aharonov-Bohm effect to exist, vacuum topology must define an appropriate group space, a space that cannot be singly connected [23]. The fundamental topological arguments have been applied to date mainly to the original Aharonov-Bohm effect [23,32—37] and have assumed that the group space of vacuum electromagnetism is the Abelian U(1) = O(2). In both the weak and strong field limits, however, there exists a magnetic field, $\boldsymbol{B}^{(3)}$, perpendicular to the Abelian plane defining $\boldsymbol{A}^{(1)}$ and $\boldsymbol{A}^{(2)}$. Self-consistently, therefore, relations such as Eq. (2) must be considered in group theory as being *non*-Abelian, because they contain three space indices, (1), (2) and (3). Proceeding on this basis, Eq. (2) was derived in Vol. 2 [2] from general O(3) gauge theory [23], with ideas adapted from general relativity.

Within the O(3) group space in vacuo the field equations of Evans and Vigier [1,2] supplement the Maxwell equations in vacuo with terms quadratic in A_μ, of which Eq. (2) is an example. The theory [2] shows that $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ is gauge invariant within the O(3) group space. Within the U(1) = O(2) group space [23] $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ is still gauge invariant but is not put equal to a magnetic field in axis (3). This is a key conceptual difference between Evans-Vigier and Maxwell electrodynamics in vacuo, and leads to many interesting ramifications in the theory of finite photon mass [1,2]. It also has consequences in the elementary (i.e., fundamental) nature of charge, e, because in the O(3) gauge e can be a *field* property. We have illustrated this by accelerating a charged fermion infinitesimally close to the speed of light, whereupon it takes the properties of a photon, F.A.P.P. This idea has been neatly illustrated by Jackson [47] in classical electrodynamics, and can be traced [81] to the time of Bragg, Bateman and contemporaries.

In the conventional U(1) gauge theory of electromagnetism in vacuo [22—25], there is also to be expected an OAB, because $\boldsymbol{A} \times \boldsymbol{A}^*$ and $\boldsymbol{A} \cdot \boldsymbol{A}^*$ exist in U(1) as well as O(3). As we have shown, however, the U(1) group space does not allow the left and right hand sides of Eq. (2) to be identified. This is because fields are defined in a plane in U(1), and $\boldsymbol{B}^{(3)}$ is a field that is obviously not in this plane, being perpendicular to it. However, there is nothing in fundamental symmetry [1] or dimensionality that prohibits the identity in Eq. (2), which asserts, quite naturally, that magneto-optical effects are magnetic in nature. The equation (2) emerges rigorously as a *field equation* [2] in the group space O(3) and so $\boldsymbol{B}^{(3)}$ is a property of this group space. Experimentally, the magnetic effect of $\boldsymbol{B}^{(3)}$ can be distinguished from that of $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ in the strong field limit, because the effect becomes proportional to the square root of I in this limit, as argued in Vol. 1 and 2 [1,2]. Evidence for photon mass [82] is also evidence for $\boldsymbol{B}^{(3)}$. In the weak field limit, the magnetic effect of $\boldsymbol{B}^{(3)}$ must be understood in terms of $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$, and is proportional to beam intensity I, as known experimentally [16—21]. All magneto-optic experiments to date have been carried out well within the weak field limit and the I proportionality always found in consequence.

Therefore we expect that the OAB (as yet unobserved) will be due to $\boldsymbol{B}^{(3)}$ defined in the weak field limit by Eq. (2) because this is the only experimentally accessible limit at present. The only consistent way to view a positive result would be to assert that vacuum electromagnetism is non-Abelian in nature, and described within the O(3) group space, because Aharonov-Bohm effects are always ascribed to a *magnetic* field's vector potential [32—37]. The theory of vacuum topology [23] must allow this result therefore. Topological arguments are existence arguments, giving very general conditions which must be fulfilled in order that solutions similar to $\boldsymbol{B}^{(3)}$, such as solitons or stable vortices [23] can exist. Vacuum topology is thought to set the stage upon which any subsequent argument must be played out. They show for example [23] that string-like solutions to spontaneously broken gauge theories [2] cannot exist in a group space such as SU(2) because $\pi_1(S^3)$ is trivial, where the group space of SU(2) is S^3 and where π_1 is the first homotopy group. The mapping $\pi_1(S^3)$ is trivial because every closed curve S^1 on S^3 may be shrunk to a point: boundary conditions may be shrunk to a trivial constant condition ϕ = const., and no stable vortices [23] exist. It follows that no field equations can produce the stable vortex $\boldsymbol{B}^{(3)}$ within the group space of SU(2); and conversely if $\boldsymbol{B}^{(3)}$ exists in vacuo, the group space needed to sustain it cannot be that of SU(2).

The O(3) group on the other hand is doubly connected topologically [23]. There is a 2 : 1 mapping of SU(2) on to O(3), and the group space of O(3) is obtained from that of SU(2) by identifying opposite points on the three space, S^3, since these points correspond to the same O(3) transformation. The group space of O(3) being doubly connected, there are two types of closed path S^1, those homotopic to a point and those homotopic to a line. The O(3) group space sustains, in consequence, one non-trivial, stable, vortex [23], which in the Evans-Vigier field equations becomes [2] $\boldsymbol{B}^{(3)}$. The O(3) group space is a vacuum group space, so $\boldsymbol{B}^{(3)}$ is sustained topologically in vacuo.

The vacuum in this contemporary view is a complicated entity, anything but trivial in nature.

The Evans-Vigier field equations [1,2] are also those of particles, through the de Broglie principle which is the basis of matter waves [83]. The O(3) group is that of particles with integral isospin, the SU(2) that of particles with half-integral isospin. In the field equations of Vol. 1 and 2, the isospin indices are identified with the circular indices, (1), (2) and (3), of three dimensional space itself. The non-trivial vortex line sustained by O(3) is $\boldsymbol{B}^{(3)}$ itself and there is a non-Abelian OAB effect, meaning that solitons [23] exist in O(3). Therefore the existence and stability of $\boldsymbol{B}^{(3)}$ is ensured by the doubly connected O(3) vacuum topology. The particles with integral isospin become photons with mass, and the Evans-Vigier field equations become equations describing three dimensional photons with mass, such as the Proca field-particle equation [1,2]. Thus, $\boldsymbol{B}^{(3)}$ is a solution of the vacuum Proca equation [1,2].

In the conventional U(1) framework, in which photons are asserted to have identically zero mass, and to propagate in vacuo identically at c, solitons are also considered to create a vortex line *perpendicular* [23] to the U(1) plane. This view is obviously inconsistent with itself if $\boldsymbol{B}^{(3)}$, a physical field, can be described in terms of this vortex line in vacuo. The Lagrangian used conventionally in building up this result is that of the Higgs model [2,23] with spontaneous symmetry breaking (SSB). It is structurally identical with the famous Landau-Ginzburg Lagrangian that leads to the Proca wave equation itself. The soliton flux, if identifiable with $\boldsymbol{B}^{(3)}$, must be a *quantized* flux in vacuo, or in matter must be exemplified by observable quantized flux lines such as those of Abrikosov in type II superconductors. The *same* Lagrangian is used for superconductor and vacuum flux, but it is asserted [23], arbitrarily in our view, that the superconductor flux (Abrikosov line) exists but the vacuum flux ($\boldsymbol{B}^{(3)}$) does not. Ryder, for example, gets out of this quandary by stating blandly that a superconductor is not a vacuum [84]. The problem, we suggest, is that fields exist in both media and must be described consistently in both, especially if we are using the same Lagrangian to do so.

In summary therefore, the conventional theory, although a flat U(1) theory, produces the Proca equation, which is the wave equation for a boson with mass. It is nevertheless asserted that the boson known as the photon is massless identically, an assertion which is inconsistent with the existence of a stable vortex line in vacuo *perpendicular* to the U(1) plane. The O(3) Evans-Vigier field equations in contrast are free of this logical strain in that they recognize that $\boldsymbol{B}^{(3)}$ can be an intrinsic component of vacuum electromagnetism, whatever that may turn out to be in future. Finally, Roy and Evans [50] have adapted the Dirac condition [48,49] to show that the existence of photon mass can be reconciled straightforwardly with gauge invariance of the second kind (which we accept because it is the result of a fundamental conservation law of physics, the Noether theorem). As described already the Roy-Evans condition leads to

$$A_\mu A_\mu \xrightarrow[F.A.P.P.]{} 0, \tag{161}$$

and this condition can be satisfied by $\boldsymbol{B}^{(3)}$ in *both* the high and low field limits. There appears to be no internal inconsistency in the theory of $\boldsymbol{B}^{(3)}$.

3.2 THE ORIGIN OF THE OAB

In Chap. 1 and 2 the Dirac equation was used to show that there is a novel interaction energy (En_2) generated between $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ and the intrinsic fermion spin, represented as usual by a Pauli spinor, $\boldsymbol{\sigma}^{(3)}$. This interaction produces two energy levels between which resonance can be made to occur. The energy En_2 divided by ω has the units of angular momentum, which are also the units of action [1,2]. The action En_2/ω divided by $\hbar$ is unitless, and as we shall show, is responsible for the OAB through a phase shift in the matter wave of a fermion such as an electron. This phase shift occurs in regions where there is no electromagnetic radiation and no $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$, and is due to a gauge transformation of $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ into the structured vacuum, a vacuum whose topology is not trivial. Accordingly, the expected phase shift is proportional to I/ω^3 where I is intensity and ω is beam angular frequency, because En_2 is proportional to I/ω^2 as we have seen. The OAB should therefore be much more readily observable at low frequencies and depends for its existence on non-locality in the conjugate product $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ that defines the Evans-Vigier field in the accessible weak field limit. Therefore electromagnetic radiation acts at a distance because of the structure of the vacuum: the OAB is caused by a gauge transformation, in other words, of the type

$$\boldsymbol{A}^{(1)} \rightarrow \boldsymbol{A}^{(1)} + \nabla^{(1)}\chi, \qquad \boldsymbol{A}^{(2)} \rightarrow \boldsymbol{A}^{(2)} + \nabla^{(2)}\chi, \tag{162}$$

into regions of the vacuum in which the original $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ is excluded experimentally. A narrowly collimated microwave beam, for example, which is circularly polarized and directed in the shadow of two interfering fermion beams, should produce an OAB due to the multi-valued (periodic) nature of the gauge transformed $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$, i.e., due to periodicity [23] in $\chi^{(1)}$ and $\chi^{(2)}$ of Eq. (162). This periodicity is possible only because the group space of O(3) is doubly-connected: the vacuum topology needed to sustain $\boldsymbol{B}^{(3)}$ also sustains an OAB due to $\boldsymbol{B}^{(3)}$. Due to its implication of action at a distance in electromagnetic waves, the observation of the OAB might catalyze work within the interesting framework of deterministic quantum mechanics proposed by Bohm [85] after discussions with Einstein.

The original Aharonov-Bohm effect [37] was inferred theoretically from a quantum relativistic theory, in which action, S, was expressed in terms of A_μ for static fields. In the Dirac equation, the wave functions are four-spinors which can be described as plane waves [22—25] with phase exponent $\exp(S/\hbar)$ where S is a relativistically defined action, so a fully relativistic theory of the Aharonov-Bohm effect requires a description in these terms of the observed phase shift [32—37]. In a simplified view of the original effect [37], the observable shift in a fringe pattern due to interfering fermion matter waves is attributed [23] to a change in the phase,

$$\alpha_1 \rightarrow \alpha_1 + \frac{1}{\hbar}\boldsymbol{p}\cdot\boldsymbol{r}, \tag{163}$$

where $\boldsymbol{p}$ is the non-relativistic limit of the fermion linear momentum at a point $\boldsymbol{r}$ in three dimensional space. The phase α_1 is an action or angular momentum divided by the quantized unit of action or angular momentum, $\hbar$. The intrinsic angular momentum ($\boldsymbol{S}^{(3)} = (\hbar/2)\boldsymbol{\sigma}^{(3)}$) of the fermion does not appear in the picture because $\boldsymbol{S}^{(3)}$ is a relativistic concept. Reinstating $\boldsymbol{S}^{(3)}$ produces the *spin phase of the fermion*

$$\alpha_S = \frac{1}{\hbar}\boldsymbol{S}^{(3)}\cdot\boldsymbol{\sigma}^{(3)}, \tag{164}$$

which exists due to topology. In Eq. (164), the angular momentum $\boldsymbol{S}^{(3)}$ is analogous with the linear momentum $\boldsymbol{p}$ in Eq. (163), and the coordinate $\boldsymbol{\sigma}^{(3)}$ is analogous with the coordinate $\boldsymbol{r}$.

The OAB can now be traced to the effect of a circularly polarized electromagnetic field on the spin phase σ_S of one fermion's matter wave. Following Talin *et al.* [54] we write the field-fermion interaction energy En_2 in the form

$$En_2 = \omega J = \omega S, \tag{165}$$

which originates in the conservation laws of angular momentum (J) and energy. The energy of a free photon is $\hbar\omega$ by definition, and this is a limit in which J has been replaced by the quantized unit of angular momentum (or action) of an electromagnetic or matter wave, the Dirac constant $\hbar$. In the presence of an electromagnetic field the spin phase of one fermion is changed to

$$\alpha_S \rightarrow \alpha_S + \frac{En_2}{\hbar\omega}, \tag{166}$$

which is the action En_2/ω generated by the field-fermion interaction divided by $\hbar$. This is the change in energy, En_2, caused by fermion-field interaction divided by the photon $\hbar\omega$, the quantized, elementary, unit of electromagnetic or matter-wave energy. Alternatively the phase change can be viewed as $J/\hbar$, i.e., as the change in angular momentum, J, generated by the interaction of the fermion and field divided by $\hbar$, the quantized, elementary, unit of electromagnetic or matter-wave angular momentum. It can be checked that if there is no change in energy or angular momentum, i.e., no *interaction* energy or angular momentum, there is no phase shift, and vice-versa. The OAB therefore requires an interaction between a fermion property and a *gauge transformed* field property if the electromagnetic field is excluded from direct interaction with the interfering fermion matter waves.

For one fermion, the change in phase is, from Eq. (1),

$$\Delta\alpha_S = \frac{ie^2}{2m_0\hbar\omega}\boldsymbol{\sigma}^{(3)}\cdot\boldsymbol{A}^{(1)}\times\boldsymbol{A}^{(2)}, \tag{167}$$

and is proportional to the conjugate product in the weak field limit, or in the transverse gauge. In the strong field limit this result is modified as discussed in section (3.1). Finally, the phase change (167) is expressible as

$$\Delta\alpha_S = \mp\left(\frac{e^2c\mu_0}{2m_0\hbar}\right)\frac{I}{\omega^3}, \tag{168}$$

and is proportional to I/ω^3 as we set out to show. If there is no direct field-fermion interaction, $\boldsymbol{A}^{(1)}\times\boldsymbol{A}^{(2)}$ in Eq. (167) must have been gauge transformed from an original electromagnetic beam. If there is direct field-fermion interaction the field property $\boldsymbol{A}^{(1)}\times\boldsymbol{A}^{(2)}$ is used without gauge transformation.

3.3 GAUGE TRANSFORMATION AND THE OAB

Equation (168) shows that a circularly polarized electromagnetic field changes the spin phase of a fermion by an amount which is proportional to I/ω^3. The effect can be positive or negative depending on the spin state of the fermion. The inverse cubed dependence on ω means that the OAB should be orders of magnitude greater for a given I at radio or microwave frequencies than at visible frequencies. The OAB is inversely proportional to the fermion mass and the lightest accessible fermion, the electron, should be used experimentally to maximize the effect. Theoretically, the OAB also exists for other fermion matter waves, such as those of protons, neutrons and certain atoms and molecules with net-odd total of protons, neutrons and electrons (e.g. ^{3}He but not ^{4}He). An array of OAB (and AB) effects is possible in principle using various interfering beams and resonance techniques. Clearly, the simple one fermion theory will not be adequate in detail for interfering atomic and molecular beams and must be developed appropriately. The OAB is closely related to RF-NMR and other magnetic effects of light as described in Chaps. 1 and 2.

In order to demonstrate the OAB experimentally it is necessary to show that the phase shift occurs in regions where there is no direct field-fermion interaction. The only physical possibility of an OAB in this experimental configuration is through gauge transformation. Since $\boldsymbol{A}^{(1)}$ and $\boldsymbol{A}^{(2)}$ are both vector potentials of the electromagnetic field in the complex representation they each undergo, by definition [22—25], type two gauge transformation as follows,

$$\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} \rightarrow \boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} + \nabla^{(1)}\chi \times \nabla^{(2)}\chi, \tag{169}$$

and since the original cross product $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ is physical, the gauge transformed cross product, $\nabla^{(1)}\chi \times \nabla^{(2)}\chi$, must be such that gauge invariance of the original $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ is maintained. This is analogous with the well known requirement of gauge invariance in the curl, $\nabla \times \boldsymbol{A}$, of a physical magnetic field. It is therefore convenient to think of the OAB as the original AB effect [37] with the magnetic field of that effect replaced by $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$. The Evans-Vigier field $\boldsymbol{B}^{(3)}$ plays this role precisely if defined as in Eq. (2) in the experimentally accessible weak field limit.

The OAB is therefore an AB effect due to the Evans-Vigier field.

Gauge transformation into the vacuum [23] can therefore be symbolized in exactly the same way as in the original AB theory, bearing in mind that $\boldsymbol{B}$ of the original theory is replaced by $\boldsymbol{B}^{(3)}$ of Eq. (2). The OAB can equally well be understood in terms of the gauge transformation into the vacuum of the *physical* conjugate product, physical because it is known [16—21] to produce physical magneto-optic effects. In order for an electromagnetic conjugate product (OAB) or a static magnetic field (AB) to be physical, they must each be gauge invariant. This is the requirement that leads to non-trivial vacuum topology [23] in both cases. The only contemporary difference is that the AB has been observed [32—37] while the OAB has not yet been observed.

In the original AB effect [23] gauge transformation into the vacuum can be summarized by

$$\boldsymbol{O} \rightarrow \boldsymbol{O} + \nabla\chi, \tag{170}$$

because initially there is no magnetic field or vector potential present in the vacuum, thus *0* on the left hand side. The right hand side of Eq. (170) symbolizes the fact [23] that for an AB effect to occur $\nabla \times \nabla\chi$ must be *non-zero in the vacuum*. (In the

vacuum there is no $\boldsymbol{B}$ and no $\boldsymbol{A}$ because $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ by definition.) In order for $\nabla \times \nabla\chi$ to be non-zero, χ must be periodic as discussed by Ryder [23] and this can only be so if the vacuum itself possesses a non-trivial topology. In classical electrodynamics of the pre-AB era, $\boldsymbol{B}$ was physical, and $\boldsymbol{A}$ was mathematical, because it was thought that the type two gauge transformation,

$$\boldsymbol{A} \rightarrow \boldsymbol{A} + \nabla\chi, \tag{171}$$

changed $\boldsymbol{A}$ to $\boldsymbol{A} + \nabla\chi$ but left $\boldsymbol{B}$ unchanged. This assumed implicitly that the quantity $\nabla \times \nabla\chi$ was always zero, implying a trivially structured (or non-structured) vacuum.

Using a combination of the quantum hypothesis $\nabla^{(1)} := i\boldsymbol{p}^{(1)}/\hbar$ and the minimal prescription $\boldsymbol{p}^{(1)} := e\boldsymbol{A}^{(1)}$ we obtain

$$\nabla^{(1)} = \nabla^{(2)*} = \frac{i}{\hbar} e\boldsymbol{A}^{(1)}, \tag{172}$$

and the analogous argument for the OAB can be developed conveniently as follows. We consider the eigenequations,

$$\nabla^{(1)}\chi = i\frac{e}{\hbar}\boldsymbol{A}^{(1)}\chi, \qquad \nabla^{(2)}\chi = i\frac{e}{\hbar}\boldsymbol{A}^{(2)}\chi, \tag{173}$$

where the wavefunction is

$$\chi = \chi_0 e^{i\phi}, \qquad \chi_0 = i\frac{\hbar}{e}, \tag{174}$$

in which ϕ is the electromagnetic phase [1] itself. Therefore

$$\nabla^{(1)}\chi_0 \times \nabla^{(2)}\chi_0 = -\frac{e^2}{\hbar^2}\chi_0^2 \boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} = \boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}, \tag{175}$$

and $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ is invariant under gauge transformation as required by its physical nature. This result is consistent with Noether's theorem [1,2,23] which implies that invariance of action under space-time translation or rotation produces conservation of energy-linear momentum and energy-angular momentum respectively. The action in our context is

$$S = \frac{ie^2}{2m_0\omega} \boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} \cdot \boldsymbol{\sigma}^{(3)*}, \tag{176}$$

and, self-consistently, is invariant under type two gauge transformation if and only if $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ is so invariant. If so, χ must be periodic as in Eq. (174), leading again to the concept of non-trivial vacuum topology as in the AB effect. Furthermore, the periodicity in χ is supplied by the electromagnetic phase itself, and the OAB becomes a phase shift, which in quantum mechanics is a phase shift in the wave function χ. Since S in Eq. (176) is the magnitude of an angular momentum, the OAB is the result of the necessity for conservation of angular momentum when $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ is gauge transformed into the vacuum. Finally, therefore, if there is no OAB, there can be no conservation of angular momentum and no Noether theorem. It is very likely on these grounds that the OAB will be observed experimentally.

3.4 EXPERIMENTAL INVESTIGATIONS OF THE OAB

There are several elegant experimental methods [32—36] now available to detect the original AB [37] and each can be adapted in principle for the OAB provided the necessary experimental configuration can be attained in practice. Phase shifts in the original Aharonov-Bohm effect can be observed with high precision using a Wien filter. Hasselbach and co-workers [86] have developed very accurate instruments based on electron interferometry. The Wien filter is a device that uses magnetic and electric fields mutually perpendicular to interfering electron beam paths. The electric force on the electron is balanced by the magnetic force when the Wien filter is compensated. Hasselbach *et al.* [86] have discussed the role played by the electric and magnetic Aharonov-Bohm phase shifts in such a device, and have shown that the Aharonov-Bohm effects are responsible for the experimentally verifiable fact that the fringe system does not change its appearance with increasing deflection angle. Thus, the original Aharonov-Bohm quantum phase shifts [37] are responsible for the practical utility of electron-optical elements. The magnetic field in the conventional Wien filter causes an Aharonov-Bohm phase shift due to the difference in magnetic flux enclosed by the two coherent electron beams. Therefore the novel Evans-Vigier field defined in Eq. (2), a physical magnetic flux density, is expected to produce an OAB if the Wien filter is modified to accommodate a circularly polarized radio frequency field mutually perpendicular to the electron beam paths and electric field.

Such an experiment would be of fundamental importance because it would imply that the group space of the electromagnetic sector is O(3), and not U(1), in contemporary string and superstring theory [2]. The OAB can be understood entirely equivalently through the conjugate product $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$, which affects the relativistic spin phase of the electron wave function, as discussed already. In the modified Wien filter, the shift in electronic spin phase caused by $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$, or equivalently $\boldsymbol{B}^{(3)}$, is observable because it is different for the two interfering electron beams. In other words, the magnetic flux density $\boldsymbol{B}^{(3)}$ enclosed by the two interfering electron beams is different, and this flux density is $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ multiplied by $-ie/\hbar$. Therefore the phase shift per electron due to $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$, Eq. (42), is non-zero and observable after appropriate path integration.

The Wien filter thus modified to take a circularly polarized electromagnetic field instead of a magnetic field can now be used to detect the OAB. The experimental method follows the description in Fig (3) of Ref. 86a. Electron wave packets emerge from two coherent sources and reach the modified Wien filter, in which an electric field and circularly polarized radio frequency electromagnetic field are arranged mutually perpendicular to the electron beam baths. In the absence of the modified Wien filter, an electron fringe pattern is observed which is symmetric about the axis of propagation. Application of the electric field renders this pattern asymmetric because the two electron beams traverse different electric field potential regions. The central maximum of the pattern is shifted off axis, and this effect is exactly equivalent to moving back one of the sources by n wavelengths. The central axis can be brought back on to the symmetry axis by application of the magnetic field in the conventional Wien filter, or by switching on the circularly polarized radio frequency field in the modified Wien filter. This automatically shows the presence of $\boldsymbol{B}^{(3)}$ in the radio frequency beam, or equivalently the presence of $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$.

An alternative procedure is to take a conventional Wien filter, consisting of crossed electric and magnetic fields, and to supplement the magnetic field with an additional, circularly polarized electromagnetic field. The Wien filter is first compensated with the electric and magnetic fields as usual, but then the r.f. field is applied. The latter should again make the fringe pattern unsymmetrical, because the total magnetic flux density being applied has been increased by $\boldsymbol{B}^{(3)}$. This effect can be observed in principle with high precision, because shifts of this kind can be measured to small fractions of a wavelength [86]. The primary interference pattern in the two electron beams should also be deflected by the OAB, i.e., the spin phase of the matter wave should be changed. This is a consequence of Eq. (168) for one electron. As for the electric and magnetic AB effects [86], the OAB spin phase shift should keep pace exactly with the path length difference caused by deflection, and the electron fringe system should not change its appearance with increasing deflection angle caused by the radio frequency field. In other words, the difference in magnetic flux density, $\boldsymbol{B}$, enclosed between the coherent beams compensates the geometric phase shift for every wavelet making up the wave fronts. The additional path length Δ caused by the bending of the electron beams by the additional radio frequency field applied to a compensated Wien filter, and the corresponding change in phase Δ/λ, where λ is the wavelength, are exactly cancelled by the counteracting OAB effect. With increasing deflection angle therefore there should be observed *no* continuous change from bright to dark of a single fringe of the pattern with increasing deflection angle caused by the additional r.f. field.

In order to demonstrate the quantum relativistic nature of the OAB, it is necessary in addition to these experiments, precise as they are, that resonance can be made to occur between the two phase states of Eq. (168), negative and positive respectively. These two states originate in the nature of the Pauli spinor $\boldsymbol{\sigma}^{(3)}$, and this has no counterpart in classical or non-relativistic quantum physics. A complete appreciation of the OAB requires therefore a resonance experiment. It has been demonstrated theoretically in Chaps. 1 and 2 that a circularly polarized radio frequency field produces fermion resonance akin to NMR and ESR. For an electron, the resonance frequency is

$$\omega_{res} = 2\Delta\alpha_S\omega, \tag{177}$$

i.e., is twice the OAB phase shift of Eq. (168) multiplied by the angular frequency of the resonance causing (pump) field. The resonance frequency for a 100 watt cm^{-2} microwave beam interacting with one electron is at about 150 cm^{-1} in the far infra red, and detection of this frequency would also be a demonstration of the OAB and of the Evans-Vigier field $\boldsymbol{B}^{(3)}$. A complete appreciation of the OAB therefore requires a combination of the elegant methods based on the Wien filter [86] and resonance technique. The latter specifically demonstrates the quantum relativistic origins of the OAB.

The OAB as proposed in this chapter would allow action at a distance in electromagnetism, and would signal the existence of the Evans-Vigier field $\boldsymbol{B}^{(3)}$. The Bohm model of quantum mechanics [72] has been much discussed [1—15] as an alternative to the Copenhagen interpretation, but requires at the outset action at a distance. The experimental observation of the OAB might go some way towards explaining action at a distance through vacuum topology, and towards reconciling two strikingly different interpretations of quantum theory, leading to a deeper understanding of both. The discovery of the non-zero $\boldsymbol{B}^{(3)}$ and its implication of non-zero photon mass is another feature which may become useful in a realist interpretation of quantum mechanics, or in efforts to reconcile the realist and Copenhagen points of view.

3.5 OBSERVATIONAL CONDITIONS FOR $\boldsymbol{B}^{(3)}$ IN GENERAL

This chapter has been concerned with the delicate and interesting optical Aharonov-Bohm effect and outlines the conditions under which experiments should be pursued. Recently, an experiment [74] has been reported which did not detect the optical Faraday effect (OFE) [79,80] in liquid benzene. This negative result was used to conclude that the Evans-Vigier field does not exist [74]. The existence, however, of magneto-optic effects [16—21] is well supported by data and the OFE had already been observed [79] under appropriate conditions. The claim that $\boldsymbol{B}^{(3)}$ does not exist contradicts the Dirac equation as developed in Chaps. 1 and 2, and seems vanishingly improbable. In this

section a summary is provided of the conditions under which $\boldsymbol{B}^{(3)}$ interacts with a fermion in the weak and strong field limits. We hope that this summary will be useful in defining appropriate experimental conditions.

We have seen in earlier chapters that the interaction of a fermion with a circularly polarized electromagnetic field is described by the following Dirac equation after averaging over many field cycles,

$$\begin{aligned}\big((En - mc^2 + ecA_0)(En + mc^2 + ecA_0) - c^2(\boldsymbol{p} + e\boldsymbol{A}^*)\cdot(\boldsymbol{p} + e\boldsymbol{A})\\ - ie^2c^2\boldsymbol{\sigma}\cdot\boldsymbol{A}\times\boldsymbol{A}^*\big)\psi = 0.\end{aligned} \tag{178}$$

In the weak field limit it has been shown that this equation reduces to

$$W\psi = (En - mc^2)\,\psi \sim \left(\frac{e^2}{2m}\big(\boldsymbol{A}^*\cdot\boldsymbol{A} + i\boldsymbol{\sigma}\cdot\boldsymbol{A}^*\times\boldsymbol{A} - ecA_0\big)\right)\psi \tag{179}$$

and in the strong field limit it reduces to

$$W\psi = \left(ec\boldsymbol{\sigma}\cdot\frac{i}{A^{(0)}}\boldsymbol{A}^*\times\boldsymbol{A}\right)\psi. \tag{180}$$

The Evans-Vigier field from Eq. (179) is defined by Eq. (2), and this definition makes the term in $\boldsymbol{\sigma}\cdot\boldsymbol{A}^*\times\boldsymbol{A}$ an ordinary Zeeman effect term, with fermion half integral spin eigenvalue $\pm\hbar/2$ as usual. The Evans-Vigier field from the strong field limit (180) is defined by

$$\boldsymbol{B}^{(3)*} := -i\frac{\kappa}{A^{(0)}}\boldsymbol{A}\times\boldsymbol{A}^* = -\frac{i}{B^{(0)}}\boldsymbol{B}\times\boldsymbol{B}^*, \tag{181}$$

and is fundamental because $B^{(0)}$, $\boldsymbol{B}$ and $\boldsymbol{B}^*$ are fundamental. The concept of $\boldsymbol{B}^{(3)}$ is analogous in some ways to that of Poynting's vector [1—15], another fundamental vector, and another field cross product. The origin of $\boldsymbol{A}\times\boldsymbol{A}^*$ however is to be found in the founding axioms of special relativity itself — the Einstein equation of motion is *quadratic* in energy and momentum, and so therefore is Dirac's equation of motion. Assuming *only* that $\boldsymbol{A}$ is complex in general, so that it is not equal to $\boldsymbol{A}^*$, the (quadratic) term in $\boldsymbol{A}\times\boldsymbol{A}^*$ follows from either the Einstein or Dirac equation, and this term defines the Evans-Vigier field $\boldsymbol{B}^{(3)}$. The latter is therefore also a consequence of special relativity, and being a field, can be observed only by field-matter interaction.

Equation (2) is transformed into Eq. (181) in the limit,

$$\frac{\kappa}{A^{(0)}} \rightarrow \frac{e}{\hbar}, \tag{182}$$

and Eq. (181) applies only when $I \gg 7.72 \times 10^{-9}\, \omega^2$ for the electron in the field or $I \gg 0.026\, \omega^2$ for the proton in the field. In these limits the momentum magnitude $eA^{(0)}$ transferred from field to fermion is, F.A.P.P., the momentum magnitude, $\hbar\kappa$, of the free photon itself. Under these conditions the photon gives up essentially all its energy and momentum to the fermion. Its limits of application were first shown in Ref. 1, Eqs. (411) — (414), using the classical but relativistic Hamilton-Jacobi equation.

Rikken [74] based his claim of non-existent $\boldsymbol{B}^{(3)}$ on an experiment in which the peak intensity was only $5.5 \times 10^{12}\ W\ m^{-2}$ at a Nd-YaG frequency of 10,640 cm^{-1} ($\omega = 1.77 \times 10^{16}\ rad\ s^{-1}$). Equations (156) and (157) both show that he would have been working well within the weak field limit, a limit in which $\boldsymbol{B}^{(3)}$ must be defined, according to the Dirac equation, by Eq. (2). His sample (benzene) did not have free spin, however, so his conditions were entirely inappropriate for comparison with one fermion theory. Unfortunately, Rikken appears not to have been aware of Ref. 1 or of any recent reference to work on $\boldsymbol{B}^{(3)}$, and interpreted the negative result in terms of Eq. (181) of the strong field limit as discussed earlier in this chapter. He appears then to have asserted that $\boldsymbol{B}^{(3)}$ is zero under all conditions, whereas the precise opposite is indicated by the Dirac equation. Finally, the emergence of $\boldsymbol{B}^{(3)}$ from the Dirac equation means that earlier criticisms [1] of $\boldsymbol{B}^{(3)}$ based on symmetry and so forth are incorrect if the Dirac equation is accepted as a working hypothesis in relativistic quantum theory [22—25]. The Dirac equation is by no means perfection [22] but for our purposes it is perfectly adequate.

Chapter 4. Properties of $\boldsymbol{B}^{(3)}$ in the Strong Field Limit

In the previous chapter it was shown that in the strong field limit and therefore in the vacuum, the $\boldsymbol{B}^{(3)}$ field is defined through an equivalence principle $eA^{(0)} = \hbar\kappa$ that describes the complete transfer of photon momentum to fermion momentum in a photon-fermion collision. In this chapter this relation is developed systematically and a total of ten forms of $\boldsymbol{B}^{(3)}$ are derived and tabulated (Table 3). It has been shown in earlier chapters that the $\boldsymbol{B}^{(3)}$ field is the fundamental magnetizing field of relativistic magneto-optics: phenomena such as the inverse and optical Faraday effects the optical Zeeman effect, the OAB, RF-NMR and related technology, all depend on its existence in the vacuum. In the strong field limit these phenomena are experimentally proportional to the square root of beam intensity (I) and the ratio of κ to $A^{(0)}$ goes to the limiting constant $e/\hbar$. In this condition the fermion has been accelerated infinitesimally near the speed of light, c, so that its charge has become a field property. Based on these considerations, the fine structure constant can be interpreted classically, and the vector potential in vacuo, $\boldsymbol{A}$ becomes physically significant.

The existence of $\boldsymbol{B}^{(3)}$ was first inferred [1—15] from the *experimentally* verified existence of the conjugate product, not in the form $\boldsymbol{A} \times \boldsymbol{A}^*$, or $\boldsymbol{B} \times \boldsymbol{B}^*$, but in the form $\boldsymbol{E} \times \boldsymbol{E}^*$ originally suggested phenomenologically by Pershan [41]. It has been clear for some time [16—21] that this cross product has magnetic symmetry, but the conventional viewpoint asserted that this symmetry does not imply the existence of a physical magnetic field. It is now understood that this view is insupportable because it would mean that the left and right hand sides of Eq. (2), for example, could not be equated; or that $\boldsymbol{B}^{(3)}$ could not appear in the Dirac equation while $i\boldsymbol{A} \times \boldsymbol{A}^*/A^{(0)}$ *does* appear in the same Dirac equation (Eq. (180)). It is tantamount to asserting that a physically observable quantity on one side of an equation cannot be equated to a quantity on the other side of that equation with the same units and symmetry. When the quantity on the right hand side is a physical observable, that on the left hand side must be the same observable. The development [1,2] of cyclically symmetric relations based on the above conjugate products showed, therefore, that they produce in vacuo a physical and observable magnetic field, $\boldsymbol{B}^{(3)}$. It was shortly afterwards realized [1—15] that these cyclical relations have the same O(3) symmetry [1,2] as rotation generators (or angular momenta) in three dimensional space, or their equivalents [1,2] in four dimensional space-time. This reinforced the initial inference that $\boldsymbol{B}^{(1)}$, $\boldsymbol{B}^{(2)}$, and $\boldsymbol{B}^{(3)}$ are three physical fields in vacuo. The same symmetry arguments [1,2] show that there is no physical $E^{(3)}$, the cyclical symmetry of the field relations demands, however, an imaginary $iE^{(3)}$ whose scalar square modulus $E^{(0)2} = -i^2 E^{(3)} \cdot E^{(3)}$ is real and which can contribute to the total field energy. The fact that there is no real $E^{(3)}$ means that there is no free space Faraday induction [1—15], i.e., the time derivative of $B^{(3)}$ does not produce the curl of a real $E^{(3)}$ unless there is a sample present, as in the inverse Faraday effect. Experiments [87] to date confirm this expectation and the Dirac equation shows that the fermion spin interacts always with a magnetic field, there being no interaction term in the imaginary and unphysical electric field that emerges (Chap. 1) from the same analysis [26]. Finally, this conclusion can be shown to hold in the classical limit, when the Dirac equation becomes a Hamilton-Jacobi equation (Chap. 2). As described in earlier chapters, interesting new resonance effects are expected due to $B^{(3)}$. There can be no reasonable doubt, therefore, that $B^{(3)}$ is a novel and useful physical magnetic flux density arising from the fundamentals of special relativity, both in classical and quantum mechanics.

4.1 FERMION ACCELERATED INFINITESIMALLY CLOSE TO c PRODUCES $B^{(3)}$

The pre-$B^{(3)}$ approach to magneto-optical phenomena typified by the inverse Faraday effect therefore relied on the conjugate product in the form $E \times E^*$. This quantity multiplied by the vacuum permittivity ϵ_0 has, formally, the units of an electromagnetic torque density, Tq_V, or torque per unit volume carried by electromagnetic radiation in vacuo. The torque density is also given by $-iB \times B^*/\mu_0$, where μ_0 is the vacuum permeability, and, as shown in Chap. 7, can be expressed in terms of wavelength. The quantity $c|Tq_V|$ is the antisymmetric part of the tensor of light intensity, a tensor whose scalar part is

$$I = \frac{c}{\mu_0} B^{(0)2}, \tag{183}$$

an equation used in Chap. 1 to 3. Similarly, the energy density of the electromagnetic radiation in vacuo is given by the relation $B^{(1)} \cdot B^{(2)}/\mu_0$, and the intensity is c times the energy density.

The $B^{(3)}$ field emerges from Tq_V as follows:

$$B^{(3)} = \frac{\mu_0}{B^{(0)}} Tq_V, \tag{184}$$

and, like Tq_V, is real and physical. Its existence in vacuo implies that of the magnetic dipole moment whose origin is rotating charge e,

$$m^{(3)} = \frac{1}{\mu_0} B^{(3)}. \tag{185}$$

Similarly, Maxwell's rotating electric field strength $E^{(1)}(Vm^{-1})$ in electromagnetic radiation propagating through the vacuum can be expressed in terms of the same rotating electric charge in vacuo,

$$E^{(1)} = \frac{e}{\epsilon_0 V} r^{(1)}, \tag{186}$$

where dimensionality requires the rotating radius vector, $r^{(1)}$. The latter is complex in general because $E^{(1)}$ is complex. This analysis, based on fundamental S.I. units and definitions, introduces the charge e into electromagnetic radiation in the vacuum, and shows the origin of the $\hat{C}$ negative symmetry of the scalar amplitude $E^{(0)}$ [1]. It is therefore possible to describe electromagnetic radiation as a charge rotating around the origin. The length of $r^{(1)}$ can be shown as follows to be

$$r^{(0)} = |r^{(1)}| = \frac{1}{\kappa} = \frac{\lambda}{2\pi} = \left(\frac{V}{4\pi\alpha}\right)^{1/3}, \tag{187}$$

where λ is the wavelength of the radiation contained in a volume V, and where α is the fine structure constant. If V is the radiation volume, it is easily checked that Eq. (187) is dimensionally and physically consistent in the theory of electrodynamics [47] in S.I. units. The electromagnetic field propagates in the axis perpendicular to the plane of the rotating charge, which draws out a helical path at the speed of light in vacuo. In analogy with the solenoid, this movement produces the magnetic field $B^{(3)}$. If an electron is placed in the rotating $E^{(1)}$ of an electromagnetic field in the strong field limit, its classical trajectory would be exactly the same as that of our rotating charge and would be described by the Hamilton-Jacobi equation of Chap. 12 of Vol. 1 in this limit. This also leads to the $B^{(3)}$ field.

Under these conditions we have shown in previous chapters that

$$eA^{(0)} = \hbar\kappa, \tag{188}$$

where $\hbar$ is a constant which must have the units of angular momentum or action. It can be shown as follows that this definition *implies* that $\hbar$ must be the Dirac constant, because it *implies* the Planck-Einstein condition. The latter is therefore a direct consequence of accelerating an electron in the strong field limit.

From Eq. (186) the scalar magnitude of the rotating Maxwellian $\boldsymbol{E}^{(1)}$ is

$$E^{(0)} = \frac{er^{(0)}}{\epsilon_0 V}. \tag{189}$$

The classical electromagnetic energy in the volume V is

$$En = \epsilon_0 E^{(0)2} V, \tag{190}$$

where the following definition is used of average volume

$$V := \int_0^V dV, \tag{191}$$

and so the energy in volume V is expressible in terms of the radius $r^{(0)}$,

$$En = eE^{(0)} r^{(0)} = \frac{e^2 r^{(0)2}}{\epsilon_0 V}. \tag{192}$$

Using the usual relation between $E^{(0)}$ and $A^{(0)}$,

$$A^{(0)} = \frac{E^{(0)}}{\omega}, \tag{193}$$

it is found that

$$En = \left(eA^{(0)} r^{(0)}\right)\omega, \tag{194}$$

i.e., *the electromagnetic energy is proportional to the electromagnetic angular frequency*, and this is the Planck-Einstein relation provided we make the identity

$$\hbar = eA^{(0)} r^{(0)}. \tag{195}$$

Therefore the first principle of quantum theory, the Light Quantum hypothesis, has emerged from classical theory in a new way. This is a consequence of thinking of an electron being accelerated in the strong field limit infinitesimally close to the speed of light c, and this point is vividly underlined if we identify $r^{(0)}$ with $\boldsymbol{\kappa}^{-1}$, the inverse of the wavevector, which for radiation propagating at c is the quantity c/ω. The radius $r^{(0)}$ is, in this condition, the ratio of the forward and angular velocities of the radiation. The identification $r^{(0)} = \kappa^{-1}$ produces from Eq. (195) the equivalence condition (188), i.e.,

$$\hbar = \frac{eA^{(0)}}{\kappa}, \tag{196}$$

which in Vol. 2 [2] we also obtained from O(3) gauge theory of vacuum radiation. In Chap. 3 we saw that this condition is precisely that required to transform the weak field solution of the Dirac equation into the strong field solution, in which the fermion is accelerated infinitesimally near c. Equation (196) in Eq. (194) produces, self-consistently, the familiar

$$En = \hbar\omega, \tag{197}$$

of the famous Light Quantum hypothesis [88] first proposed heuristically by Planck and then by Einstein.

4.2 PHOTON RADIUS AND QUANTUM OF LIGHT ENERGY

Using Eq. (188), with the equation

$$A^{(0)} = \left(\frac{c}{\epsilon_0 \omega^2 V}\right) e, \tag{198}$$

the quantum of electromagnetic energy $\hbar\omega$ is defined as

$$\hbar\omega = \frac{e^2}{\epsilon_0 \kappa^2 V}, \tag{199}$$

and is proportional to e^2. This result can now be expressed in terms of the fine structure constant [89,90],

$$\alpha := \frac{e^2}{4\pi c\epsilon_0\hbar}, \tag{200}$$

of quantum electrodynamics. From Eqs. (199) and (200),

$$\alpha = \frac{V}{4\pi r^{(0)3}}, \tag{201}$$

and so we arrive at Eq. (187) for the radius of the rotating charge, as we set out to show. The quantum of light energy is therefore

$$En = \hbar\omega = (4\pi\alpha V)^{3/2}\frac{e^2}{\epsilon_0}, \tag{202}$$

and becomes expressible in terms of $r^{(0)}$. The concept of photon radius $r^{(0)}$ in classical electrodynamics has linked the Planck-Einstein hypothesis (197) and the new equivalence condition (188), and has shown that the photon, whatever it is, can be produced by accelerating an electron in the strong field limit. Significantly, Eq. (187) defines the Thompson radius [91] of the photon, and so the latter, if particulate, has a finite radius c/ω that becomes longer at lower frequency.

This view is consistent with special relativity and Maxwell's vacuum equations and is derived by re-expressing the usual transverse fields $E^{(1)}$ and $B^{(1)}$ in terms of elementary charge e, a procedure equivalent to imagining an electron accelerated in the strong field limit infinitesimally near to c. This procedure results in $En = \hbar\omega$ if $\hbar = eA^{(0)}/\kappa$. The quantity $eA^{(0)}/\kappa$ is therefore a constant angular momentum, as required, but an *electronic* angular momentum of classical electrodynamics. In the quantum theory it is of course the universally constant angular momentum of one photon, the latter being the quantum of energy, $\hbar\omega$. *Provided* that we use the new equivalence condition $eA^{(0)} = \hbar\kappa$ the Light Quantum hypothesis is a logical outcome of classical electrodynamics.

Furthermore, the re-expression of $E^{(1)}$ as a rotating electric charge translating very near c results in the field $B^{(3)}$, which can be thought of as the outcome of helical

charge motion. Conversely the $B^{(3)}$ field drives an electron in a helical trajectory. This is consistent with the fact that $B^{(3)}$ from the Dirac equation in the strong field limit produces the equivalence condition (196), and consistent with the classical analysis given in Vol. 1 based on the Hamilton-Jacobi equation of e in the electromagnetic field.

The usual idea of a photon as being uncharged and therefore its *own anti-particle* is shown by Eq. (202) to have the narrowest of meanings. The photon can now be thought of as a *classical* amount of energy proportional to the square of e, and is not uncharged. The equivalence condition (196) shows that the origin of the Dirac constant $\hbar$ is the electronic charge e multiplied by $A^{(0)}/\kappa$, where $A^{(0)}$ is within a constant c the scalar potential of the classical wave in the strong field limit. Re-expressing $A^{(0)}$ through Eq. (199) leads to Eq. (201), which produces the fine structure constant of quantum electrodynamics in the form of a simple ratio of volumes, the ratio of V to $V^{(0)} = (4/3)\pi r^{(0)3}$, the Thompson sphere. The fine structure constant is

$$\alpha = \frac{1}{3}\frac{V}{V^{(0)}}. \tag{203}$$

Equation (187) shows that the magnitude of the classical Maxwellian wavevector is defined by the volume V through

$$\kappa = \left(\frac{4\pi\alpha}{V}\right)^{1/3}, \tag{204}$$

and because α is a universal constant [2,22—25], the de Broglie photon momentum becomes

$$p = \hbar\kappa = \left(\frac{4\pi\alpha\hbar^3}{V}\right)^{1/3} = \left(\frac{e^2\hbar^2}{\epsilon_0 c V}\right)^{1/3}. \tag{205}$$

The Planck-Einstein photon therefore becomes expressible as

$$En = \hbar\omega = \hbar\kappa c = \left(\frac{e^2\hbar^2 c^2}{\epsilon_0 V}\right)^{1/3}. \tag{206}$$

These equations show that the quantum of radiation energy $\hbar\omega$ and the quantum of radiation momentum, $\hbar\kappa$, are both defined in terms only of V and the universal fine structure constant α. This is consistent with the fact that they are fundamental quanta of energy and momentum, and with the fact that the magnitudes of these quanta vary *only* with the volume V introduced in Eq. (186). On the most fundamental level in this Maxwellian theory, everything depends ultimately on e. In generalized gauge theory [2], from which the equivalence condition (188) emerges, e simultaneously plays the role of the elementary charge and a gauge scaling constant. Therefore e can be thought of as being topological in nature, so that an angular momentum such as $\hbar$ becomes understandable in terms of a geometrical entity, e, an elementary geometrical measure of the known universe. In electromagnetic radiation in the vacuum (or free space), e becomes subsumed into the definition of $\boldsymbol{E}^{(1)}$ through the radius $\boldsymbol{r}^{(1)}$ as in Eq. (186). The enigmatic photon, if it is a particle, must always have a finite radius.

The mass of this particle [1,2], if it exists, must be concentrated very near to the origin if the particle has a finite radius and is rotating about this origin. Otherwise it would rotate about a center of mass somewhere between the origin and the negative charge. What is known as photon mass may therefore be residual electron mass. Almost all of the latter's rest energy mc^2 must be transformed into other forms of energy as the electron is accelerated infinitesimally close to c in its helical trajectory, leaving a small residual mass, the photon mass, which we consider here to be elementary (Chap. 2) and irremovable in nature. If the electron mass is about 10^{-31} kgm and the photon mass no greater than about 10^{-44} kgm [92], a large fraction of the electron rest energy mc^2 is lost as other forms of energy; but not all. In the Maxwellian picture, radiation travels through a simple vacuum at c precisely, so there is no mass specified and no rest frame. The Maxwell equations in vacuo then show that the charge e rotates about an origin that is translating at c. The orbit of rotation is the Thompson radius, which becomes shorter at higher frequencies. The next section develops these ideas through the classical Hamilton-Jacobi equation of one electron in the strong field limit, and shows that the relativistic electron momentum $\boldsymbol{p}$ becomes $\hbar\kappa$ in this limit.

4.3 DERIVATION OF THE PLANCK-EINSTEIN CONDITION FROM THE RELATIVISTIC HAMILTON-JACOBI EQUATION

In this Section the equivalence condition (196) is used to derive the Planck-Einstein condition (197) from the classical Hamilton-Jacobi equation of a charge e in the electromagnetic field. In the limit where all the momentum of an incoming photon is transferred to an electron, the latter is accelerated to a state where it becomes indistinguishable from a photon, i.e., is accelerated infinitesimally near the speed of light, in which condition its concomitant electromagnetic fields become indistinguishable from those of Maxwell's vacuum theory, as discussed by Jackson [47].

By considering the classical, but relativistic, motion of an electron of mass m in the electromagnetic field, the Hamilton-Jacobi (HJ) equation can be used [1] to show

that the induced orbital angular momentum of the electron is given through $\boldsymbol{B}^{(3)}$ as follows (in which spinors are missing)

$$\boldsymbol{J}^{(3)} = \left(\frac{ec}{\omega}\right)^2\left(\frac{B^{(0)}}{(m^2\omega^2+e^2B^{(0)2})^{1/2}}\right)\boldsymbol{B}^{(3)}, \tag{207}$$

where ω is the angular frequency of the field and of the electron in equilibrium with the field. The relativistic factor in Eq. (207) is

$$\gamma := \frac{c}{\omega}(m^2\omega^2+e^2B^{(0)2})^{1/2}, \tag{208}$$

and using $B^{(0)} = \omega A^{(0)}/c$ this becomes

$$En^2 = c^2\gamma^2 = m^2c^4 + c^2(e^2A^{(0)2}), \tag{209}$$

where En is total energy. The units of both sides in this equation are those of energy squared because γ has the units of linear momentum. Using the equivalence condition (196) is tantamount to assuming

$$ecA_0 \rightarrow ecA^{(0)} \gg mc^2, \tag{210}$$

because Eq. (209) is obtained in the strong field limit (Chap. 3). Therefore Eq. (209) becomes

$$En \rightarrow ecA^{(0)} = \hbar\kappa c = \hbar\omega, \tag{211}$$

which is the Planck-Einstein condition (197). If it is assumed that mass in equation (209) remains finite, Eq. (208) gives the de Broglie matter-wave equation

$$\omega^2 = \frac{m^2c^4}{\hbar^2} + c^2\kappa^2, \tag{212}$$

which is well known to be the Einstein equation after application of the quantum hypotheses,

$$En = \hbar\omega, \quad p = \hbar\kappa. \tag{213}$$

If Eq. (212) is to be regarded as a matter wave equation for the electron of mass m then the electron must have acquired a linear momentum $\hbar\kappa$ from the quantized electromagnetic field. The classical momentum $eA^{(0)}$ introduced by the gauge has been identified with the quantized momentum $\hbar\kappa$, and in the limit $ecA^{(0)} \gg mc^2$ the relativistic factor of the classical relativistic Hamilton-Jacobi equation becomes the quantum relativistic Planck-Einstein condition. If we ignore the electron rest energy mc^2, it can be accelerated to such a degree as to become a photon, because a photon is *defined* by the Planck-Einstein condition. The electron, in order to be accelerated into a conventional photon, must lose all its rest energy in the form of radiation, but must retain its charge, e, fully intact.

Using the equivalence condition (196) the HJ equation (207) can be rewritten as

$$J^{(3)} \rightarrow \frac{\hbar\kappa}{(m^2c^2 + \hbar^2\kappa^2)^{1/2}} \frac{ec^2}{\omega^2} B^{(3)}, \tag{214}$$

in which the field $B^{(3)}$ drives the electron in an orbit at an angular frequency ω. All the other quantities in this equation are particulate in nature, describing an electron that has acquired a linear momentum $\hbar\kappa$ from the quantized field. In the strong field limit,

$$\hbar\kappa \gg mc, \tag{215}$$

and Eq. (214) reduces to

$$J^{(3)} \rightarrow \frac{ec^2}{\omega^2} B^{(3)}, \tag{216}$$

showing that the orbital angular momentum acquired by the electron is directly proportional to $B^{(3)}$. This produces a characteristic $I^{1/2}$ dependence in the strong field limit, as discussed in Chap. 3 and earlier volumes [1,2]. In this limit, the wavevector becomes indistinguishable from ω/c, and using $B^{(0)} = \kappa A^{(0)}$, we obtain

$$J^{(3)} \rightarrow \frac{ec^2 B^{(0)}}{\omega^2} e^{(3)} = \frac{eA^{(0)}}{\kappa} e^{(3)}. \tag{217}$$

The equivalence condition (196) finally reduces this equation to

$$J^{(3)} \rightarrow \hbar e^{(3)}, \tag{218}$$

in which $J^{(3)}$ is the angular momentum of the photon as we set out to show. The maximum orbital angular momentum that the electron can attain is $\hbar$, a universal constant, and this is so only if the rest energy mc^2 is regarded as negligible in comparison with the energy $ecA^{(0)}$ acquired from the quantized electromagnetic field. In this limit, $eA^{(0)}$ of the field becomes $\hbar\kappa$ of the electron, and if mc^2 is negligible, this becomes the total quantized linear momentum of the electron.

If the $B^{(3)}$ field were zero, no orbital angular momentum could be transferred from the photon to the electron, and this is something that contradicts conservation of angular momentum. There would be no magneto-optic effects, in contradiction with experience [16—21]. A similar analysis from the Dirac equation shows that if $B^{(3)}$ were zero, the intrinsic spin of the electron could not interact with the field through a Pauli spinor. In other words, there would be no optical anomalous Zeeman effect [1,2]. The field $B^{(3)}$ is the fundamental field responsible for magneto-optical effects and also for the interaction of fermions with the electromagnetic field.

The major unsolved problem, or so it seems at present, is to define more precisely the circumstances under which the rest energy of the electron can be rendered negligible. This is not a trivial question because if the electron is given a mass m of $\sim 10^{-31}$ kgm, its energy mc^2 is a constant of special relativity unless it is transmuted into radiation energy.

4.4 THE DERIVATION OF $B^{(3)}$ FROM A ROTATING CHARGE e IN VACUO

A magnetic field is due to a rotating charge, and in vacuo the field is the vacuum permeability multiplied by the magnetic dipole moment caused by the rotating charge. The S.I. unit of magnetic dipole moment is C m^{-1} s^{-1} and so it is possible to define a magnetic dipole moment in vacuo,

$$|m^{(3)}| = \frac{erv}{V}, \tag{219}$$

where e is the elementary charge rotating around the origin at the end of a radius r with tangential speed v in a volume V of electromagnetic radiation. It is straightforward to show, as follows, that the magnetic field $B^{(3)} = \mu_0 m^{(3)}$ from Eq. (219) is precisely equation (2). This derivation is given firstly, and the discussion of the validity of equation (219) secondly.

The magnetic field $B^{(3)}$ from Eq. (219) is

$$B^{(3)} = \frac{\mu_0 e r v}{V} e^{(3)}, \tag{220}$$

and V can be defined as the average volume (191) without loss of generality. The radius r is the Thompson radius κ^{-1} as discussed already, and the tangential velocity v is by definition

$$v = \omega \times r, \tag{221}$$

where ω is the angular velocity. Therefore,

$$vj = \omega k \times \frac{c}{\omega} i, \tag{222}$$

where i, j, and k are Cartesian unit vectors, and so $v = c$. Therefore,

$$B^{(3)} = B^{(0)} e^{(3)} = \left(\frac{\mu_0 e c^2}{\omega V} \right) e^{(3)}, \tag{223}$$

is the magnetic field due to the rotating charge e used earlier in this chapter.

The validity of this result can be checked using the equivalence condition (188), which can be written as

$$\frac{c^2}{\omega} = \frac{\hbar \omega}{e B^{(0)}}. \tag{224}$$

Using this in Eq. (223) produces

$$\hbar \omega = \frac{1}{\mu_0} B^{(0)2} V, \tag{225}$$

which is the well known expression [47] for the photon in vacuo in terms of $B^{(0)}$ and the radiation volume V. The photon in equation (225) is the usual quantum of electromagnetic energy in vacuo.

Using Eq. (225) in Eq. (223) produces (*cf.* Eq. (4)),

$$B^{(3)} = \frac{e}{\hbar} \cdot \frac{c^2}{\omega^2} B^{(0)2} e^{(3)} = \frac{e}{\hbar} A^{(0)2} e^{(3)}, \tag{226}$$

which can be re-expressed as

$$B^{(3)} = -i \frac{e}{\hbar} A^{(1)} \times A^{(2)}, \tag{227}$$

and this is equation (2). Therefore Eq. (224) links Eq. (223) and Eq. (227), showing them to be *equivalent for radiation in vacuo*. This result was first derived in Vol. 1 and 2. The representation of electromagnetic radiation in vacuo by the rotating elementary charge e, propagating both tangentially and forward (along Z) at c, is therefore entirely self-consistent with the existence of the vacuum $B^{(3)}$. This result is consistent, furthermore, with O(3) gauge theory [2], which produces equation (2) using methods adapted from general relativity, and is also consistent with analysis based on rotation generators and developed elsewhere [1—15]. Thirdly, it is consistent with the equivalence equation (188), which was first shown in Vol. 2 [2], and referred to there as the *charge quantization condition* because it makes e proportional to $\kappa / A^{(0)}$ through $\hbar$. Fourthly, the result is consistent with Jackson's analysis [47] of a radiating electron accelerated infinitesimally close to c, whereupon the concomitant fields become vacuum electromagnetic fields.

If it asserted that $B^{(3)}$ is zero [62—65], then e must be zero, a reduction to absurdity.

Mass is absent from the above demonstration, which is relativistic in nature because we are describing radiation propagating in vacuo exactly at c. Obviously, the calculation is valid only in the vacuum, as soon as $B^{(3)}$ interacts with a fermion the Dirac equation shows that the two definitions of $B^{(3)}$ interlinked in vacuo by equation (188) remain valid, but become *inequivalent* because Eq. (188) is no longer valid. They must be used in the appropriate limits as discussed already, e.g. equation (2) is valid only in the weak field limit as shown by the Dirac equation. Equally obvious is the fact that $B^{(3)}$ can never be observed in vacuo, it can be observed only when it influences, or interacts with, a fermion. The same is true of any field [22—25].

4.5 ORIGIN OF THE ROTATING VACUUM CHARGE IN SIMPLE RADIATION THEORY

In this section we follow closely the classical (and classic) discussion by Jackson [47], of simple radiative processes. The origin of radiation is oscillating charge and current density, ρ and J, i.e., charge per unit volume and current per unit area. In the Maxwell equations, these are the famous source terms. In the vacuum Maxwell equations, the source terms are set to zero, because the *density* of charge and current is vanishingly small, the source of vacuum electromagnetic radiation can be thought of as being infinitely far away, but this is a mathematical ideal rather than anything that can be accepted as being physically meaningful. The important point is that charge conservation (the Noether theorem) demands that e cannot disappear from the analysis, so that even when the source terms are missing, charge conservation must be taken account of in the vacuum Maxwell equations, and therefore in the vacuum fields *themselves*. This means that the charge conjugation symmetry [1] ($\hat{C}$ symmetry), of the field amplitudes $E^{(0)}$ and $B^{(0)}$ (and also of $A^{(0)}$) must be negative. Since A is expressed [47] in terms of J, which is expressed in terms of e, the latter must appear in the vacuum fields themselves. This section demonstrates this conclusion mathematically using simple radiation theory. The classical picture therefore replaces the early nineteenth century concept of action at a distance (Coulomb interaction between two charges) by interaction via electromagnetic radiation propagating at c in vacuo. The d'Alembertian derivative [1,2] of the potential four-vector A_μ is *equated* to oscillating charge-current density J_μ (S.I. units),

$$\Box A_\mu = -\frac{J_\mu}{\epsilon_0}, \tag{228}$$

and since J_μ is defined in terms of e, so must A_μ, and since e is elementary and universal, it cannot disappear. Therefore electromagnetic fields in vacuo can be described through e as in the opening sections of this chapter. The charge e moves through the vacuum in a helical trajectory at c, depending on the radiation's wavelength, the propagation being made possible by the interaction of magnetic and electric fields as in Maxwell's vacuum equations. The source is an oscillating (or rotating) electron and the process of radiation can be thought of as transmutation of part of the electron energy to photon energy. This means that the photon carries with it the elementary charge e as in Eq. (202). In this process, as in any other, energy, momentum and charge must be conserved.

In S.I. units the general solution of Eq. (228) is given by Jackson [47],

$$A(r,t) = \frac{1}{4\pi\epsilon_0 c^2}\int d^3r'\int dt\frac{J(r',t')}{|r-r'|}\,\delta\left(t'+\frac{|r-r'|}{c}-t\right), \tag{229}$$

where the current density is defined through a Dirac delta function,

$$J = ev\delta(r-r'). \tag{230}$$

Here r is a coordinate vector. Equation (229) shows that *radiation* is a product of e with a space-time factor. For sinusoidally varying sources [47],

$$A(r) = e\left[\frac{1}{4\pi\epsilon_0 c^2}\int v\frac{\delta(r-r')}{|r-r'|}e^{i\kappa|r-r'|}d^3r'\right]. \tag{231}$$

Therefore the fields E and B of classical electromagnetic radiation are directly proportional to e, even though they propagate through the vacuum. This is precisely in accord with earlier sections of this chapter. This conclusion must carry through into relativistic quantum field theory [22—25] through the usual methods of canonical quantization, or alternative methods such as those proposed by Weinberg [93]. The photon as quantum of electromagnetic energy is proportional to e^2 (Eq. (202)) and can be thought of as *uncharged*, as in the literature on elementary particles [67], only with the utmost conceptual strain. Quantized fields, however, are creation and annihilation operators [1,2,22—25], and are proportional to e at first order. Consistently, electromagnetic energy (Eq. (225)) is again proportional to the square of field amplitude. An overemphasis on photon as *uncharged particle* leads to very great confusion, as evidenced in some recent theoretical papers [62—65] which attempt to assert that $B^{(3)}$ in vacuum is zero. As we have seen, this can only be so if e is zero, reductio ad absurdum.

In the radiation zone [47], the condition holds that $r \gg d$, where d is the dimension of the source. The oscillating electromagnetic field components become transverse to the direction of propagation. In much simpler language, and in a circularly polarized electromagnetic wave, the charge e describes a helix as it propagates in vacuo at c. The radius of the helix is Thompson's radius, κ^{-1}, and as discussed in the previous section, $B^{(3)}$ is formed by this motion, in analogy with a solenoid. In the latter, of course, the charge is carried by electrons, in the vacuum, Eq. (231) shows that electron mass is not present, but e clearly *is* present in free space. In the radiation zone [47], and in the standard dipole approximation,

$$\boldsymbol{A}(\boldsymbol{r}) = -e\left[\frac{1}{4\pi\epsilon_0 c^2}\int \nabla\cdot\boldsymbol{v}\,\delta\left(\boldsymbol{r}-\boldsymbol{r}'\right)e^{i\phi}\,d^3r'\right], \tag{232}$$

an equation that can be re-expressed as

$$\boldsymbol{A}(\boldsymbol{r}) := \boldsymbol{A}_0 e^{i\phi}, \tag{233}$$

where ϕ is the phase of the electromagnetic radiation,

$$\phi = \omega t - \boldsymbol{\kappa}\cdot\boldsymbol{r}. \tag{234}$$

Therefore electromagnetic radiation in vacuo can be described entirely, and very simply, in terms of the motion of elementary charge, e. If e spirals through the vacuum with circularly polarized plane waves, we have

$$\boldsymbol{B} = \frac{1}{c}\boldsymbol{k}\times\boldsymbol{E}, \tag{235}$$

and, as in Eq. (186), the transverse electric field component is

$$\boldsymbol{E}^{(1)} = \frac{e}{\epsilon_0 V}\boldsymbol{r}^{(1)}. \tag{236}$$

The transverse magnetic field component from Eq. (235) is

$$\boldsymbol{B}^{(1)} = \frac{e}{\epsilon_0 c V}\boldsymbol{k}\times\boldsymbol{r}^{(1)}, \tag{237}$$

whose magnitude is $B^{(0)}$, given by

$$B^{(0)} = \frac{er}{\epsilon_0 c V} = \frac{e\mu_0 rc}{V}. \tag{238}$$

This is also the magnitude of $B^{(0)}$ from Eq. (220), a result which is consistent with the cyclical field relations,

$$\boldsymbol{B}^{(1)}\times\boldsymbol{B}^{(2)} = iB^{(0)}\boldsymbol{B}^{(3)*}, \text{ et cyclicum}, \tag{239}$$

derived independently in Vol. 1 and 2. Again we find precise self-consistency in our theory. Experimentally, electromagnetic radiation is known to cause magnetization [16—21], and this magnetization can be pictured as circulating charge, motion caused in matter by an incoming electromagnetic field which is itself helical motion of e through the vacuum. A source electron at one end of the universe influences a matter electron at the other end through a helical journey of e through free space. This process conserves energy, charge and momentum, and the range of electromagnetic radiation is very great because there is very little mass associated with the radiation's e.

As we have seen in Chap. 1 and 2, $\boldsymbol{B}^{(3)}$ can be detected only as it *interacts* with matter, and this interaction is controlled in the Dirac equation entirely by $\boldsymbol{A}$ and $\boldsymbol{A}^*$, and not directly by the fields themselves. This lends support to the feeling that the four-potential A_μ is something *more fundamental* than the derivative fields, a line of thought that is supported by the Aharonov-Bohm effects [32—37,39] as described in Chap. 3. If this point is not appreciated, confusion will multiply, as in the experiment by Rikken [74]. This author's conclusion that $\boldsymbol{B}^{(3)}$ is zero must mean that e is zero, and if there is no e, there is no radiation at all. Conversely therefore, $\boldsymbol{B}^{(3)}$ is *non* zero whenever there is circularly polarized radiation present. Linearly polarized radiation is a superposition of left and right circular components, and $\boldsymbol{B}^{(3)}$ is again present, but equal and opposite for these components. Similarly, $\boldsymbol{B}^{(3)}$ is non-zero in elliptical polarization.

Finally in this section we emphasize a little known, but important, conclusion of special relativity, that two charges co-translating at c do not attract or repel. This point has been made clearly by French [70], and is used in a recent paper by Mac Gregor [69]. Therefore charged fields on neighboring photons (traditionally of zero mass and propagating at c) do not attract or repel in vacuo. As soon as field-matter interaction occurs, there is charge-charge interaction because the matter charge velocity is less than c and the incoming electromagnetic field influences an electron as demonstrated by Hertz in the nineteenth century. Photon-photon scattering appears to take place without Coulomb interaction because both beams propagate very near c.

4.6 BIOT-SAVART LAW FOR $\boldsymbol{B}^{(3)}$ IN THE VACUUM

The origin of magnetic fields in moving charges was demonstrated [47] in the early nineteenth century by Biot and Savart, whose work was extended greatly by Ampère. Therefore if $\boldsymbol{B}^{(3)}$ is a magnetic field, it must be described classically by the Biot-Savart-Ampère (BSA) law as usual. In this section this expectation is supported by the following analysis in vacuo. If classical electromagnetism in vacuo is simply the spiralling motion of e at c, then our derivation follows clearly. The source of $B^{(3)}$ is a moving charge, e, but this is *not* the electron, because mass is not present in the Maxwellian point of view, it is in that view the elementary charge propagating without mass. If the photon has mass, then this is thought [1,2] to be at least thirteen or fourteen orders of magnitude less than that of the electron. The BSA law for $B^{(3)}$ follows from the fact that classical electromagnetism is the spiralling motion of e in vacuo, and the law can be expressed as

$$B^{(3)*} = -\frac{i}{c^2} v^{(1)} \times E^{(2)}, \tag{240}$$

in S.I. units. Here $v^{(1)}$ is the transverse velocity of the charge in vacuo, and as shown earlier in this chapter, the magnitude of this velocity must be c. Equation (240) is also a direct consequence of special relativity. In the circular basis ((1), (2), (3)) [1,2] the field $E^{(2)}$ is a rotating electric field, and so $v^{(1)}$ is also a rotating transverse velocity, that of the spiralling charge e in vacuo.

The self-consistency of Eq. (240) can be checked by expressing $E^{(2)}$ as

$$E^{(2)} = i\omega A^{(2)} = -\frac{\partial A^{(2)}}{\partial t}, \tag{241}$$

in terms of the rotating vector potential $A^{(2)}$. Similarly, the complex conjugate $E^{(1)}$ (whose physical, real, part is the same as $E^{(2)}$ by definition) can be expressed as

$$E^{(1)} = -i\omega A^{(1)} = -\frac{\partial A^{(1)}}{\partial t}. \tag{242}$$

The BSA law for $B^{(3)}$ therefore becomes expressible as

$$B^{(3)*} = \frac{\kappa}{c} v^{(1)} \times A^{(2)}. \tag{243}$$

The factor $\kappa v^{(1)}/c$ in Eq. (243) has the units of a wave-vector, and can be expressed in quantum mechanics as a momentum operator ($\hbar\kappa = ip$), giving the result

$$B^{(3)*} = -\frac{i}{\hbar} p^{(1)} \times A^{(2)}. \tag{244}$$

Re-expressing $p^{(1)}$, finally, as a del operator, $p^{(1)} = -i\hbar\nabla^{(1)}$, we recover the fundamental definition of a magnetic field in terms of the curl of a vector potential. As demonstrated in Vol. 1 and 2 [1,2], there is no physical $A^{(3)}$ or $E^{(3)}$, and so in the Dirac equation, the field and fermion interact through $A^{(1)}$ and $A^{(2)}$ only, these being the only physical components of the vector potential in vacuo. If classical electromagnetism is spiralling charge, then there is no longitudinal electric field in vacuo, in analogy with a solenoid. The latter produces a longitudinal $B^{(3)}$ but no longitudinal $E^{(3)}$. The only conceptual difference is that in a solenoid, the charge is carried by a massive electron in the windings of the solenoid, in the vacuum the charge is massless, F.A.P.P. There is no Faraday induction in vacuo of a putative $E^{(3)}$ from $B^{(3)}$ because for a given frequency ω, there is no experimental method available of changing the spiralling motion of e in vacuo. Its radius is fixed at c/ω and its forward and transverse velocities at c. In contrast, the ordinary Faraday induction law in vacuo is one of the Maxwell equations, and deals with the physical transverse electric and magnetic fields.

4.7 EQUIVALENT FORMS OF $B^{(3)}$ IN VACUO

The fundamental and widely known axioms of the quantum theory assert that

$$En = \hbar\omega, \quad p = \hbar\kappa, \quad p \rightarrow -i\hbar\nabla, \quad En \rightarrow i\hbar\frac{\partial}{\partial t}, \tag{245}$$

and so it is possible to write transverse momenta in the circular basis as wave-vectors and del operators. For example

$$p^{(1)} = \hbar\kappa^{(1)} = -i\hbar\nabla^{(1)}, \quad p^{(2)} = \hbar\kappa^{(2)} = -i\hbar\nabla^{(2)}. \tag{246}$$

The origin of these transverse momenta is, as we have seen, simply the spiralling motion of e at c. The famous axioms (245) identify the particulate and undulatory nature of radiation and matter, and make all particles waves and vice-versa. This is the idea originally proposed by de Broglie [94]. The classical forms of the Evans-Vigier field $B^{(3)}$ show that it is a physical observable, directly so in the strong field limit, and show that the group space of electromagnetism is that of O(3). The electromagnetic sector of unified field theory is therefore O(3), perhaps with far reaching consequences. Some of these have been sketched in Vol. 2 [2]. In other words, vacuum electromagnetism is elementary and universal charge, e, spiralling through free space, and the often complicated mathematical machinery of field theory is subsidiary to, and dependent upon, this very simple view. Nature at the fundamental level appears to be inherently simple provided the source of understanding is found.

The most intricate derivation of $B^{(3)}$ must therefore depend on the simple, spiralling e, and all the expressions given for $B^{(3)}$ in this chapter must be equivalent. By using condition (188) in wave-vector form,

$$p^{(1)} = eA^{(1)} = \hbar\kappa^{(1)}, \tag{247}$$

two more equations for $B^{(3)}$ can be derived. In Eq. (247), $\kappa^{(1)} = \kappa^{(2)*}$ is a rotating, transverse, wave-vector in the circular basis [1,2], a vector whose magnitude is κ. Using Eq. (188) we obtain

$$B^{(3)*} = -i\frac{B^{(0)}}{\kappa^2}\kappa^{(1)} \times \kappa^{(2)} = -i\frac{B^{(0)}}{p^{(0)2}}p^{(1)} \times p^{(2)}. \tag{248}$$

The various forms of $B^{(3)}$ are collected in Table 3. They can all be interpreted physically to mean that e spirals forward at the speed of light, the radius of the spiral being c/ω. The elementary charge appears explicitly in the O(3) form, and implicitly in the nine others. All balance $\hat{C}$, $\hat{P}$, and $\hat{T}$ symmetry [1]. The BSA and curl A forms are standard, classical expressions for a magnetic flux density. Condition (188) produces a *Dirac form* directly from the O(3) gauge form. Therefore the equivalence condition (188) is confirmed as a fundamental part of vacuum electromagnetism. The same conclusion is obtained from the relativistic HJ equation of e in A_μ. The BSA form is obtainable from the double A form using it, and the BSA form is the earliest classical understanding of the source of a magnetic field. The source of $B^{(3)}$ is found in the propagating electromagnetic plane waves, which can be understood in circular polarization simply as spiralling elementary charge e. Each photon can be thought of as producing its own quantum of $B^{(3)}$, which we have alluded to as the photomagneton [4].

TABLE 3

Forms of the Evans-Vigier Field $B^{(3)}$

Form	$B^{(3)*}$
Double A	$B^{(3)*} = -i\frac{\kappa}{A^{(0)}}A^{(1)} \times A^{(2)}$
O(3) Gauge	$B^{(3)*} = -i\frac{e}{\hbar}A^{(1)} \times A^{(2)}$
Biot-Savart-Ampère	$B^{(3)*} = -\frac{1}{cp^{(0)}}p^{(1)} \times E^{(2)}$
AE	$B^{(3)*} = -\frac{1}{cA^{(0)}}A^{(1)} \times E^{(2)}$
Curl A	$B^{(3)*} = -\nabla^{(1)} \times A^{(2)}$
Dirac	$B^{(3)*} = -\frac{i}{\hbar}p^{(1)} \times A^{(2)}$
Double B	$B^{(3)*} = -\frac{i}{B^{(0)}}B^{(1)} \times B^{(2)}$
Double E	$B^{(3)*} = -\frac{i}{cE^{(0)}}E^{(1)} \times E^{(2)}$
Double p	$B^{(3)*} = -i\frac{B^{(0)}}{p^{(0)2}}p^{(1)} \times p^{(2)}$
Double κ	$B^{(3)*} = -i\frac{B^{(0)}}{\kappa^2}\kappa^{(1)} \times \kappa^{(2)}$

With the benefit of hindsight, the double E, B and A forms of Table 3 have already shown that $iB^{(0)}B^{(3)}$ has been observed [16—21] experimentally many times using visible frequency radiation, and once with 3 GHz microwave radiation [17c]. Examples include the inverse Faraday effect [17], optical Faraday effect [79,95], and light shifts in atomic spectra [18]. Less clearly, but definitively, light induced NMR shifts have been reported [20] as described in Chap. 1 and 2. It is now clear from the work of Chap. 3 that these are phenomena of the weak field limit, and are proportional to intensity, I. The conditions of the strong field limit appear not to have been approached yet, as discussed in earlier volumes [1,2]. Electric and magnetic fields are space and time derivatives of vector and scalar potentials, and the latter determine the way in which electromagnetic fields interact with matter. This is the minimal prescription, a consequence of gauge theory and the conservation theorems. The interaction of $\boldsymbol{B}^{(3)}$ with a fermion is therefore determined by the fermion's interaction with $\boldsymbol{A}$ and $\boldsymbol{A}^*$, a consideration which shows that our first, heuristic, theory of $\boldsymbol{B}^{(3)}$ [1—15] is the strong field limit of this chapter. In the weak field limit, Eq. (2) applies.

The existence in the vacuum of $\boldsymbol{B}^{(3)}$ neatly demonstrates the acute weaknesses inherent in the view that electromagnetism is a flat U(1) sector of unified field theory. The $\boldsymbol{B}^{(3)}$ field is both physical and perpendicular to the plane used to define the U(1) group. Furthermore $\boldsymbol{B}^{(3)}$ is the immediate result of a spiralling e, which emerges from classical radiation theory [47]. The Euclidean little group E(2) [1,2,23] must be replaced by an O(3) little group. For this reason the photon as particle can no longer be thought of as massless, because in that view the little group is the unphysical [23] E(2) and fields in vacuo are transverse. Our overall conclusion is that there are no massless particles in nature.

4.8 CONSERVATION OF ENERGY-MOMENTUM IN VACUO

It is well known that the electrodynamical conservation laws are fundamental to any consideration of light in free space or the interaction of light with matter. The momentum and energy of a radiation pulse totally contained within a finite volume V [52] has the same Lorentz transformation properties as a material point particle, and so the laws of conservation of energy and momentum must be similar to those of a particle. The latter is conventionally asserted to have no mass, and is the *standard* photon. The previous section concluded, however, that there can be no massless particle in nature, so the standard photon is a flawed concept. In the received view [52] the energy-momentum conservation law of classical field theory can be expressed as

$$\frac{\partial T^{\mu\nu}}{\partial x^{\mu}} = 0, \tag{249}$$

where $T^{\mu\nu}$ is the energy momentum four-tensor and x^{μ} the metric. This implies that the quantity

$$G^{\mu}G_{\mu} = W^2 - c^2G^2 = 0, \tag{250}$$

is a Lorentz invariant, where W denotes classical electromagnetic energy density and where G denotes classical electromagnetic linear momentum density

$$\hbar\omega = \int W dV = c\int G dV. \tag{251}$$

These relations remain valid in the presence of the magnetic field $\boldsymbol{B}^{(3)}$, because the same concept that produces (249) and (251) produces $\boldsymbol{B}^{(3)}$. This concept is simply that of e spiralling in vacuo at c. In this section it is shown that the validity of the Rayleigh Jeans and Planck laws remain unaffected by $\boldsymbol{B}^{(3)}$. This must be so because the source of $\boldsymbol{B}^{(3)}$ (e spiralling through the vacuum) is the same as that of the conventionally accepted transverse fields $\boldsymbol{B}^{(1)}$, $\boldsymbol{B}^{(2)}$, $\boldsymbol{E}^{(1)}$ and $\boldsymbol{E}^{(2)}$.

For example, the value of the Planck constant remains the same, because $\boldsymbol{B}^{(3)}$ simply causes the available electromagnetic energy density to be re-distributed among three space indices rather than two. The radiation oscillator must therefore be associated at each angular frequency ω by

$$U_{new} = \frac{1}{\mu_0}\left(\boldsymbol{B}^{(1)}\cdot\boldsymbol{B}^{(1)*} + \boldsymbol{B}^{(2)}\cdot\boldsymbol{B}^{(2)*} + \boldsymbol{B}^{(3)}\cdot\boldsymbol{B}^{(3)*}\right), \tag{252}$$

instead of the traditional

$$U = \frac{1}{\mu_0}\left(\boldsymbol{B}^{(1)}\cdot\boldsymbol{B}^{(1)*} + \boldsymbol{B}^{(2)}\cdot\boldsymbol{B}^{(2)*}\right), \tag{253}$$

When the sense of circular polarization (handedness) is switched from right (+) to left (-),

$$\boldsymbol{B}_+^{(1)} = \boldsymbol{B}_+^{(2)*} \rightarrow \boldsymbol{B}_-^{(1)} = \boldsymbol{B}_-^{(2)*}, \qquad \boldsymbol{B}_+^{(3)} \rightarrow \boldsymbol{B}_-^{(3)}, \tag{254}$$

and the total field is defined in both right and left circular polarization by three circular indices,

$$B_+ = B_+^{(1)} + B_+^{(2)} + B_+^{(3)}, \quad \Downarrow \quad B_- = B_-^{(1)} + B_-^{(2)} + B_-^{(3)}, \tag{255}$$

n terms of the electric components

$$U_{new} = \epsilon_0 \left(E^{(1)} \cdot E^{(1)*} + E^{(2)} \cdot E^{(2)*} + iE^{(3)} \cdot (iE^{(3)})^* \right), \tag{256}$$

eplaces the conventional

$$U = \epsilon_0 \left(E^{(1)} \cdot E^{(1)*} + E^{(2)} \cdot E^{(2)*} \right). \tag{257}$$

The premultiplier (factor of two) in the density of states calculated from the Rayleigh-eans law is therefore *two* in the presence of $B^{(3)}$, because there are still two senses of ircular polarization (+ and - subscripts).

The Rayleigh-Jeans law determines the number density of oscillators with requencies in the range ω to $\omega + d\omega$. For each oscillator there are two senses of ircular polarization, *each* of which is described by three circular indices. The two enses are physically distinct, and for each there exists the cyclically symmetric algebra 239). In left circular polarization the helical motion of e through free space at c is he mirror image of the motion in right circular polarization. These two physically listinct components determine the premultiplying value of two in the Rayleigh-Jeans law or the density of states,

$$dN = \frac{8\pi\nu^2}{c^3} d\nu, \tag{258}$$

vhere ν is the frequency of the wave. The existence of $B^{(3)}$ does not therefore affect Eq. (258) for the density of states, but $B^{(3)}$ does of course introduce the third space ndex to the analysis. When the sense of circular polarization of the beam is witched, $B^{(3)}$ changes its direction. In the strong field limit this effect might become experimentally observable. In the weak field limit, the interaction is determined by $B^{(3)}$ at second order, through $A^{(1)} \times A^{(2)}$.

The Planck hypothesis of November 1900 [61] asserts that a radiation oscillator can possess only discrete energies, measured in quanta, $\hbar\omega$, of radiation energy. The quantum is the dictionary definition [61] of the photon. The effect of $B^{(3)}$ is to change

the mathematical expression of the Planck hypothesis for one photon from

$$\hbar\omega = \frac{1}{\mu_0} \int \left(B^{(1)} \cdot B^{(1)*} + B^{(2)} \cdot B^{(2)*} \right) dV, \tag{259}$$

to

$$\hbar\omega = \frac{1}{\mu_0} \int \left(B^{(1)} \cdot B^{(1)*} + B^{(2)} \cdot B^{(2)*} + B^{(3)} \cdot B^{(3)*} \right) dV_1 . \tag{260}$$

Thus, classical electromagnetic energy is expressed as a sum over three circular indices instead of two, but the total available energy density is the same. The numerical values of the right hand sides of equations (259) and (260) are the same, and the Planck constant $\hbar$ is unchanged. The photon also remains the same and there is no change in the Planck radiation law,

$$dU = \frac{8\pi h\nu^3}{c^3} \left(\frac{e^{-h\nu/kT}}{1 - e^{-h\nu/kT}} \right) d\nu . \tag{261}$$

The total energy density of black body radiation is obtained by integrating dU over all frequencies [61]. Since $\hbar\omega$ is redistributed among three circular indices, so is black body radiation, but it is not possible to isolate the specific effect of $B^{(3)}$ from that of $B^{(1)} = B^{(2)*}$ simply through measurement of black body radiation with heat detectors. The specific effect of $B^{(3)}$ can be measured in the strong field limit, however, by isolating the expected $I^{1/2}$ profile in magneto-optic effects, as described in Vol. 1 and 2 [1,2].

The equilibrium between radiation and matter was shown by Einstein to be made up of several distinguishable processes, described by the well known Einstein coefficients [61]. The rate of absorption of electromagnetic radiation is described in terms of the Einstein B coefficient,

$$W_{f \leftarrow i} = B_{if}\rho , \tag{262}$$

from an initial quantum state $|i\rangle$ to a final quantum state of higher energy $|f\rangle$. Here ρ is the energy density of states at the absorption frequency ν_{fi}, and ρ must be evaluated at the transition frequency. For electric dipole transitions $W_{f \leftarrow i}$ is proportional to $En\,\hat{\rho}_N / V$ where En is the energy of the field at frequency ν and $\hat{\rho}_N$ is the frequency density of states for a given volume V occupied by the electromagnetic radiation. The specific effect of $B^{(3)}$ in these calculations is to cause the electromagnetic energy to be redistributed among three space indices (1), (2) and (3). The rate of absorption (262) is therefore also unaffected.

We conclude that the absorption of a photon of energy $\hbar\omega$ by an atom at a frequency ν_{fi} defined by a transition from $|i>$ to $|f>$ is affected by $B^{(3)}$ only insofar that the definition of $\hbar\omega$ is modified by $B^{(3)}$ as in Eq. (260). The angular momentum conservation rules that apply in the absorption of a photon are not changed by the existence of $B^{(3)}$, which is generated by the angular momentum of the photon about the propagation axis. The specific $I^{1/2}$ effect of $B^{(3)}$ has evaded detection because the strong field limit has not been attained.

4.9 CONSERVATION OF CHARGE

Charge is always conserved experimentally, and this feature of the natural world has become understood in terms of a continuity equation,

$$\nabla \cdot \boldsymbol{J} + \frac{\partial \rho}{\partial t} = \nabla \cdot \boldsymbol{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0. \tag{263}$$

The scalar potential ϕ of electromagnetic radiation in vacuo due to a point charge $ee^{i\omega t}$ [96] is identical to the equivalent electrostatic potential except that $t - R/c$ is substituted for t where R is the source to observer distance. *The potential corresponds to a state of the charge at a previous time* $t - R/c$. Therefore the appearance of e in the vacuum equations in previous sections can be understood as corresponding to its state in the source volume at a time R/c earlier. If the charge e had been circling in the source at this time, then it would have formed a longitudinal magnetic field according to the Biot-Savart law. At a time R/c later this field appears in the vacuum and is the curl of a vector potential which can be defined in general through the retarded current, $[\boldsymbol{J}]$,

$$\boldsymbol{A} = \frac{\mu}{4\pi}\int_{\infty}\frac{[\boldsymbol{J}]}{R}dV. \tag{264}$$

This is the well known mechanism of Liénard and Wiechert [96] and it is a mechanism which conserves e. The vacuum $B^{(3)}$ field corresponds to its state in the source a time R/c earlier, a source in which it was formed by circulating charge. This is also true of the transverse oscillating fields, and the circulating charge does not produce an axial electric field because the Biot-Savart law does not do so in the source. The phase

velocity of light in vacuo is c for the conventionally massless photon, and we see distant stars as they were, millions of years ago, due precisely to the Liénard-Wiechert mechanism. The complex nature of $A^{(1)}$ and $A^{(2)}$ allow therefore for phase differences in the source, as they were a time R/c ago [96].

Therefore all observable field components propagating in vacuo also correspond to a state of charge a time R/c earlier, and this is how e must be interpreted in vacuo. A circulating charge in the source transmits its influence, via the vacuum $B^{(3)}$, to produce a circulating movement of charge in matter, i.e., a magnetization known as the inverse Faraday effect. The machinery of interaction of field and matter is governed, however, through the Dirac equation using the potentials, and not directly through the fields themselves. This is a statement of the minimal prescription [22—25], which is itself a consequence of gauge invariance and the conservation of charge. This leads to the weak and strong field limits discussed in Chap. 3.

At a time t the field $B^{(3)}$ corresponds to a state of circulating charge a time R/c earlier; at time t' the field $B^{(3)}$ corresponds to a state of circulating charge a time R'/c earlier; and so on. So it is clear that $B^{(3)}$ propagates in vacuo at the phase velocity c. It propagates at the same phase velocity as the transverse components $B^{(1)}$ and $B^{(2)}$, the rotating transverse magnetic components that correspond also to a state of charge a time R/c earlier. Since $B^{(3)}$ is the curl of a vector potential by definition, its influence on matter is determined by the influence of that vector potential on matter, and this occurs through the use of the minimal prescription in the Dirac equation. The radiative corrections of quantum electrodynamics can now be incorporated in this classical picture, and it is clear that $B^{(3)}$ violates none of the principles of electromagnetic theory.

Chapter 5. The Classical Radiation Theory of $\boldsymbol{B}^{(3)}$

There exists in free space a primordial $\boldsymbol{B}^{(3)}$ field (Chap. 8). Its interaction with matter depends on the intensity and frequency of the electromagnetic field that transmits $\boldsymbol{B}^{(3)}$ from source to sample. In this chapter we first prove this theorem using a simple development of the radiation theory summarized in the last chapter. The effects of the free space $\boldsymbol{B}^{(3)}$ on radiation theory (boson ensembles) are then developed following the methods first proposed by Dirac for quantizing the electromagnetic field.

5.1 RADIATION OF $\boldsymbol{B}^{(3)}$ FROM A CIRCLING ELECTRON

In this section it is shown that a circling source electron radiates $\boldsymbol{B}^{(3)}$ to a sample electron, which is set in circular motion according to the reciprocity theorem [96]. The latter states that the current in a detector divided by the voltage applied at the source remains constant when source and detector are interchanged, as long as the frequency and impedances are left unchanged. This theorem applies to any electromagnetic field, and therefore applies to $\boldsymbol{B}^{(3)}$. The energy emitted as radiation by the circling source electron is given up to the sample electron, which is thereby set in circular motion. This process can be understood entirely in terms of the field $\boldsymbol{B}^{(3)}$, which is radiated through the vacuum by the Liénard-Wiechert potentials $\boldsymbol{A}^{(1)}$ and $\boldsymbol{A}^{(2)}$, which are transverse to the direction of propagation. The circular motion set up in the sample electron is magnetization, and this process is controlled by the Dirac equation as described in Chap. 1 and 2, using only the transverse potentials $\boldsymbol{A}^{(1)}$ and $\boldsymbol{A}^{(2)}$ through the minimal prescription. The overall process is simply the influence of one circling electron on another, initially static, sample electron.

It can be demonstrated qualitatively in the laboratory using a model consisting of an ideally massless charged object (A) on the end of a rotating rod. The amount and sign of charge and angular velocity can be varied experimentally. Another, initially stationary, ideally massless and charged object (B) is constrained to rotate in the same plane on the perimeter of a circular loop whose diameter is just larger than that of the rod carrying A. If A is highly charged and rotating very slowly, B is either attracted or repelled by A (depending on whether it is charged with the opposite or same sign respectively), and the angular velocity of B is approximately the same as that of A, and is the *same* in direction independently of the sign of charge. The charged object B is either pushed (repelled) or pulled (attracted) in an orbit by the rotating A, clockwise if A is moving clockwise, or vice-versa. The motion of B is the result of the magnetic field set up by A. If this macroscopic laboratory model is replaced by a system A consisting of a source electron moving in a circle, and a system B consisting of a sample electron, the sample electron will move in a circle due to the magnetic field set up by the source electron. This is the field $\boldsymbol{B}^{(3)}$, and the source and sample electrons can be separated by millions of light years because electromagnetic radiation from the former reaches the latter through the *radiated* $\boldsymbol{B}^{(3)}$. In the strong field limit, the angular frequency of the radiation is low (source electron moving slowly) and its intensity is high. In this limit, as shown by the Dirac equation, the circling motion produced in the sample electron is proportional to $\boldsymbol{B}^{(3)}$ and to the square root of beam intensity.

In the weak field limit the massless object A is weakly charged (low field intensity) and is spinning very rapidly (high frequency). The effect on B will be indirect, it will take many revolutions of A before B starts moving, and it will be out of synchronization with A. In the weak electromagnetic field limit, the Dirac equation shows that the effect on the sample electron is second order in the magnitude of $\boldsymbol{B}^{(3)}$ and first order in field intensity. Intermediate cases will show a dependence on both I and square root I as discussed in Chap. 12 of Vol. 1 [1].

In both cases discussed above, the sample electron (charge) is set in motion in a circle by the source electron (charge), and this is magnetization. The magnetic field $\boldsymbol{B}^{(3)}$ is simply the rotation of charge. In the magneto-optics literature the magnetization is known as the inverse Faraday effect [16—21] if the circling motion is caused by a circularly polarized electromagnetic field. It becomes clear that the radiated $\boldsymbol{B}^{(3)}$ field is due to the rotation of the potentials $\boldsymbol{A}^{(1)}$ and $\boldsymbol{A}^{(2)}$, which according to the Liénard-Wiechert law refer to a state of charge [96] a time R/c earlier, as discussed in Chap. 4 (R is used in this section to distinguish the observer to source distance from the radius, r, of a charge circling in a source a time R/c earlier.) At a time R/c prior to that at which $\boldsymbol{B}^{(3)}$ is observed in the vacuum, the state of the charge was its state in the source of radiated $\boldsymbol{B}^{(3)}$. The radiated $\boldsymbol{A}^{(1)}$ and $\boldsymbol{A}^{(2)}$ are transverse to the direction of propagation, but they represent a state of circling charge a time R/c earlier. This circling charge produces the $\boldsymbol{B}^{(3)}$ field, and the magnetization of the inverse Faraday effect. In order to observe the direct effect of $\boldsymbol{B}^{(3)}$ at first order, it is necessary to observe the square root intensity dependence of the magnetization [1,2]. The same rotating charge that produces $\boldsymbol{A}^{(1)}$ and $\boldsymbol{A}^{(2)}$ must inevitably produce $\boldsymbol{B}^{(3)}$, and since $\boldsymbol{A}^{(1)}$ and $\boldsymbol{A}^{(2)}$ exist in vacuo (being Liénard Wiechert potentials) so does $\boldsymbol{B}^{(3)}$. By causality, it is not possible to obtain electromagnetic radiation without a source of radiation having been present a time R/c earlier.

Conversely, if $\boldsymbol{B}^{(3)}$ is zero in vacuo, there are no potentials present, and no electromagnetic radiation. Any argument [62—65] that attempts to show that $\boldsymbol{A}^{(1)}$ and $\boldsymbol{A}^{(2)}$ are non-zero while $\boldsymbol{B}^{(3)}$ is zero is therefore incorrect, because that argument violates the Biot-Savart law. This conclusion holds independently of the details of the argument because, as we show next, the rotating elementary charge, e, radiates $\boldsymbol{A}^{(1)}$, $\boldsymbol{A}^{(2)}$ and $\boldsymbol{B}^{(3)}$ simultaneously. In other words, the fact that e radiates the Liénard-Wiechert potentials means that it must also radiate $\boldsymbol{B}^{(3)}$, because without $\boldsymbol{B}^{(3)}$ there is no $\boldsymbol{A}^{(1)} = \boldsymbol{A}^{(2)*}$.

5.2 RADIATION FROM THE ROTATING ELEMENTARY CHARGE e

Corson and Lorrain [96] have provided a clear summary of classical radiation theory, which produces electromagnetic fields in vacuo through the scalar and vector potentials, ϕ and $\boldsymbol{A}$ respectively. In S.I. units these can be written as a single potential four-vector,

$$A_\mu = (c\boldsymbol{A}, \phi). \tag{265}$$

The Aharonov-Bohm effects [32—39] show that the potentials are physically meaningful in classical electrodynamics and the interaction of radiation with matter takes place through the minimal prescription, i.e., through the four-vector A_μ and not through the fields $\boldsymbol{E}$ and $\boldsymbol{B}$. Electric and magnetic fields are therefore secondary concepts. In particular, a magnetic field is essentially a moving charge, and it is shown in this section that the circular motion of one elementary charge, e, radiates $\boldsymbol{A}^{(1)} = \boldsymbol{A}^{(2)*}$ simultaneously with $\boldsymbol{B}^{(3)}$, meaning that if $\boldsymbol{B}^{(3)}$ were zero there would be no source for the electromagnetic field, and by causality, no field. This conclusion is easily seen from the cyclic relations (154) by setting $\boldsymbol{B}^{(3)} = 0$. This results in $\boldsymbol{B}^{(1)} = \boldsymbol{B}^{(2)} = 0$ and the complete loss of electromagnetic radiation in vacuo.

The electromagnetic potentials satisfy the nonhomogeneous wave equations,

$$\nabla^2\phi - \epsilon\mu\frac{\partial^2\phi}{\partial t^2} = -\frac{\rho}{\epsilon}, \qquad \nabla^2\boldsymbol{A} - \epsilon\mu\frac{\partial^2\boldsymbol{A}}{\partial t^2} = -\mu\boldsymbol{J}, \tag{266}$$

where ρ and $\boldsymbol{J}$ are the source charge and current densities respectively. The solutions of these equations for ϕ and $\boldsymbol{A}$ at time t are expressible in terms of ρ and $\boldsymbol{J}$ defined at the earlier time $t - R/c$. The time R/c is that taken by an electromagnetic wave to travel the distance R between the element of volume where ρ and $\boldsymbol{J}$ are evaluated to the point of observation [96]. In the laboratory this interval of time is very short, and the interaction between charges appears to be instantaneous. On a cosmological scale the interval can be millions of light years, but the ρ and J must still be defined as they were in the source. If there *was* no source there *is* no radiation. For electric dipole radiation for example [96], a transverse spherical wave radiates away from the dipole, and $E^{(0)}/B^{(0)}$ is the same as for a plane wave in the vacuum. Nearer the source, the radiation is more complicated in form. In essence, the transverse plane wave is a charge spiralling forward in the vacuum, and this single charge produces both $A^{(1)} = A^{(2)*}$ and $B^{(3)}$. The latter are therefore descriptions derived from the same moving charge, and it is not possible to assert that $B^{(3)}$ is zero for non-zero $A^{(1)}$ because at an instant t they are descriptions of the same thing, a circling elementary e defined a time R/c earlier. Being different ways of describing the same thing, they must be equivalent and so one cannot be zero while the other is non-zero. These points are clarified by the following simple example.

Consider an idealization of a source of electromagnetic radiation, an idealization consisting of the elementary charge e describing a circle of radius r at an angular frequency ω. In the basis ((1), (2), (3)) this motion defines the dipole moments [96],

$$p^{(1)} = ere^{i\omega t}e^{(1)}, \qquad p^{(2)} = ere^{-i\omega t}e^{(2)}, \tag{267}$$

at the instant t. The real parts of these two equations are the same, and describe a charge e rotating clockwise in a plane. As shown in Chap. 4, this motion produces the magnetic flux density,

$$B^{(3)} = \mu_0 m^{(3)} = \frac{\mu_0 erve^{(3)}}{V}, \qquad v = \omega \times r. \tag{268}$$

Without loss of generality, the volume V can be defined as $(4/3)\pi r^3$, i.e., as a sphere of radius r. The magnitude $B^{(0)}$ of $B^{(3)}$ is therefore

$$B^{(0)} = \frac{3\mu_0 ev}{4\pi r^2}, \tag{269}$$

which has the right units of J s C^{-1} m^{-2}, i.e., tesla or Wb m^{-2} i.e., V s m^{-2}. So $B^{(0)}$ is a magnetic flux $(3/4)\mu_0 ev$ in *Wb* divided by the area πr^2. This is another way (more complicated and more obscure) of saying that e circles with a radius r and angular velocity ω. The magnetic flux in *Wb* is proportional to the product ev. In terms of the dipoles,

$$ere^{(3)*} = -\frac{i}{|p|}p^{(1)} \times p^{(2)}. \tag{270}$$

The transverse current (C rad s^{-1}) is $I = e\omega$. Therefore,

$$B^{(0)} = \frac{3\mu_0 I}{4\pi r}, \tag{271}$$

and the magnitude of the $B^{(3)}$ field is proportional to the current divided by the radius. Putting these definitions together gives

$$B^{(3)*} = -\frac{3\mu_0\omega}{4\pi r^3 e} ip^{(1)} \times p^{(2)}, \tag{272}$$

which shows that $B^{(3)}$ is proportional in the source to the conjugate product of $p^{(1)}$ with $p^{(2)}$. This example clarifies what is meant physically in electrodynamics by any conjugate product, it represents, essentially, a circling motion of charge and therefore a magnetic flux density. The magnetic flux density $B^{(3)}$ has been defined in the source in terms of circling charge, and in so doing the exponents $e^{i\omega t}$ and $e^{-i\omega t}$ have been used in a conjugate product.

This source radiates because the charge e is accelerating. The radiation is measurable at a point in space at an instant t if ωt is replaced in Eq. (272) by $\omega(t - R/c)$. The $B^{(3)}$ in free space is no longer static and reappears as

$$B^{(3)*} = -\frac{3\mu_0\omega}{4\pi r^3 e} i[p^{(1)}] \times [p^{(2)}], \tag{273}$$

where $[p^{(1)}]$ and $[p^{(2)}]$ signify [96] that the state of charge is defined at a time R/c earlier. This may be millions of light years or may be nanoseconds. As we have seen in earlier volumes [1,2] and chapters, Eq. (273) may be redefined as

$$B^{(3)*} = -i\frac{\kappa}{A^{(0)}}A^{(1)} \times A^{(2)}, \tag{274}$$

where $A^{(1)} = A^{(2)*}$ is the radiated transverse vector potential. Far enough away from the source, the only vector potentials present are the transverse potentials, as in classical radiation theory [96]. The transverse vector potential is defined in terms of the transverse current I as

$$A^{(1)} = \frac{3\mu_0}{4\pi}[J^{(1)}] := \int \frac{\mu_0}{V_0}[J^{(1)}]\,dV, \tag{275a}$$

$$[J^{(1)}] = Ie^{i(\omega t - \kappa R)}e^{(1)}, \tag{275b}$$

where V_0 can be taken to be a sphere or any other volume. Similarly, its complex conjugate is

$$A^{(2)} = \frac{3\mu_0}{4\pi}[J^{(2)}] := \int \frac{\mu_0}{V_0}[J^{(2)}]dV = \frac{3\mu_0 I}{4\pi}e^{-i(\omega t - \kappa R)}e^{(2)}, \tag{276}$$

and this is the result of standard radiation theory [96] applied to a rotating and radiating e in a source. Gradually, the whole of the energy of the rotating e will be dissipated by radiation and it will stop radiating. The energy loss per unit time is given up as radiative power [96] and this can be expressed in terms of $B^{(3)}$. Finally, using the free space relation $B^{(0)} = \kappa A^{(0)}$, we obtain from Eq. (275)

$$B^{(3)} = \frac{\mu_0 ec^2}{\omega V_0}e^{(3)}, \tag{277}$$

which is equation (223). Therefore $B^{(3)}$ originates in the radiating e circling at radius r a time R/c prior to the instant, t, at which $B^{(3)}$ is detected in free space. The conjugate product $A^{(1)} \times A^{(2)}$ represents exactly the same thing, but in a different mathematical language, and since $A^{(1)}$ and $A^{(2)}$ are the ordinary radiated vector potentials, $B^{(3)}$ is also radiated. This is an inference which is missing from conventional radiation theory, but the fact that $B^{(3)}$ is radiated simply means that there was a charge e circling at r a time R/c ago in the source of that radiation. A circling charge does not produce an electric field in the axis perpendicular to the plane in which the charge is circling, and there is no radiated $E^{(3)}$. Finally, the interaction of $B^{(3)}$ with matter is controlled by A_μ interacting with matter, and is the reverse of the radiative process. The correct way to describe this interaction is through the Dirac equation, as described in Chap. 1 and 2. Qualitatively, the interaction can be understood in terms of a mechanical model which can be set up in the undergraduate laboratory and which has been described in this chapter.

In summary, the field $B^{(3)}$ in the source is static and is

$$B^{(3)*} = -\frac{\mu_0\omega}{eV_0}ip^{(1)} \times p^{(2)}, \tag{278}$$

where V_0 is a volume defined by the radius of the rotating elementary charge, e. The radiated $B^{(3)}$ in the vacuum is propagating and is

$$B^{(3)*} = -\frac{\mu_0\omega}{eV_R}i[p^{(1)}] \times [p^{(2)}], \tag{279}$$

where V_R is the volume of radiation being considered, e.g. the sphere defined by the Thompson radius $\lambda/2\pi$. Since $B^{(3)}$ is non-zero in the source, it is non-zero in the vacuum.

5.3 CONSERVATION OF ANGULAR MOMENTUM AND ENERGY

The quantum theory asserts that the angular momentum of the photon is $\hbar$. The classical angular momentum magnitude of the rotating charge in the radiating source is $eA^{(0)}r$, where $A^{(0)}$ is the amplitude of $A^{(1)}$ in the source. In the radiated field, $r = \kappa^{-1}$, and we have, from Eq. (182),

$$\hbar = e\frac{A^{(0)}}{\kappa} = e\frac{c^2}{\omega^2}B^{(0)}. \tag{280}$$

The angular momentum radiated by the circling charge e is therefore the field angular momentum,

$$J^{(3)} = e\frac{c^2}{\omega^2}B^{(0)}e^{(3)} = \frac{1}{\mu_0\omega}B^{(0)2}Ve^{(3)}, \tag{281}$$

from Chap. 4. This means that

$$B^{(0)} = \frac{\mu_0 ec^2}{\omega V}, \tag{282}$$

as in Eq. (277) or (223).

The magnetic dipole moment per unit volume of the radiated field is obtained with $r = \kappa^{-1}$ and $v = c$ as

$$m^{(3)} = \frac{erv}{V}e^{(3)} \rightarrow \frac{ec^2}{\omega V}e^{(3)}. \tag{283}$$

Therefore the magnetic dipole moment of the field is

$$Vm^{(3)} = \frac{ec^2}{\omega}e^{(3)}. \tag{284}$$

Equations (281), (282) and (284) give the field's magnetogyric ratio

$$\gamma_f = \frac{\omega^2 V}{\mu_0 e c^2}, \tag{285}$$

such that

$$Vm^{(3)} = \gamma_f J^{(3)}. \tag{286}$$

It is easily verified that the units of γ_f are C kgm^{-1}, the units of the conventional magnetogyric ratio [45], $e/2m$, where m is mass. The equation

$$\gamma_f := \frac{e}{2m}, \tag{287}$$

gives

$$m = \frac{\mu_0 e^2 c^2}{2\omega^2 V}, \tag{288}$$

or,

$$\chi' = \frac{V}{\mu_0}, \tag{289}$$

where χ' is the susceptibility,

$$\chi' = \frac{e^2c^2}{2m\omega^2}, \tag{290}$$

used in Chap. 12 of Vol. 1 [1] for the electron. Here it is being used for the electromagnetic field itself, and m is the equivalent mass of the field [52].

Therefore the circling elementary charge radiates, among other things: an angular momentum, $J^{(3)}$; a magnetic dipole moment, $m^{(3)}$; a magnetic field, $B^{(3)}$; a field mass, m; a field susceptibility, χ'; and a field energy, $\omega J^{(3)}$. All these quantities are present in free space at the instant t and refer to the state of circling charge [96] a time R/c earlier in the source, where R is the source to observer distance.

Introducing the fine structure constant [22—25],

$$\alpha = \frac{e^2}{4\pi\hbar\epsilon_0 c} = \frac{e^2\mu_0 c}{4\pi\hbar}, \tag{291}$$

it follows from Eq. (288) that

$$mc^2 = \frac{3}{2}\alpha\hbar\omega. \tag{292}$$

This equation links the field energy mc^2 to the photon $\hbar\omega$, the quantum of electromagnetic energy. Since α is unitless, the equation is consistent, and should be compared with Eq. (1) of Vol. 1 [1], the de Broglie Guiding theorem.

$$m_0 c^2 = \hbar\omega, \tag{293}$$

where m_0 is the mass of the photon. Comparing equations (292) and (293),

$$m = \frac{3}{2}\alpha m_0. \tag{294}$$

If the photon mass were zero, as in much of the literature [23], then m would be zero from Eq. (294), because α is made up of the universal constants. Since m can be expressed as

$$m = \left(\frac{3}{8\pi}\mu_0 e^2 c\right)\omega, \tag{295}$$

it is zero if and only if ω is zero, in which case there is no classical radiation present. Therefore the photon mass is non-zero if $\boldsymbol{B}^{(3)}$ is non-zero, because the gyromagnetic ratio from which m is calculated is defined in terms of the magnetic dipole moment

$$V\boldsymbol{m}^{(3)} = \frac{V}{\mu_0}\boldsymbol{B}^{(3)} = \chi'\boldsymbol{B}^{(3)}. \tag{296}$$

The appearance of the fine structure constant in this analysis is probably deeply significant [6], because α is the hallmark of quantum electrodynamics, or QED [22—25].

The conservation of angular momentum and energy can be demonstrated using Eq. (402) of Vol. 1 [1], derived from the relativistic Hamilton-Jacobi equation of the classical electron in the classical field. If we replace the electron by a sample consisting of the elementary charge, e, and assume that any mass associated with the charge is very small, we obtain from Eq. (402) of Vol. 1 [1],

$$\boldsymbol{J}^{(3)} = \frac{e^2c^2}{\omega^2}\left(\frac{B^{(0)}}{\left(m_0^2\,\omega^2 + e^2 B^{(0)2}\right)^{1/2}}\right)\boldsymbol{B}^{(3)}, \tag{297}$$

$$\xrightarrow[m_0 \to 0]{} \frac{ec^2}{\omega^2}\boldsymbol{B}^{(3)} = \hbar \boldsymbol{e}^{(3)}. \tag{298}$$

This result is a clear demonstration of the transfer, under the right conditions, of all the field angular momentum ($\hbar$ per photon) to the sample. This process conserves angular momentum and the elementary charge e rotates on a radius c/ω with angular velocity ω. (If it had mass, the sample charge would be made to spiral, because the field has linear momentum.) Energy is similarly conserved through $\hbar\omega$, the photon energy, given up entirely to the sample charge e. The process of absorption can be thought of in these terms, because the light quantum hypothesis asserts that absorption takes place through the photon of energy $\hbar\omega$. This process is now seen to be controlled by $\boldsymbol{B}^{(3)}$ through equation (297) or its quantized equivalent, the Dirac equation.

From Eqs. (297) and (298) it is clear that if $\boldsymbol{B}^{(3)}$ were zero, there would be no field-charge interaction, contrary to experience. Equation (297) shows that the interaction is quadratic both in e and $B^{(0)}$, and the angular momentum given to the charge by the field is independent of the sign of either e or $B^{(0)}$, even in the limit $m_0 \to 0$, the strong field limit. A rotating charge pushes around a charge of the same sign (clockwise, for example), and pulls around a charge of opposite sign in the same sense, clockwise. The magnetization caused by $\boldsymbol{B}^{(3)}$ (the inverse Faraday effect [16—21]) is proportional to $\boldsymbol{J}^{(3)}$ and is independent of whether $\boldsymbol{B}^{(3)}$ is radiated by a negative charge $(-|e|)$ or positive charge $(|e|)$. In experiments such as the inverse Faraday effect, m_0 in Eq. (297) is always non-zero, both in the weak and strong field limits, because a real material sample must be made up of massive particles. Therefore the observed magnetization does not change sign with the sign of $\boldsymbol{B}^{(3)}$.

Unfortunately, there is a book review [97] in the literature that asserts without proof that $\boldsymbol{J}^{(3)}$ changes sign with the sign of $B^{(0)}$, but the dependence of $\boldsymbol{J}^{(3)}$ on $B^{(0)2}$ in Eq. (297) shows that this assertion is erroneous. The error is compounded by the assertion [97] that $\boldsymbol{B}^{(3)}$ is zero because it is claimed that there is no experimental evidence for this assumed change of sign. Equation (297) shows *that no such change of sign occurs* in relativistic field theory. A related article [62] completely misapplies $\hat{C}$ symmetry [1] in an attempt to show that $\boldsymbol{B}^{(3)}$ is zero. Any such conclusion violates the Biot-Savart law, that a magnetic field is rotating charge. Two related papers [63,64] appear to assert (obscurely) that $\boldsymbol{B}^{(3)}$ is zero, but as argued self-consistently here and elsewhere [1,2] this would mean the disappearance of all radiation. These papers display a fundamental lack of understanding of electrodynamics and relativity. Similar papers [98] on topics other than $\boldsymbol{B}^{(3)}$ but based on related symmetry arguments are probably incorrect, as argued elsewhere [4]. Their conclusions are at best fortuitously in accord with experience, at worst misleading.

In summary of this section it has been shown that a source consisting of a circling, elementary charge, e, with unspecified mass, radiates $\boldsymbol{B}^{(3)}$, which induces circular motion in a sample consisting of the elementary charge e, again with unspecified mass. If the target mass is assumed to be very small, the angular momentum of the radiated field is given up entirely to the charge, and this is a process which conserves angular momentum and energy. The rotating charge e in the source produces a $\boldsymbol{B}^{(3)}$ field through the Biot-Savart law, a field which induces rotation in the e of the sample. The whole process can be described in terms of the motion of the elementary charge e. In the source it circles at ω on a radius r, which is identified with the Thompson radius c/ω in free space radiation, which is the spiralling motion of e at the speed of light c. The charge e in the sample is put into motion when the spiralling e of the field interacts with it. The sample charge is pushed or pulled into circular motion by repulsion or attraction of the field charge. The language of fields, potentials and relativity which permeates these volumes is an attempt to describe this simple process.

5.4 MAXWELL'S CONCEPT OF THE ELECTROMAGNETIC FIELD

The electromagnetic field according to Maxwell's point of view is *an agent of interaction between two charges*. The existence of the $B^{(3)}$ field is a consequence of the Maxwellian picture, which leaves mass unspecified in the sense that mass does not appear directly in the Maxwell equations or the wave equations. The electromagnetic field is an example of the philosophical concept of fields in physics, a concept which has been clearly described by Barut [53], whose discussion we follow here.

External scalar or electromagnetic fields produce an *action at a distance* force on a sample particle or charge. More fundamentally, a field is a self-contained mechanical system in its own right, and can be described by equations of motion constructed in the appropriate mathematical limits [53]. The field carries energy, and has a continuously infinite number of degrees of freedom. It fills the whole of space like a fluid and has physical reality in relativistic field theory. The potential four-vector, A_μ, of the electromagnetic field is therefore directly proportional to the energy-momentum four-vector through the scaling constant e, which is also the elementary charge. Chapter three has shown that A_μ is physically meaningful in contemporary classical field theory because of the topology of the vacuum. In relativistic particle theory [53], the classical system is the mass point and the fields are auxiliary, phenomenological quantities. In relativistic field theory [53] the physical system is the field itself and physical laws are obeyed by the fields themselves. Interaction between fields and particles is described by an extra term in the appropriate Lagrangian [1,2], a term based on the minimal prescription as used in Chap. 1 and 2 for example. Therefore the field interacts with a particle through the vector and scalar potentials. This is why there occur weak and strong field limits.

The electromagnetic field is an energy carrying physical system in its own right [53], a system which produces interaction between charges. Therefore the field is an agent of interaction. Instantaneous action at a distance is replaced in special relativity by a signal velocity, or phase velocity, which is c in vacuo for an assumed massless photon. If there were no charges there would be no field, so the latter is described fundamentally in terms of the former, as we have shown in this and the preceding chapters. As described by Barut [53] the field is produced by a charged particle whose existence is assumed as the starting point of the analysis. The field is measured through the acceleration it produces when acting upon another charged particle. The field cannot be detected if this acceleration cannot be detected and all depends ultimately on the existence of the elementary e in the primordial universe. The equations which describe the motion of e are the Maxwell equations, or d'Alembert wave equations to which they are equivalent. The electromagnetic field propagates (moves forward) in vacuo because the relevant unit of time is $t - R/c$ as we have seen. Static electric and magnetic fields do not propagate and are equivalent to the limit $R = 0$, so the unit of time is t and there are no moving waves. The $B^{(3)}$ field propagates for precisely the same reason, it can be defined in terms of exponents in which appear the unit $t - R/c$.

The $B^{(3)}$ field becomes static, as for any other static field, when the exponents contain the time unit t and $R = 0$. This point has been illustrated in Eqs. (278) and (279). The fact that radiation of any kind occurs at all is due to the empirically supported equivalence of $\Box A_\mu$ and $-J_\mu/\epsilon_0$ in the d'Alembert equation. Therefore radiation is energy transfer from charge to charge and as we have seen in Chap. 4, the quantum of energy of the field, the photon, is proportional to the square of e. This is consistent with the fact that if there were no e there would be no photon. The interaction energy-momentum between field and charge is eA_μ, which is again proportional to the square of the elementary charge e. This gives the general rule that all equations of electrodynamics must conserve $\hat{C}$ symmetry, and that the $\hat{C}$ symmetry of fields and potentials is always negative.

These arguments are inherently relativistic, as is well known. For example Faraday induction is a relativistic phenomenon as pointed out clearly by Barut [53]. If we move a closed circuit in a magnetic field with velocity v_0, a charge in the loop will be acted on by a Lorentz force $ev_0 \times B$ with the same units as eE where E is an electric field. However, if v_0 is parallel to B, there is no force at all. Since $B^{(3)}$ is always parallel to the direction (Z) of propagation of electromagnetic radiation in vacuo, there is no electric force due to $v_0 \times B^{(3)}$ and no Faraday induction. Relativistically and equivalently, if we hold the circuit loop fixed and move the magnetic field, the same occurs, no Faraday induction. There is no Faraday induction due to $B^{(3)}$ and there is no $E^{(3)}$ [1,2]. This has been verified by available experiments [1,2,99] consisting of chopping a circularly polarized electromagnetic beam passed through a Faraday induction coil. These experiments can be repeated to any degree of precision and there will be no Faraday induction in free space due to $B^{(3)}$. (When $B^{(3)}$ encounters matter within an induction coil, there will occur the inverse Faraday effect [16—21].)

In the next section, these remarks are put into quantitative form by considering a radiating charge held in a circular orbit by a static magnetic field. This is a standard problem developed by Landau and Lifshitz [75] but shows with clarity why there is a radiated $B^{(3)}$.

5.5 RADIATION FROM A CIRCLING CHARGE HELD IN A STATIC MAGNETIC FIELD AT $R = 0$

Consider a particle with charge e and mass m constrained to move in a circle of radius r by a static magnetic field at $R = 0$. This means that the particle circles around the axis (Z) in which R is defined, i.e., if R were in Z, the charge circles in the XY plane. As ably demonstrated by Landau and Lifshitz [75], the transverse acceleration on the particle in its circular orbit is

$$\boldsymbol{w} = \frac{e}{m}\left(1 - \frac{v^2}{c^2}\right)^{1/2} \boldsymbol{v} \times \boldsymbol{B}^{(3)}, \tag{299}$$

where $\boldsymbol{v}$ is its transverse velocity. This is the result of the Lorentz force equation. The relativistic Lorentz force defined by Eq. (299) is therefore

$$\boldsymbol{F} = m\boldsymbol{w}\left(1 - \frac{v^2}{c^2}\right)^{-1/2} = e\boldsymbol{v} \times \boldsymbol{B}^{(3)}, \tag{300}$$

and can be expressed in the basis ((1), (2), (3)) [1,2] as

$$\boldsymbol{F}^{(1)*} = -ie\boldsymbol{v}^{(2)} \times \boldsymbol{B}^{(3)}. \tag{301}$$

This particle radiates because of the existence of a non-zero $\boldsymbol{w}$, a centripetal acceleration of charge. If the static magnetic field were switched off, the charge would continue to radiate away energy until eventually only the rest energy, mc^2, would be left. This process is a well known part of standard classical radiation theory [75]. The $\boldsymbol{B}^{(3)}$ field in Eq. (299) is static because it is calculated with $R = 0$, so that the unit of time being used is t. In the absence of radiation, this means that the charge is describing an orbit in a plane, and this orbit is a circle, not a helix, because there is no forward motion of charge along the Z direction.

As radiation occurs, however, $\boldsymbol{F}$ decreases in Eq. (300) due to a radiated $\boldsymbol{B}^{(3)}$ field; the transverse acceleration $\boldsymbol{w}$ tends always to decrease due to radiation but is kept constant by the applied magnetic field. By considering the limit $|\boldsymbol{v}| \to c$, it is easily demonstrated as follows that the radiation is the propagating $\boldsymbol{B}^{(3)}$ field discussed earlier and in Chap. 4.

The angular frequency of the charged particle in the static magnetic field is given by [75]

$$\omega = \frac{e}{m}\left(1 - \frac{v^2}{c^2}\right)^{1/2} B^{(0)}, \tag{302}$$

where $B^{(0)}$ is the amplitude of the field in tesla. In the limit $|\boldsymbol{v}| \to c$, the strong field limit, we can apply the equivalence principle, Eq. (280), in the form

$$e\frac{c^2}{\omega^2}B^{(0)} \xrightarrow[|\boldsymbol{v}| \to c]{} \hbar, \tag{303}$$

which shows that in this limit,

$$mc^2\left(1 - \frac{v^2}{c^2}\right)^{-1/2} \xrightarrow[|\boldsymbol{v}| \to c]{} \hbar\omega. \tag{304}$$

Similarly, the angular acceleration in this limit becomes

$$|\boldsymbol{w}| \xrightarrow[|\boldsymbol{v}| \to c]{} \omega c, \tag{305}$$

and using $\omega = v/r$, the radius becomes c/ω. The rotating charge in this limit is described by

$$|\boldsymbol{r}| \to \frac{c}{\omega}, \quad |\boldsymbol{v}| \to c, \quad |\boldsymbol{w}| \to \omega c, \tag{306}$$

and is radiating, because of the presence of non-zero transverse acceleration.

Equation (306) has all the properties of electromagnetic radiation except that we have not yet introduced forward motion along Z, we have considered the $|\boldsymbol{v}| \to c$ limit of the charge circling in a plane. The property of propagation at c is now *introduced* through the familiar equation,

$$\kappa = \frac{\omega}{c}, \tag{307}$$

where κ is the wave vector. The photon forward momentum is thus identified as $eA^{(0)} \to \hbar\kappa$. The radiation process is now describable by replacing the unit of time t by the unit of time $t - R/c$. The introduction of κ, and its identification with ω/c, *is sufficient to describe the radiation process*. Here R is the source to observer distance introduced earlier in this chapter. The dynamics of the system are now to be evaluated [96] at the instant $t - R/c$, and in so doing, we refer to a state of circling charge at observer point R a time $t_1 = R/c$ earlier in the source, situated at $R = 0$. Since all is unaffected except t, the quantities $\boldsymbol{r}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ remain as they were originally, meaning that e at point R must be the charge described earlier at point $R = 0$. Similarly the charge at a point R_1 further along the Z axis is the same as that described at point R a time $(R_1 - R)/c$ earlier and so on. The charge e spirals forward at c through the vacuum. The magnetic field $B^{(3)}$ is radiated from its position at $R = 0$ to its position at R a time R/c later, and the forward propagation has been introduced through the equation $\kappa = \omega/c$ linking the wavevector to the angular frequency. (If $B^{(3)}$ were not propagating there would be no wavenumber, no spiralling motion of e forward at c, and no radiation.)

To summarize therefore, the analysis has been based on the idea of a charge e attached to a particle of mass m circling at $R = 0$. The circling motion is caused by an applied static magnetic field. The circling motion of charge produces the static magnetic field

$$B^{(3)*} = -\frac{3\mu_0\omega}{4\pi r^3 e} i p^{(1)} \times p^{(2)}, \tag{308}$$

through the Biot-Savart law, as shown in Eq. (278). In this expression the time parameter is t. Conversely, therefore the field $B^{(3)}$ is the same as that needed to keep the charge in its circular orbit. In the limit $|v| \to c$, it has been demonstrated that the circling charge has all the characteristics of electromagnetic radiation except for forward motion (propagation at c in vacuo). The latter is *introduced* through the concept of wavevector $\kappa = \omega/c$ and this is sufficient to describe the phenomenon of radiation of $B^{(3)}$. The charge e spirals forward, and no longer remains in a plane. The radiated $B^{(3)}$ at position R a time R/c later is the same as in Eq. (308), except that

$$t \to t - \frac{R}{c}, \qquad |v| \to c. \tag{309}$$

The radiated $B^{(3)}$ is therefore

$$B^{(3)} = \left(\frac{3\mu_0 \omega e}{4\pi r}\right) e^{(3)} := B^{(0)} e^{(3)}, \tag{310}$$

and is proportional to e. If there were no e there would be no radiation, in accord with the philosophy of the electromagnetic field summarized briefly in the previous section.

The radiated $B^{(3)}$ is accompanied by the radiated plane waves $B^{(1)}$ and $B^{(2)}$, through which $B^{(3)}$ is defined, and the radiated $B^{(1)}$ and $B^{(2)}$ are proportional to the radiated $E^{(1)}$ and $E^{(2)}$. As demonstrated in Chap. 4, all these fields can be thought of in terms of a spiralling charge e. In the limit $|v| \to c$, the range of the radiation is effectively infinite, and the electromagnetic wave at the point R must be evaluated at the instant $t - R/c$. This remains true of $B^{(3)}$ because the latter is evaluated in terms of a product of conjugate phase exponents, each containing $t - R/c$. Therefore the circling e at $R = 0$ is the disturbance that initiates the travelling wave, and transmits the energy associated with the disturbance.

In the limit $|r| \to c/\omega$ Eq. (310) becomes

$$B^{(3)} \xrightarrow[|r| \to c/\omega]{} \frac{3\mu_0 e}{4\pi c}\omega^2 e^{(3)}, \tag{311}$$

showing that the intensity associated with the radiated $B^{(3)}$ is

$$I = \frac{c}{\mu_0} B^{(0)2} = \left(\frac{9\mu_0 e^2}{16\pi^4 c}\right)\omega^4, \tag{312}$$

and proportional to the fourth power of angular frequency. The intensity is inversely proportional to the fourth power of the wavelength

$$I = \frac{9\mu_0 e^2 c^3}{\lambda^4}. \tag{313}$$

It is interesting that the long wavelength limit of the Planck distribution,

$$I(\lambda, T) = \frac{2\pi h c^2}{\lambda^5}\left(e^{hc/(\lambda kT)} - 1\right)^{-1}, \tag{314}$$

is

$$I(\lambda, T) \xrightarrow[\lambda \to \infty]{} \frac{4\pi^2 \hbar c k T}{\lambda^4}, \tag{315}$$

and is also proportional to the inverse fourth power of wavelength. Therefore the result (313) is for one photon before thermal averaging in an ensemble of photons. In deriving Eq. (313) the volume occupied by the photon has been defined as a sphere of radius $(4/3)\pi r^3$, but the result can be generalized to any type of volume.

In essence, therefore, electromagnetic radiation is the replacement of time t by $t - R/c$, which is equivalent to the introduction of the relation (307) between wavenumber and angular frequency. In general, radiation is controlled by the d'Alembert wave equation, whose solutions in the plane wave approximation we have been considering in this section. In this context, the conventional literature asserts that radiation in the plane wave approximation is transverse to the direction of propagation, and this remains true in our analysis, the radiated potentials $A^{(1)}$ and $A^{(2)}$ are transverse, but their existence in vacuo means that there is a $B^{(3)}$ in vacuo. Under the right conditions, the characteristic square root intensity dependence of $B^{(3)}$ becomes observable experimentally [1,2].

5.6 THE LIÉNARD-WIECHERT POTENTIALS FOR $B^{(3)}$

The Liénard-Wiechert potentials are discussed in many excellent texts [7,52,53,96] to which we refer the interested reader. They describe radiation from a charge in an arbitrary trajectory. Radiation at observer point R is described at the instant $t - R/c$, and so the Liénard-Wiechert potentials are related to the retarded potentials. In general, and in S.I. units, the scalar and vector potentials radiated from the source are

$$\phi = \frac{ec}{4\pi\epsilon_0 R}\left(1 - \boldsymbol{k}\cdot\frac{[\boldsymbol{v}]}{c}\right)^{-1}, \tag{316}$$

and

$$\boldsymbol{A} = \frac{e\mu_0}{4\pi R}[\boldsymbol{v}]\left(1 - \boldsymbol{k}\cdot\frac{[\boldsymbol{v}]}{c}\right)^{-1}, \tag{317}$$

where $\boldsymbol{k}$ is the unit vector $\boldsymbol{R}/|\boldsymbol{R}|$, and where $[\boldsymbol{v}]$ is the velocity of the elementary charge e in the source at the instant $t - R/c$. The potentials are a direct outcome of special relativity [47,96]. Equations (316) and (317) are generally valid because they are manifestly covariant under Lorentz transformation, and therefore hold in any Lorentz frame. The theory of radiation is developed in many texts in terms of situations of interest and for various approximations to Eqs. (316) and (317). The fact that there is a source being considered means that the radiation emanates from that source, and this satisfactorily causal in character. If the charge in the source were not moving at $t - R/c$,

$$\phi = \frac{e}{4\pi R\epsilon_0}, \qquad \boldsymbol{A} = \boldsymbol{0}, \tag{318}$$

which is Coulomb's law.

In this section we use the Liénard-Wiechert potentials to describe the radiation of $B^{(3)}$ from a charge e circling at radius r with angular frequency ω on a mass m. This can be a circling electron, for example. The tangential (or transverse) velocity of the charge can be represented in complex form $\boldsymbol{v}^{(1)} = \boldsymbol{v}^{(2)*}$ in the circular basis [1,2] ((1), (2), (3)). There is always present an inward seeking (centripetal) acceleration because the charge is constrained in a circular orbit, for example by a static magnetic field as in the previous section. The complex vector potentials radiated by the centripetal acceleration of the charge e are, in general, the complex Liénard-Wiechert potentials,

$$\boldsymbol{A}^{(1)} = \boldsymbol{A}^{(2)*} = \frac{e\mu_0 c}{4\pi R}[\beta^{(1)}]\left(1 - \boldsymbol{k}\cdot[\beta^{(1)}]\right)^{-1}, \tag{319}$$

where $\beta^{(1)} := [\boldsymbol{v}^{(1)}]/c$. *It is important to note that mass does not appear specifically in these potentials*, and that the velocities are evaluated at $t - R/c$. There is energy flow associated with the radiation. The radiated $B^{(3)}$ field is therefore

$$\boldsymbol{B}^{(3)*} = -i\frac{e}{\hbar}\boldsymbol{A}^{(1)}\times\boldsymbol{A}^{(2)}, \tag{320}$$

is described in terms of these potentials and is always non-zero in radiation from a circling electron. The radiated $B^{(3)}$ is described *without the mass of the sample being specified.* The $B^{(3)}$ field in the sample depends on the mass as in Eq. (299). We will return to this key point later in this section.

The usual transverse radiated fields are [53], in S.I. units,

$$\boldsymbol{B}^{(1)} = \boldsymbol{B}^{(2)*} = \frac{e\mu_0}{4\pi R}\frac{\left((\boldsymbol{k} - [\beta^{(1)}])\times[\dot{\beta}^{(1)}] + \boldsymbol{k}\left(\boldsymbol{k}\cdot[\beta^{(1)}]\times[\dot{\beta}^{(1)}]\right)\right)}{\left(1 - \boldsymbol{k}\cdot[\beta^{(1)}]\right)^3}, \tag{321}$$

and

$$E^{(1)} = E^{(2)*} = \frac{e\mu_0 c}{4\pi R}\frac{\left(k \times \left((k - [\beta^{(1)}]) \times [\dot{\beta}^{(1)}]\right)\right)}{\left(1 - k \cdot [\beta^{(1)}]\right)^3}, \tag{322}$$

and these are generally valid equations without approximation. They illustrate the geometrical and dynamical intricacies associated with the many phenomena of radiation, e.g. from antennae [96], and at electron velocities close to c (synchrotron radiation).

When the source electron is circling slowly, then $\beta < 1$ [53], and in this limit the radiated vector potential at point R becomes directly proportional to the transverse velocity at time $t - R/c$,

$$A^{(1)} \xrightarrow[\beta < 1]{} \frac{e\mu_0}{4\pi R}[v^{(1)}]. \tag{323}$$

The radiated $B^{(3)}$ field therefore becomes

$$B^{(3)*} \xrightarrow[\beta < 1]{} -i\frac{e}{\hbar}\left(\frac{e\mu_0}{4\pi R}\right)^2 [v^{(1)} \times v^{(2)}], \tag{324}$$

where the conjugate product of velocities $[v^{(1)} \times v^{(2)}]$ must as usual be evaluated at the instant $t - R/c$. The radiated $B^{(3)}$ field is governed by $A^{(1)}$ and $A^{(2)}$ therefore, and so its interaction with matter is described by the interaction of $A^{(1)}$ and $A^{(2)}$ with matter, e.g. one electron. The equations of motion to be used to describe this interaction are the Dirac or Hamilton-Jacobi equations, based on the minimal prescription. This leads to the strong and weak field limits as described in Chap. 3. The radiated $B^{(3)}$ field adds nothing to the Poynting vector (energy flux) because it adds nothing to the linear momentum of radiation. Because it is radiated, it does however produce a novel rotational contribution to the vacuum energy density [1—15],

$$En = \frac{1}{\mu_0}\int B^{(3)} \cdot B^{(3)*} dV, \tag{325}$$

so that the radiation laws should be adjusted to take this into account, as described in Chap. 4. This simply means adjusting the effective volume of radiation V, leaving the value of $\hbar$ the same.

For low source electron velocities, the usual transverse radiated fields (321) and (322) become

$$B^{(1)} = B^{(2)*} \xrightarrow[\beta < 1]{} \frac{e\mu_0}{4\pi Rc} k \times [\dot{v}^{(1)}], \tag{326}$$

$$E^{(1)} = E^{(2)*} \xrightarrow[\beta < 1]{} \frac{e\mu_0}{4\pi R} k \times (k \times [\dot{v}^{(1)}]), \tag{327}$$

$$B^{(1)} \xrightarrow[\beta < 1]{} \frac{1}{c} k \times E^{(1)}, \tag{328}$$

and are radiated plane waves because Eq. (328) characterizes plane waves. There is a relation between the Liénard-Wiechert and retarded potentials which is satisfied by representing the source transverse velocity by

$$[v^{(1)}] = \frac{v_0}{\sqrt{2}}(ii + j)\exp\left(-i\omega\left(t - \frac{R}{c}\right)\right), \tag{329}$$

in which v_0 is the transverse speed of the source electron in its circular orbit. The transverse acceleration $[\dot{v}^{(1)}]$ is,

$$[\dot{v}^{(1)}] = -i\omega[v^{(1)}], \tag{330}$$

giving the center-seeking, or centripetal, acceleration of charge,

$$k \times [\dot{v}^{(1)}] = \omega[v^{(1)}], \tag{331}$$

which is responsible for the radiation of $B^{(3)}$. From these equations we find

$$B^{(1)} = \kappa A^{(1)} := \frac{\omega}{c}A^{(1)}, \tag{332}$$

which is Eq. (11) of Vol. 1, an equation which describes a plane wave in vacuo as expected, provided $\kappa := \omega/c$.

The amplitude of the radiated $B^{(3)}$ from this analysis is given in terms of v_0, ω and R,

$$B^{(0)} = |B^{(3)}| = e\frac{\mu_0 \omega v_0}{4\pi Rc}, \tag{333}$$

and is the same as the amplitude of the radiated $B^{(1)} = B^{(2)*}$. However, the radiation in vacuo must propagate at c, and must propagate as plane waves, as we see from Eq. (328), so from the analysis of Chap. 4 this means that in the vacuum,

$$v_0 \rightarrow c, \qquad \frac{\omega}{c} \rightarrow \kappa, \qquad r \rightarrow \frac{c}{\omega}. \tag{334}$$

From Eq. (220), however, we know that

$$B^{(0)} = e\mu_0 \frac{rv}{V}, \tag{335}$$

where V is formally, a volume of radiation. It is therefore possible to find an expression for V/R in the limit where the radiation becomes a plane wave propagating at c in vacuo. This volume in the present analysis can be thought of as being the volume of a cylinder [96] of length R and radius r, which in vacuo becomes the Thompson radius $1/\kappa$ as discussed in Chap. 4. The cylinder's volume is

$$V := \pi r^2 R, \tag{336}$$

and in the limit of plane waves propagating at c in vacuo we find

$$\frac{V}{R} \rightarrow \left(\frac{\lambda}{2}\right)^2. \tag{337}$$

This is an area, proportional to the square of the wavelength, through which passes the magnetic flux in weber associated with $B^{(3)}$, whose units are tesla, i.e., weber m^{-2}. The magnetic flux density at observer point R is $B^{(3)}$. Equation (337) shows that whatever the magnitude of V and R, the area defining the radiated flux density $B^{(3)}$ is always $\lambda^2/4$. This can be thought of as the area of a circularly polarized laser or microwave beam, or the area of starlight collected in a telescope. The source of the starlight may be many millions of light-years distant, so that the light being collected by the observer represents the state of the star many millions of years ago. More generally, the radiation from a source is radiated in a sphere, in which case spherical geometry must

be used to calculate the radiated energy density and flux density per unit solid angle in three dimensions [47,52,53,96].

In summary, therefore, the source of the radiated $B^{(3)}$ is the elementary charge e in a circular orbit of radius r. The angular frequency of this motion is ω and the transverse orbital speed is v_0; the motion is that of a particle *with mass*, m, such as an electron. The radiated $B^{(3)}$ is charge e in a spiralling orbit with Thompson radius $\lambda/(2\pi)$ and both forward and transverse speed c. *There is no mass specified with the charge e in this motion in free space.* If there is mass associated with this radiation, it is the mass of the photon [1,2], which is many orders of magnitude less than that of the electron. Therefore the radiative process (transfer of essentially massless energy) extends the range of e enormously, while leaving essentially all the mass of the radiating particle in the source. In this process, e itself is unchanged, or conserved [1,2, 22—25], but ceases to be localized in the source. This process is described through charge (ρ) and current (J) *densities* using the continuity equation [96],

$$\nabla \cdot J = -\frac{\partial \rho}{\partial t}. \tag{338}$$

Thus, a circling charge at one end of the universe influences another charge at the other end through the electromagnetic field. When the latter, carrying the radiated $B^{(3)}$, meets the target charge, (matter consisting of one fermion), the latter is made to circle according to the appropriate equations of motion. The radius and transverse speed of this circling motion depend on the *mass* of the fermion. The Hamilton-Jacobi equation, for example, shows that if the fermion is massless (a limit not encountered experimentally), it is made to circle at the Thompson radius with transverse speed c, and is therefore indistinguishable from a photon. This is the strong field limit. Otherwise the radius and transverse speed depend on the target fermion mass, and the circling of charge (magnetization) is proportional to a mixture of terms in I and $I^{1/2}$ where I is the field intensity.

The radiated $B^{(3)}$ field is therefore associated with a *very small amount of mass*. This distinguishes it from an ordinary induction field, such as the static magnetic field in a solenoid. The latter is caused by circling electrons and is not a radiated field. The $B^{(3)}$ field, being essentially massless, has, F.A.P.P., no rest frame, and can never be static in any frame of reference. It can be observed only through it interaction with matter, an interaction that must be described through $A^{(1)}$ and $A^{(2)}$ through which $B^{(3)}$ is defined.

5.7 THE VACUUM POYNTING THEOREM FOR $B^{(3)}$

We have seen in Eq. (202) that the quantum of electromagnetic energy in vacuo, the photon $\hbar\omega$, is proportional to e^2, and therefore the propagation of $\hbar\omega$ past the observer at point R is due to an acceleration of charge at the earlier instant $t - R/c$. The $B^{(3)}$ field in the source is due to a conjugate product involving exponents $i\omega t$ and $-i\omega t$; the $B^{(3)}$ field in vacuo is the same precisely, but with t replaced by $t - R/c$, the unit e being maintained constant. This section develops the Poynting theorem for $B^{(3)}$, the energy due to which is defined in Eq. (325). This is rotational energy, and the Poynting vector due to $B^{(3)}$ is proportional to an *angular* momentum density of radiation in vacuo. The vacuum $B^{(3)}$ does not contribute therefore to the well known Poynting vector N [1,2,47,52,53,96] which is a *linear* momentum density of radiation in vacuo. The novel Poynting vector due to $B^{(3)}$, and the energy density due to $B^{(3)}$ form a four-vector of special relativity, an angular momentum-angular energy four-vector.

In S.I. units the Poynting theorem in vacuo is

$$\nabla \cdot N = -\frac{\partial U}{\partial t}, \tag{339}$$

where N, Poynting's vector, is the electromagnetic power per unit area, i.e., the electromagnetic energy flux, and U is the electromagnetic energy per unit volume, the energy density. Since $B^{(3)}$ is a novel vacuum field, it generates the novel energy density [1,2],

$$U_J = \frac{1}{\mu_0} B^{(3)} \cdot B^{(3)*} \left(= \frac{1}{\mu_0} B^{(1)} \cdot B^{(1)*} \right), \tag{340}$$

which is rotational energy density. If U_J is constant in a given radiation volume V, then $\partial U_J / \partial t$ is zero, meaning that

$$\nabla \cdot \left(\frac{J^{(3)}}{V} \right) = -\frac{\partial U_J}{\partial t} = 0, \tag{341}$$

so that the Poynting vector associated with U_J is divergentless and independent of X, Y, and Z. Since there is no $E^{(3)}$, there is no contribution to N from a putative cross product of $B^{(3)}$ and $E^{(3)}$, as discussed on page 4 of Vol. 1. From Eq. (281) of this volume we know, however, that there is an angular momentum, $J^{(3)}$, of vacuum radiation,

$$J^{(3)} = \frac{1}{\mu_0 \omega} V B^{(0)} B^{(3)}, \tag{342}$$

so that its angular momentum density is proportional to $B^{(3)}$,

$$\frac{J^{(3)}}{V} = \left(\frac{B^{(0)}}{\mu_0 \omega} \right) B^{(3)}. \tag{343}$$

The radiation's angular energy density is

$$U_J = \frac{En}{V} = \frac{1}{\mu_0} B^{(3)} \cdot B^{(3)*}, \tag{344}$$

and dividing Eq. (343) by Eq. (344) gives

$$En = \omega \, | J^{(3)} |. \tag{345}$$

We recover from this the definition of the photon if the angular momentum of radiation is the angular momentum, $\hbar$, of one photon, so Eq. (345) becomes the familiar Planck-Einstein hypothesis $En = \hbar\omega$.

The Poynting theorem for $B^{(3)}$ is therefore

$$\nabla \cdot J^{(3)} + \frac{\partial En}{\partial t} = 0, \tag{346}$$

and the four-vector defined by this theorem is

$$J_\mu := \left(J^{(3)}, \frac{En}{\omega} \right), \tag{347}$$

which is Lorentz covariant. Finally, the classical definition of $J^{(3)}$, as given, for example, by Jackson [47], is

$$\boldsymbol{J}^{(3)} = \left(\frac{\epsilon_0}{\kappa}\int|\boldsymbol{E}^{(1)}\times\boldsymbol{B}^{(2)}|dV\right)\boldsymbol{e}^{(3)}, \tag{348}$$

and is seen to be closely related to $\boldsymbol{N}$ integrated over V,

$$\boldsymbol{P}^{(3)} = \epsilon_0\int\boldsymbol{E}^{(1)}\times\boldsymbol{B}^{(2)}dV. \tag{349}$$

The ratio of Eqs. (349) and (348) is κ, which for one photon in the quantum field theory is the ratio of $\hbar\kappa$ to $\hbar$.

It is possible, therefore, to define Poynting's theorem entirely in terms of $\boldsymbol{B}^{(3)}$.

5.8 Q.E.D., BOSON ENSEMBLES, AND $\boldsymbol{B}^{(3)}$ IN VACUO

Dirac [26] was among the first to consider the application of the new quantum theory to radiation, initiating the subject of quantum electrodynamics [22—25]. It is convenient in this final section of this chapter on the irradiation of $\boldsymbol{B}^{(3)}$ to sketch an introduction to the methods by which it can be quantized in free space, leading to the concept of the photomagneton operator [4], the phase independent magnetic flux density operator equivalent to the classical $\boldsymbol{B}^{(3)}$. We first consider the question of the complex nature of the vector potential in the classical theory, a seemingly mundane property but one which is responsible for travelling waves from Maxwell's vacuum equations. In so doing we follow the clear discussion by Jackson [47] of plane wave solutions, in particular,

> "A basic feature of Maxwell's equations is the existence of *travelling wave solutions* which represent the transport of energy from one point to another. The simplest and most fundamental electromagnetic waves are transverse plane waves."

The travelling wave equation has the form [47]

$$\nabla^2 u - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = 0, \tag{350}$$

where $v = c/(\mu\epsilon)^{1/2}$ is a constant, and where ϵ and μ are material permittivity and permeability respectively. The plane wave solutions of Eq. (350) are [47]

$$u = e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)}, \tag{351}$$

where $k = \omega/v$ and the fundamental solution is

$$u(x, t) = Ae^{i(kx - \omega t)} + Be^{-i(kx + \omega t)}, \tag{352}$$

which represents waves travelling to the right, with complex exponent $x - vt$ and to the left with complex exponent $x + vt$. The phase angle $t - z/u$ is a constant, and the wave fronts are perpendicular to the axis of propagation. In the classical theory, causality leads [96] to the neglect of the term in $x + vt$, leaving a fundamental solution of the form

$$u(x, t) = Ae^{ik(x - vt)}. \tag{353}$$

Additionally [47], electromagnetic plane waves have a vector character and can be circularly polarized. This means that circularly polarized transverse electric and magnetic fields rotate as they advance.

The novel $\boldsymbol{B}^{(3)}$ field satisfies the above criteria of electromagnetic plane waves, the reasons being as follows.

1) The $\boldsymbol{B}^{(3)}$ field is a travelling and spinning solution of the vacuum Maxwell equations, because it is defined in terms of a conjugate product that includes the exponent product $e^{-i\phi}e^{i\phi}$, where $\phi := \omega(t - R/c)$ is characteristic of a travelling wave as we have just seen. Thus, $\boldsymbol{B}^{(3)}$ is situated at a unique point R, which propagates through the vacuum, and at no other point. At some later instant $\boldsymbol{B}^{(3)}$ will be situated at another point in space, and thus propagates, or travels, with the wave front. Roughly speaking, it is carried along by the transverse plane waves. It is entirely wrong to think of $\boldsymbol{B}^{(3)}$ as a uniform field stretching throughout the whole of space, at a given instant it is at a given locality and no other. Similarly, the photon is situated at a point in space, and as in the Compton effect, can act as a particle. The $\boldsymbol{B}^{(3)}$ field is proportional to the angular momentum of this particle and is a pseudo vector.

2) There is a Poynting theorem for $\boldsymbol{B}^{(3)}$ as described in the preceding section, and so $\boldsymbol{B}^{(3)}$, as required, carries rotational energy and angular momentum through the vacuum. The field creates rotational energy density and angular momentum density. In contrast the transverse plane waves create translational energy density and linear momentum density.

3) The source of $\boldsymbol{B}^{(3)}$ at observer point R is circling charge at the instant

$t - R/c$ earlier. This is also the source of the transverse plane waves.

4) The interaction of the classical $\boldsymbol{B}^{(3)}$ with matter is described by the potentials through which it is defined. The minimal prescription, in which A_μ is complex in general, is the way in which this interaction is understood. This gives the weak and strong field limits and also describes the interaction of the transverse fields with matter. Since $\boldsymbol{B}^{(3)}$ propagates, F.A.P.P., at c this interaction is relativistic in nature, requiring a relativistic factor in the Hamilton-Jacobi and Dirac equations.

When we come to apply Dirac's method of quantization [26] of the electromagnetic field it is necessary to find a way of translating these ideas into quantum mechanics, but there is no fundamental conceptual difficulty posed by the existence of the classical $\boldsymbol{B}^{(3)}$. In Vol. 1 [1], for example, a simple scheme of quantization was adopted whereby $\boldsymbol{B}^{(1)}$, $\boldsymbol{B}^{(2)}$ and $\boldsymbol{B}^{(3)}$ were recognized as angular momentum operators of quantum mechanics, which are within a factor $\hbar$ [23] infinitesimal rotation generators of O(3). This scheme is easy to develop and removes the anomaly of the unphysical E(2) little group [1], which is replaced by the physical O(3) little group. Dirac's method [26], the original methodology leading to Q.E.D. in the Heisenberg picture, relies however on a real $\boldsymbol{A}$, and in Q.E.D. [58], fields are real in $\boldsymbol{k}$ (momentum) space. For example, Dirac [26] uses the Fourier expansion,

$$\boldsymbol{A} = \int \left(\boldsymbol{A}_{\boldsymbol{k}} + \bar{\boldsymbol{A}}_{-\boldsymbol{k}} \right) e^{-ikx} d^3\boldsymbol{k}, \tag{354}$$

which in our notation is

$$\boldsymbol{A} = \int \left(\boldsymbol{A}_{0\boldsymbol{k}}^{(1)} e^{-ikx} + \boldsymbol{A}_{0\boldsymbol{k}}^{(2)} e^{ikx} \right) d^3\boldsymbol{k}, \tag{355}$$

meaning that $\boldsymbol{A}$ is pure real. This is still classical, but quantization takes place directly from this Fourier expansion in $\boldsymbol{k}$ space by replacing the integral by a sum over a *dust* [26] of $\boldsymbol{k}$ values representing individual oscillators of the electromagnetic field. In order to get $\boldsymbol{B}^{(3)}$ from this type of Fourier expansion, it is necessary to proceed on the basis of the Poynting theorem for $\boldsymbol{B}^{(3)}$ by recognizing that

$$En = \frac{1}{\mu_0} \int \boldsymbol{B}^{(3)} \cdot \boldsymbol{B}^{(3)*} dV = \frac{1}{\mu_0} \int \left(\frac{\omega}{c} \right)^2 \boldsymbol{A}^{(1)} \cdot \boldsymbol{A}^{(2)} dV, \tag{356}$$

so that the quantized $\boldsymbol{B}^{(3)}$ can be developed in terms of cross products of complex amplitudes $\boldsymbol{A}_0^{(1)}$ and its complex conjugate $\boldsymbol{A}_0^{(2)}$ for each $\boldsymbol{k}$. For each photon,

therefore, there is a $\boldsymbol{B}^{(3)}$ field, and in quantum mechanics it is proportional directly to an angular momentum operator. (In the classical theory, Eq. (342) shows that $\boldsymbol{B}^{(3)}$ is proportional to the angular momentum, $\boldsymbol{J}^{(3)}$, of radiation in a given volume V.)

With care, therefore, the quantization of $\boldsymbol{B}^{(3)}$ using Dirac's methods [26] becomes straightforward, and there is no violation of the concepts of Q.E.D. This is intuitively acceptable because $\boldsymbol{B}^{(3)}$ violates no classical concepts and exists in the classical theory of fields.

Chapter 6. Mass of the Photon if Particulate

The existence of the $\boldsymbol{B}^{(3)}$ field in vacuo has the fundamental implication that if the photon can be considered to be a particle, it must have mass in the theory of special relativity. Since $iB^{(0)}\boldsymbol{B}^{(3)*}$ is routinely observable in magneto-optical effects, then it is no longer possible to accept the idea of a massless, particulate photon. The Wigner little group becomes the physical O(3), and not the unphysical E(2). Empirical evidence for the expected $I^{1/2}$ dependence of magneto-optic effects due to $\boldsymbol{B}^{(3)}$ will strengthen this conclusion with further data, but it has already become clear that the observation of I dependent magneto-optical effects [16—21] means the existence of $iB^{(0)}\boldsymbol{B}^{(3)*}$ and therefore of $\boldsymbol{B}^{(3)}$. This conclusion means the introduction [1,2] of various mechanisms by which gauge invariance can be reconciled with non-zero photon mass. There are several inroads already available in the literature [1,2] to this interesting area of electromagnetic theory.

The prevailing orthodoxy relies on the *assertion* that the mass of the photon is zero, because it appears convenient to do so in the Lagrangian approach to field theory [1,2]. The assertion leads directly in special relativity to the conclusion [1,2,23] that the Wigner little group is the unphysical E(2). Therefore the theory of the massless photon is unphysical and internally inconsistent in special relativity, but is nevertheless routinely presented as physical and self-consistent. To the present authors this orthodoxy is unacceptable because it is fundamentally flawed. The recent emergence of $\boldsymbol{B}^{(3)}$ as an electromagnetic field, carrying energy in vacuo according to Eq. (346), shows that there are three degrees of freedom in electromagnetic fields propagating in vacuo; there exist $\boldsymbol{B}^{(1)}$, $\boldsymbol{B}^{(2)}$ and $\boldsymbol{B}^{(3)}$ in the circular basis ((1), (2), (3)). Equivalently there are three degrees of freedom in the Cartesian frame (X, Y, Z).

6.1 SOME DIFFERENCES BETWEEN THE ORTHODOX AND $\boldsymbol{B}^{(3)}$ THEORIES OF LIGHT

It is convenient to list a selection of the major points of difference as follows.

1a) In the orthodox theory of light, the particulate photon is identically massless, and from special relativity must be associated with transverse plane waves propagating in vacuo. There can be no physical $\boldsymbol{B}^{(3)}$, it is asserted, because it is not a transverse plane wave.

1b) In the $\boldsymbol{B}^{(3)}$ theory of light, the particulate photon must be three dimensional and therefore massive, because a massless particle must be two dimensional [56] from Wigner's paper of 1939.

2a) In orthodox classical electrodynamics the Poynting vector is $\boldsymbol{N}$, which is proportional to linear momentum density, and translational electromagnetic energy in vacuo is defined in terms only of transverse plane waves.

2b) In the $\boldsymbol{B}^{(3)}$ theory there is a new Poynting theorem, Eq. (346), and a new rotational contribution to electromagnetic energy density through $\boldsymbol{B}^{(3)} \cdot \boldsymbol{B}^{(3)*}$.

3a) In the orthodox theory propagating fields depend explicitly on the phase factor $\phi = \omega(t - R/c)$.

3b) The $\boldsymbol{B}^{(3)}$ field propagates in vacuo because it is implicitly dependent on the phase through a cross product.

4a) In orthodox magneto-optics there occur conjugate products such as $\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)}$ but these are not accepted as indicating the existence of $\boldsymbol{B}^{(3)}$.

4b) In the new understanding $iB^{(0)}\boldsymbol{B}^{(3)*}$ is equal to $\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)}$, so $\boldsymbol{B}^{(3)}$ is an everyday observable of magneto-optics. The orthodox thinking cannot accept this because $\boldsymbol{B}^{(3)}$ is asserted to be zero. This is the point at which the old theory becomes internally inconsistent.

5a) In the orthodox view the mass of the photon is identically zero because otherwise gauge invariance is violated in the Lagrangian approach to field theory.

5b) In the new thinking there can be no photon whose mass is zero because of the existence of $\boldsymbol{B}^{(3)}$. By making $A_\mu A_\mu$ infinitesimally small, gauge invariance and non-zero mass can be reconciled. There are probably several other ways of achieving the same aim [1,2].

6a) In the orthodox approach magneto-optical effects are always proportional to beam power density, I, and this is corroborated by available experimental results [16—21].

6b) In the new approach, there develops under the right conditions an $I^{1/2}$ dependence which has not been observed hitherto. This has been predicted clearly and independently by Chiang [100] and by Talin *et al.* [54], but the significance of these papers has not been fully realized. Both papers indirectly indicate the existence of $\boldsymbol{B}^{(3)}$ through special relativity. The $I^{1/2}$ dependence is actually implicit in a standard text such as that of Landau and Lifshitz [75], as explained in Chap. 12 of Vol. 1 [1], but because this is an orthodox text in the classical theory of fields, $\boldsymbol{B}^{(3)}$ is not recognized explicitly.

7a) In contemporary orthodoxy, the Faraday induction law is one of the vacuum Maxwell equations and always links transverse components of the electromagnetic field.

7b) In the new theory the field $\boldsymbol{B}^{(3)}$ is dual [1,2] to the imaginary and unphysical $-i\boldsymbol{E}^{(3)}/c$ through a relation which is formally equivalent to a Faraday induction law. There is no real $\boldsymbol{E}^{(3)}$, however, and no Poynting theorem for it. In other words there is no $\boldsymbol{E}^{(3)}$ to carry electromagnetic field energy through the vacuum, only a $\boldsymbol{B}^{(3)}$.

8a) In the orthodox vacuum theory there are no cyclic relations such as

$$\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} = iB^{(0)}\boldsymbol{B}^{(3)*}, \text{ et cyclicum,} \tag{357}$$

because $\boldsymbol{B}^{(3)}$ is asserted not to exist. At this point the old theory becomes internally inconsistent because setting $\boldsymbol{B}^{(3)}$ to zero in Eqs. (357) results in $\boldsymbol{B}^{(1)} = \boldsymbol{B}^{(2)} = \boldsymbol{0}$, while it is known that this is not the case, and if so, the cross product of $\boldsymbol{B}^{(1)}$ and $\boldsymbol{B}^{(2)}$ leads back to $iB^{(0)}\boldsymbol{B}^{(3)*}$, meaning that $\boldsymbol{B}^{(3)}$ exists.

8b) In the new theory these cyclical relations indicate that $\boldsymbol{B}^{(1)}$, $\boldsymbol{B}^{(2)}$ and $\boldsymbol{B}^{(3)}$ can be understood easily as O(3) rotational generators [1], or Lorentz group generators in space-time. The theory is easily quantized because these are also angular momentum operators in quantum mechanics to within a factor $\hbar$ [1,2].

9a) The assertion $\boldsymbol{B}^{(3)} = \boldsymbol{0}$ in the old theory leads directly to the E(2) inconsistency, which is described for example by Weinberg [93] or by Ryder [23]. The E(2) little group is the planar Euclidean group and is unphysical.

9b) In the new thinking $\boldsymbol{B}^{(3)}$ is accepted as a routine experimental observable and is, thereby, not zero. The Wigner little group [1,2,22—25] becomes O(3), the rotation group, and is physical.

10a) In the orthodox view the photon is asserted to be an uncharged particle, so that it is its own anti-particle.

10b) This view is shown to be untenable by Eq. (202), and by the experimentally observable $\boldsymbol{B}^{(3)}$ field [16—21]. The energy of the photon is proportional to e^2, and there can be an anti-photon with a different sign of e but with the same mass [1] and energy but opposite $\boldsymbol{B}^{(3)}$.

11a) In the orthodox view, canonical quantization is beset by inconsistencies caused [23] by the fact that A_μ need not be completely covariant.

11b) In the new view, A_μ is a fully covariant and physical four-vector, which in free space obeys a limiting form $A_\mu A_\mu \to 0$ of the Dirac condition.

The above selection of some key differences between the old and new theories leads to the expectation that there will eventually be found (for example in Q.E.D.) numerous examples of specific effects due to $\boldsymbol{B}^{(3)}$ which can be interpreted as specific effects of finite photon mass. However these, although indicating the existence of such mass, may not be sufficient in themselves to determine it experimentally without some additional input [6]. However, $\boldsymbol{B}^{(3)}$ *is* sufficient to invalidate the concept of the identically massless photon.

6.2 THE CONNECTION BETWEEN PHOTON MASS AND $B^{(3)}$ IN THE POINCARÉ GROUP

In this section the connection is made between $B^{(3)}$ and non-zero photon mass using the symmetry of the Poincaré group [23]. The two Casimir invariants of the group are mass and spin, which are invariant under Lorentz transformations that include boost generators in special relativity. Spin corresponds to a rotation group symmetry if and only if $m^2 > 0$, where m is the mass of a particle being subjected to Lorentz transformation including boost, rotation, and space-time translation. In this view [23], discussed by Weinberg [93] for particles of any spin, the very concept of spin for a particle with mass is *defined* by the symmetry of the Wigner little group. The idea of a particle without mass, $m = 0$ identically, results in a non-compact little group, E(2), which is unphysical. Therefore the obvious inference is that the original idea itself is unphysical. The E(2) group describes rotations and translations which must be taking place simultaneously in one plane embedded in three dimensional space. The presence of the most minute amount of mass means that m is no longer identically zero, and the little group becomes the physical O(3) [1,2,23].

In the current orthodoxy, which is rapidly losing its validity, the unphysical nature of E(2) is accepted uncritically because in the Lagrangian approach to field theory a massless photon is needed to keep the term $m^2 A_\mu A_\mu$ invariant under gauge transformation. Even in this context, however, the Higgs mechanism [2] can be used to input mass into the boson known as the photon using spontaneous symmetry breaking of the vacuum. The mechanism is applied for a Lagrangian which is originally compatible with gauge invariance. These points have been pursued in Vol. 2 [2]. Weinberg [93] has shown that A_μ cannot be quantized for an identically massless photon because it corresponds to a $(1/2, 1/2)$ irreducible representation of the Poincaré group. Such a representation is not allowed, however, for $m = 0$ because in this case the helicity of the massless particle must be $\lambda = A - B$, where the irreducible representations are denoted (A, B). Thus, we encounter the familiar difficulties with canonical quantization of A_μ in for example, the Lorentz gauge: an indefinite metric, negative energies, unwelcome c numbers, and so forth. In the Coulomb gauge [23], A_μ is not completely covariant, thus compounding an already severe problem.

These difficulties are usually patched up in the orthodox theory using the Gupta-Bleuler method [1,2,22—25], but Weinberg [93] chooses to avoid completely the use of a Lagrangian formalism in favor of one based on an S-matrix. He concludes that all field equations are simply relations between spin components of the vector field. Therefore if spin is *defined* through a Wigner little group, the only physical particles are those with mass, because E(2) is unphysical. This is a complicated way of recognizing

that there must be a $B^{(3)}$ field, because otherwise, the photon has no mass. In other words if the E(2) little group of a hypothetically massless photon is unphysical, as it surely must be, the Maxwell equations themselves must also be unphysical. Expressed in a third way, if the particle spin itself is unphysical, (E(2) little group), any relation between components of the unphysical spin (i.e., the Maxwell equations) must also be so. At this point the orthodox view departs from internal consistency, because it simultaneously accepts an unphysical E(2) but a *physical* set of Maxwell equations.

This appears at the time of writing to be the most general relativistic argument for the existence of $B^{(3)}$, but there are many others as we have seen.

The defining Lie algebra (357) is cyclically symmetric, non-Abelian, compact and semi-simple [1,2,23,93]. The Lie algebra of E(2) on the other hand is not cyclically symmetric, contains an Abelian sub-algebra [93], is not compact and not semi-simple. These are troublesome features for the orthodox theory because the usual assertion $B^{(3)} = 0$ means throwing away a space dimension and an angular momentum. The E(2) group in consequence becomes non-compact because one commutator $[L_1, L_2]$ is zero, and zero is itself not a group generator. In contrast, Eqs. (357) are ordinary relations between infinitesimal generators of the O(3) group, which become commutator relations [1,2] between angular momentum operators in the quantum theory. Therefore $B^{(3)}$ is incompatible with the existence of an identically massless photon. We must abandon either the former or the latter. Since $B^{(3)}$ was unknown prior to 1992 [7] there is a vast amount of literature based on the uncritical acceptance of the orthodox theory and therefore of the unphysical E(2) group. Since $B^{(3)}$ is observable experimentally [16—21] through $iB^{(0)}B^{(3)*}$ there must be finite mass associated with the photon if particulate, and effects such as Compton scattering tend to confirm the latter point of view. We have therefore a valid chain of reasoning linking $B^{(3)}$ to non-zero m. Furthermore, these volumes have shown that $B^{(3)}$ is deeply rooted in electrodynamics.

The hypothetically massless photon on the other hand is a mathematical idea that cannot be tested experimentally because it produces concomitant electromagnetic fields which are infinite in range and not bounded by a finite universe. The range of electromagnetic radiation is known to be very great, experimentally, but cannot be shown experimentally to be infinite, because infinity is a mathematical concept. Since mass is deeply embedded in general relativity theory and the bending of space-time it cannot be expected to vanish arbitrarily as in the older, orthodox theory of electrodynamics. As we have just seen, such an idea is equivalent to throwing away a space dimension, leading to the E(2) inconsistency.

The recognition of $B^{(3)}$ has the further major advantage of rendering the little group O(3) for the photon as particle. To know *all* the representations of the Lorentz group for the photon with mass we need *only* know [23] the generators of O(3), which are directly proportional to $B^{(1)}$, $B^{(2)}$ and $B^{(3)}$. These are the three fields of the photon with mass. The experimentally verifiable [16—21] existence of $\boldsymbol{B}^{(3)}$ means that the photon's helicities are those of a boson with mass: +1, 0, and -1; and not +1 and -1 as in the orthodox point of view. If we throw away one of the $\boldsymbol{B}$ fields we no longer know all the representations of the Lorentz group for the photon with mass, meaning that the very structure of space-time itself is destroyed. This is another internal inconsistency of the orthodox point of view. Either we have a well defined space-time (metric) or we do not. An illustration of this has been sketched already, throwing away $\boldsymbol{B}^{(3)}$ in Eqs. (357) means throwing away $\boldsymbol{B}^{(1)}$ and $\boldsymbol{B}^{(2)}$, leaving nothing.

We conclude that $\boldsymbol{B}^{(1)}$, $\boldsymbol{B}^{(2)}$ and $\boldsymbol{B}^{(3)}$ are each irreducible representations of the O(3) little group of the Poincaré group, a little group which leaves the momentum-energy four-vector p_μ (or boost generator [23]) invariant under the most general type of Lorentz transformation. This invariance is an expression of energy-momentum conservation and the Noether theorem [1,2]. It is therefore *also compatible with gauge transformation*, showing that a photon with mass is compatible with gauge transformation. The Lagrangian formulation of field theory (or indeed any other, e.g. S matrix, formalism) must therefore be modified to be compatible with this very general result from energy-momentum conservation. Therefore $\boldsymbol{B}^{(3)}$ is compatible with gauge transformation and is physical.

6.3 CONSEQUENCES WITHIN THE POINCARÉ GROUP

The boost generators of the Poincaré group disappear from its little group if the latter is O(3), however minute the particle mass may be experimentally. The fundamental reason for this is that boost generators cannot form a cyclically symmetric Lie algebra akin to Eq. (357). In simple vector language, the cross product of two polar vectors is an *axial* vector, not another polar vector, whereas the cross product of two axial vectors is another axial vector. The magnetic fields $\boldsymbol{B}^{(1)}$, $\boldsymbol{B}^{(2)}$ and $\boldsymbol{B}^{(3)}$ are axial vectors, and are represented in matrix form by infinitesimal generators of O(3). The algebra of vacuum electric fields akin to (357) is [1—15]

$$\boldsymbol{E}^{(1)} \times \boldsymbol{E}^{(2)} = -E^{(0)}(i\boldsymbol{E}^{(3)})^*, \text{ et cyclicum,} \tag{358}$$

in which the longitudinal ((3)) component is pure imaginary and *unphysical*. There is no known experimental effect due to a putative $\boldsymbol{E}^{(3)}$. It has been shown [1—15] that electric fields must be proportional to boost generators for a particle with $m = 0$ identically. Its unphysical nature is overlooked in a vast number of papers that assert that the photon is massless. With the advent of $\boldsymbol{B}^{(3)}$ this assertion becomes untenable, and so does the isolated existence of plane waves. The transverse $\boldsymbol{B}^{(1)}$ and $\boldsymbol{B}^{(2)}$ do not exist in isolation of $\boldsymbol{B}^{(3)}$, which is also relativistically invariant. The unphysical nature

of E(2) is an irrecoverable fault in the theory of massless particles. If we try to associate such particles with fields, relativistic field theory also becomes unphysical. The specifically Abelian feature of the Lie algebra of E(2) shows up through the fact that the commutator of $\hat{L}_1$ and $\hat{L}_2$ does not produce a generator in the 3 axis orthogonal to the plane (1,2) of $\hat{L}_1$ and $\hat{L}_2$. This is despite the fact that the generator $\hat{J}_3$ appears in the other two commutators, and as shown elsewhere {1}, is proportional to the $\hat{\boldsymbol{B}}^{(3)}$ field in vacuo. This demonstrates the internal inconsistency of E(2) and the concomitant, orthodox and Abelian electrodynamics it represents, because $\boldsymbol{B}^{(3)}$, as argued earlier, is an experimental observable. This inference emerges throughout these three volumes. The rotation generator $\hat{J}_3$ appears in two out of three commutators of E(2), and $\hat{\boldsymbol{B}}^{(3)}$ is directly proportional to $\hat{J}_3$. Therefore $\hat{\boldsymbol{B}}^{(3)}$ also appears in these commutators, but does not appear in the first commutator [23] on the right hand side. Since $\hat{L}_1$ and $\hat{L}_2$ are two of the basic generators of E(2), ($\hat{J}_3$ being the third), this group cannot produce $\boldsymbol{B}^{(3)}$ self-consistently, and the group is, significantly, also unphysical. Thus, $\boldsymbol{B}^{(3)}$ is a *physical* field component in vacuo, as deduced throughout these volumes. The *physical* little group O(3) produces this result consistently through the defining Lie algebra (357).

6.4 COMPATIBILITY OF $\boldsymbol{B}^{(3)}$ WITH NOETHER'S THEOREM

Noether's theorem [1,2,23] relates fundamental space-time symmetries to fundamental conservation laws, and links the existence of $\boldsymbol{B}^{(3)}$ to that of the canonical energy-momentum tensor, $T_{\mu\nu}$, that appears in Einstein's field equations of general relativity. It is possible that the existence of $\boldsymbol{B}^{(3)}$ may provide a new link between electromagnetic theory on the one hand and general relativity on the other because the defining Lie algebra (357) is non-Abelian and can be linked as in Vol. 2 [2] to Yang-Mills type gauge theory. Noether's theorem [23] is fundamental to physics, because it links fundamental symmetries to fundamental conservation laws. It is therefore necessary to show that $\boldsymbol{B}^{(3)}$ is rigorously compatible with the theorem. In view of the link between the photomagneton [4] $\hat{\boldsymbol{B}}^{(3)}$ and $\hat{\boldsymbol{J}}^{(3)}$

$$\hat{\boldsymbol{B}}^{(3)} = B^{(0)}\frac{\hat{\boldsymbol{J}}^{(3)}}{\hbar}, \tag{359}$$

this can be proven through the fact that Noether's theorem implies conservation of angular momentum as a consequence of fundamental space-time symmetries. The rotation generators appearing in Eq. (357) are, within a factor $\hbar$, the three space-time components of the quantized version of the angular momentum four-tensor $M_{\mu\nu}$. Noether's theorem states that this is a conserved classical quantity,

$$\frac{dM_{\mu\nu}}{dt} = 0, \tag{360}$$

and this is in essence a statement of the new Poynting theorem, Eq. (346), for energy transmitted in vacuo by $\boldsymbol{B}^{(3)}$. This is confirmed through the fact that if x_μ is defined in Minkowski notation by (X, Y, Z, ict), then $M_{\mu\nu}$ is given by an integral over the canonical energy-momentum tensor, $T_{\mu\nu}$,

$$M_{\mu\nu} = \int \left(T_{0\mu} x_\nu - T_{0\nu} x_\mu \right) d^3x, \tag{361}$$

and by virtue of conservation of angular momentum, $T_{\mu\nu}$ must be symmetric. A symmetric $T_{\mu\nu}$ is also necessary in Einstein's field equations for gravitation, a non-Abelian field [2,23], and the non-Abelian nature of the algebra (357) forges a link between $\boldsymbol{B}^{(3)}$, $T_{\mu\nu}$ and $M_{\mu\nu}$, one which is incomplete in the orthodox view because a generator is missing. This link may well be useful in developing a unified understanding of gravitation and electromagnetism, because both theories are now non-Abelian.

The compatibility of $\boldsymbol{B}^{(3)}$ with Noether's theorem follows from the fact that the eigenvalues of $\hat{\boldsymbol{B}}^{(3)}$ are those of $\hat{J}^{(3)}$, the Z axis angular momentum operator component. This is the usual, specified [45], or observable, component with quantum numbers $M = -J, \ldots, J$. The only non-zero eigenvalues of the electromagnetic beam's $\hat{\boldsymbol{B}}^{(1)}$, $\hat{\boldsymbol{B}}^{(2)}$ and $\hat{\boldsymbol{B}}^{(3)}$ components are therefore those of $\hat{\boldsymbol{B}}^{(3)}$, a constant of motion whose classical expectation value, $\boldsymbol{B}^{(3)}$, is conserved and non-zero. Thus $\boldsymbol{B}^{(3)}$ is compatible with Noether's theorem. In the orthodox view, the expectation values of the phase dependent $\hat{\boldsymbol{B}}^{(1)}$ and $\hat{\boldsymbol{B}}^{(2)}$ vanish and there is no magnetization to first order in $B^{(0)}$. The existence of the $I^{1/2}$ profile [1,2] is missed entirely, a profile which isolates $\boldsymbol{B}^{(3)}$ within experimental uncertainty under the right conditions. It is of course important to work under these well defined [1,2] conditions, otherwise there will be a negative result, as in the experiment by Rikken [74]. It is emphasized, however, that routine magneto-optics yields data on $iB^{(0)}\boldsymbol{B}^{(3)*}$ and so $\boldsymbol{B}^{(3)}$ is a routine observable at order one in I [16—21]. The only way to deny this conclusion is to assert that $\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)}$ is *not* equal to $iB^{(0)}\boldsymbol{B}^{(3)*}$. There are no known physical, algebraic, symmetric or dimensional

arguments to support this assertion. This is the way in which $\boldsymbol{B}^{(3)}$ was discovered originally [7] and is alone sufficient to expose the inconsistency of the orthodox view. Noether's theorem is also satisfied in this view, however, because the fields $\boldsymbol{B}^{(1)}$ and $\boldsymbol{B}^{(2)}$ average to zero and zero is a conserved quantity. In the new, self-consistent theory, in which photon mass is non-zero, we obtain an exponentially decaying $\boldsymbol{B}^{(3)}$,

$$\boldsymbol{B}^{(3)} = B^{(0)} e^{-\xi Z} \boldsymbol{e}^{(3)}, \tag{362}$$

where ξ is a rest wave-number [1,2]. This is again compatible with Noether's theorem because for each Z

$$\frac{d\boldsymbol{B}^{(3)}}{dt} = 0. \tag{363}$$

6.5 $\boldsymbol{B}^{(3)}$ IN THE SEARCH FOR UNIFICATION OF ELECTROMAGNETISM AND GRAVITATION

The $\boldsymbol{B}^{(3)}$ field is compatible with Noether's theorem through conservation of angular momentum in the free electromagnetic field. It is possible to develop several interesting links between the non-Abelian Lie algebra (357) and the non-Abelian theory of gravitation in the vacuum [23]. The Lie algebra (357) implies that the gauge group of free electromagnetism must be enlarged to the non-Abelian O(3) from the Abelian U(1). As in all non-Abelian gauge structures the field may be thought of as acting as its own source, thus, the source of $iB^{(0)}\boldsymbol{B}^{(3)*}$ at the observer point R, and thus of $\boldsymbol{B}^{(3)}$, is $\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)}$ at the instant $t - R/c$, i.e., a circling charge. As we have seen in earlier chapters, the phase $\omega(t - R/c)$ is implicit in the definition of $\boldsymbol{B}^{(3)}$.

In general relativity the gravitational field carries energy, which is equivalent to mass, and is itself a source of gravitation. Thus, in the Einstein field equations, both sides are tensors whose covariant divergence vanishes. The covariant derivative in general relativity has a geometrical origin; space-time itself becomes non-Euclidean in the presence of a gravitating object. The ordinary divergence of the Einstein tensor $G^{(E)}_{\mu\nu}$ is not zero, even in the absence of matter, and this is an expression of the fact that the field couples to itself.

In analogy, an O(3) gauge symmetry for free electromagnetism results [2] in the replacement of the ordinary O(2) field tensor $\boldsymbol{F}_{\mu\nu}$ by a tensor $\boldsymbol{G}_{\mu\nu}$ which is also a vector in ((1), (2), (3)). The covariant derivative of $\boldsymbol{G}_{\mu\nu}$ vanishes [2] in vacuo, as for the Einstein tensor, but its ordinary derivative does not. The $\boldsymbol{B}^{(3)}$ field appears as an intrinsic, irremovable, and *gauge invariant* component of $\boldsymbol{G}_{\mu\nu}$, a specifically non-Abelian component,

$$\boldsymbol{B}^{(3)*} = -i\frac{e}{\hbar}\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}. \tag{364}$$

Therefore $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ is also gauge invariant, because it is part of the gauge invariant $\boldsymbol{G}_{\mu\nu}$, *provided that the gauge group is O(3)*. This point is developed in the next subsection. In analogy with general relativity, the covariant derivative in O(3) vacuum electrodynamics also has a geometrical origin [2], and the four-potentials $A_\mu^{(1)}$, $A_\mu^{(2)}$ and $A_\mu^{(3)}$ can be identified with connection coefficients, $\Gamma^{\mu}_{\lambda\nu}$, in general relativity [23]. The quantity analogous with $\boldsymbol{G}_{\mu\nu}$ of non-Abelian electrodynamics is the Riemann-Christophel curvature tensor $R^{\kappa}_{\lambda\mu\nu}$, which indicates that space-time is non-Euclidean when there is a gravitational field present. If it were possible to express $\boldsymbol{G}_{\mu\nu}$ in the same tensorial structure as $R^{\kappa}_{\lambda\mu\nu}$, then it would also be possible to say that space-time becomes curved when there is an *electromagnetic* field present. Continuing the analogy, the Bianchi identity of general relativity would become analogous with the homogeneous Maxwell equations in the O(3) gauge group [2]. Finally, our novel equivalence condition,

$$eA^{(0)} = \hbar\kappa, \tag{365}$$

becomes analogous with the Einstein field equation itself,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\frac{8\pi G^{(E)}}{c^2}T_{\mu\nu}, \tag{366}$$

so that p_μ is analogous with the Einstein tensor $G^{E}_{\mu\nu}$. The equivalence condition emerges as a direct result of the term (364) in $\boldsymbol{G}_{\mu\nu}$, and therefore as a result of O(3) gauge geometry.

6.6 THE GAUGE INVARIANT LAGRANGIAN MASS TERM ASSOCIATED WITH $\boldsymbol{B}^{(3)}$

Since $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ is gauge invariant in O(3) geometry, being, within a factor $-ie/\hbar$, the non-Abelian component [2] of the gauge invariant $\boldsymbol{G}_{\mu\nu}$, the energy density due to $\boldsymbol{B}^{(3)}$, part of the new Poynting theorem, Eq. (346),

$$\frac{En}{V} = \frac{1}{\mu_0}\boldsymbol{B}^{(3)} \cdot \boldsymbol{B}^{(3)*}, \tag{367}$$

becomes

$$\frac{En}{V} = \frac{1}{\mu_0}\frac{e^2}{\hbar^2}\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} \cdot \boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}. \tag{368}$$

Using the vector identity,

$$\boldsymbol{F} \times \boldsymbol{G} \cdot \boldsymbol{H} \times \boldsymbol{I} = (\boldsymbol{F} \cdot \boldsymbol{H})(\boldsymbol{G} \cdot \boldsymbol{I}) - (\boldsymbol{F} \cdot \boldsymbol{I})(\boldsymbol{G} \cdot \boldsymbol{H}), \tag{369}$$

the energy density becomes

$$\frac{En}{V} = \frac{1}{\mu_0}\frac{e^2}{\hbar^2}(\boldsymbol{A}^{(1)} \cdot \boldsymbol{A}^{(2)})^2. \tag{370}$$

The original expression (364) is gauge invariant in O(3), and so Eq. (370) must also be so. Using Eq. (288) for the field mass, m, associated with $\boldsymbol{B}^{(3)}$ we obtain, with Eq. (282),

$$eA^{(0)} = 2mc, \tag{371}$$

showing that the momentum $eA^{(0)}$ is equal to a mass, $2m$, multiplied by c. In the orthodox view, this mass does not exist. The energy density (370) is therefore

$$\frac{En}{V} = \frac{\epsilon_0 c^4}{\hbar^2}(2m)^2\boldsymbol{A}^{(1)} \cdot \boldsymbol{A}^{(2)} = \frac{e^2A^{(0)2}}{\mu_0\hbar^2}\boldsymbol{A}^{(1)} \cdot \boldsymbol{A}^{(2)}. \tag{372}$$

This is an interesting result because it shows that En/V is a mass term in the

Lagrangian, a term which is *gauge invariant in O(3)* because the original expression, Eq. (364) is gauge invariant in O(3). If for convenience we identify $2m$ with M, Eq. (372) becomes

$$\frac{En}{V} = \left(\frac{Mc^2}{\hbar\omega}\right)^2 \frac{1}{\mu_0} \boldsymbol{B}^{(3)} \cdot \boldsymbol{B}^{(3)*}. \tag{373}$$

This is consistent with Eq. (367) if and only if

$$\hbar\omega = Mc^2, \tag{374}$$

and this equation has the same form as de Broglie's Guiding theorem, Eq. (1) of Vol. 1 [1]. The mass that appears in de Broglie's theorem is the mass of the photon, and as seen in Eq. (294), this is proportional to M. In the orthodox view $M = 0$ identically and there is an inconsistency, because $M = 0$ means that $\omega = 0$ simultaneously to retain a finite En/V. If, however $\omega = 0$ there is no radiation. In the orthodox view there is no $\boldsymbol{B}^{(3)}$, which is inconsistent with the *experimentally verified* cyclic relations (357). These are experimentally verified because $\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)}$ is an observable [16—21].

There are several other inconsistencies in the orthodox view of the electromagnetic field in free space. For example, the energy density in the orthodox U(1) gauge group is

$$\frac{En}{V} = \frac{1}{\mu_0} \boldsymbol{B}^{(1)} \cdot \boldsymbol{B}^{(2)} = \frac{1}{\mu_0} \frac{\omega^2}{c^2} \boldsymbol{A}^{(1)} \cdot \boldsymbol{A}^{(2)}, \tag{375}$$

but if $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, the description of En/V in terms of $\boldsymbol{B}$ is unchanged if the type two gauge transformation

$$\boldsymbol{A} \rightarrow \boldsymbol{A} + \nabla\chi, \tag{376}$$

is made. However, the description of the same En/V in terms of $\boldsymbol{A}$ is *changed* by Eq. (376) to

$$\frac{En}{V} \rightarrow \frac{En}{V} + \frac{1}{\mu_0} \frac{\omega^2}{c^2} (\nabla\chi)^{(1)} \cdot (\nabla\chi)^{(2)} + \ldots, \tag{377}$$

and is not gauge invariant in U(1). This is essentially why a mass term, proportional

to $A_\mu A_\mu$, is not used in the orthodox Lagrangian density. In consequence, it is asserted [23] that the photon mass is zero. This means that there can be no $\boldsymbol{B}^{(3)}$ in vacuo, and this is inconsistent with experimental data [16—21] because of the existence of the cyclic relations (357).

In the new, consistent, theory, in contrast, we have derived a satisfactorily gauge invariant energy density, Eq. (373), which can act as a Lagrangian density. In order to do this, however, we must accept $\boldsymbol{B}^{(3)}$ as a physical field, because it is proportional to the product $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$. This means that a non-zero photon mass has been made rigorously compatible with type two gauge invariance, a most satisfactory result which persuades us to abandon the old U(1) gauge group in favor of O(3), thereby curing the anomaly of the E(2) little group. The assertion that the photon is a particle whose mass is identically zero becomes untenable in the light of this reasoning. It is therefore logical to look for signs of finite photon mass using the most precise methods available, for example the Lamb shift or measurements of the anomalous magnetic moment of the electron with radiative corrections. Although the present agreement with orthodox Q.E.D. is satisfactory to several decimal places, the photon mass is minute, (less than 10^{-45} kgm. [1,2]) and much greater precision is probably required to discern its effects on these spectra, accurately measured as they are.

In order to set the stage for these experiments, it is necessary to develop Q.E.D. with finite photon mass, because $\boldsymbol{B}^{(3)}$ means the existence of such mass, and as we have seen, the existence of $\boldsymbol{B}^{(3)}$ has been shown experimentally in magneto-optics through $i\boldsymbol{B}^{(0)}\boldsymbol{B}^{(3)*}$. The existence of photon mass itself is therefore shown through standard magneto-optics [16—21]. This opens up new areas of thought in Q.E.D. [101], for example in: mass renormalization, photon self-energy, and the ultra-violet divergence. The gauge invariant mass term (373) in the Lagrangian leads to sharply divergent terms in the propagator, and to modifications of well known expressions for phenomena such as Compton scattering, and the spectra of positronium and hydrogen. It is therefore necessary to amend the theory of radiative corrections of the Lamb shift and the anomalous magnetic moment of the electron to allow for finite photon mass and to predict the effect on the data. This may be done through Q.E.D., but also through other methods, which avoid the awkward divergences inherent in Q.E.D. The method should then be extended to Q.C.D. as systematically as possible, using all available sources of data to estimate photon mass. The latter is so minute that the precision of these superbly accurate methods will probably have to be *increased* considerably, surprising as this may seem at first.

In contrast, the orthodox view regards the existence of photon mass as being incompatible with gauge invariance. Our Eq. (373) answers this objection immediately and in its entirety, and Eq. (373) is based directly on the existence of $\boldsymbol{B}^{(3)}$. The most glaring and strident inconsistency in the orthodox view is that it is forced into asserting like a cuckoo-clock that $\boldsymbol{B}^{(3)}$ is zero. This destroys the rigorously derived and supported [1—15] cyclic relations (357) and there is no physical or mathematical ground upon which such destruction can occur.

6.7 SUMMARY OF INCONSISTENCIES IN THE U(1) THEORY

In the final section of this chapter we summarize some of the major inconsistencies that have developed in the orthodox theory, in which the gauge group of free space electromagnetism is U(1) [1,2,23].

1) The cyclic field relations (357) are relations between generators of the O(3) group in space. The O(3) group is non-Abelian, compact and semi-simple. Each component equation is dimensionally and symmetrically self-consistent. In the orthodox, U(1), theory, there can occur no non-Abelian relations such as these, and the U(1) theory cannot account for their existence.

2) The U(1) theory is forced back on the assertion that $\boldsymbol{B}^{(3)}$ does not exist [62—65], or is somehow not a magnetic field. This is an assertion that is contrary to experimental data [16—21], data which show that $iB^{(0)}\boldsymbol{B}^{(3)*}$ is a physical observable because $\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)}$ is one. If $\boldsymbol{B}^{(3)}$ did not exist $iB^{(0)}\boldsymbol{B}^{(3)*}$ would not be an observable and neither, therefore would $\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)}$.

3) The planar Euclidean group E(2) [1,2,23] is unphysical in the U(1) theory, and therefore that theory is unphysical. It follows that the idea of a particle with no mass is also unphysical. Such a particle is two dimensional in U(1) theory, but three dimensional and massive in O(3) theory.

4) As discussed by Weinberg [93], the four-potential A_μ cannot be quantized in U(1) theory. Satisfactory quantization of Eq. (357) occurs immediately because each $\boldsymbol{B}$ is an angular momentum operator of quantum mechanics.

5) In the U(1) theory no method can be found to produce a gauge invariant mass term in the Lagrangian. The theory is forced to the unphysical conclusion (see (3) above) that there exists a massless particle. Within the O(3) gauge group, however, [2], the gauge group required by the existence of $\boldsymbol{B}^{(3)}$, *a gauge invariant mass term appears* and is given by Eq. (373). This is an expression for the electromagnetic energy density, En/V, and can therefore serve as a Lagrangian density. It is a direct and simple consequence of the gauge invariant proportionality between $\boldsymbol{B}^{(3)}$ and $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ in the O(3) gauge group [2].

6) Canonical quantization of the U(1) theory is beset with difficulties, and this is well known [1,2,23]. The canonical quantization of the Proca equation, on the other hand, proceeds straightforwardly, and with Eq. (373) in mind, it is now known that it is satisfactorily gauge invariant in O(3). The particles produced by canonical quantization within the O(3) gauge group are photons with a tiny but non-zero mass.

7) The range of radiation concomitant with the identically massless photon is infinite, and this is unphysical because all infinities, or divergences, are so. The known universe is thought to be finite, and light from the most distant stars has travelled a finite distance.

8) The little group is the Poincaré sub-group that leaves energy-momentum invariant under the most general type of Lorentz transformation. This result is obtained from the fact that energy-momentum is the generator of translations within a factor $\hbar$. If energy-momentum is invariant under a Lorentz transformation, the little group must be a physical group, such as a rotation group, describing a physically meaningful rotation. However, for a particle of mass $m = 0$ this is not the case, because the little group is E(2), the group of rotations and translations in a plane.

9) Equation (373), describing a gauge invariant Lagrangian density, reveals a new inconsistency in the standard U(1) theory because the latter is based on the assertion that any Lagrangian mass term must be discarded because it is *not* gauge invariant. Ryder [23] also asserts that a photon must be massless in the Yang-Mills formalism, but this is at odds with Eq. (373) and the experimentally verified $\boldsymbol{B}^{(3)}$ field.

10) It has been shown repeatedly, e.g. by Huang [102], that a finite photon mass, or field mass, can be incorporated in electromagnetic and unified field theory, and similarly, methods for incorporating finite mass in Q.E.D. are well developed [101]. The emergence of $\boldsymbol{B}^{(3)}$, which is well defined in electrodynamics, shows that photon mass is identically *non*-zero.

11) In contemporary field theory, classical, non-linear field equations give interesting new solutions, and non-Abelian gauge theories, exemplified by our cyclic relations (357) are also non-linear in nature. If the development of these equations is confined, initially, to two space dimensions, they automatically produce a string, or vortex, in a third dimension. This can easily be understood with the non-Abelian (357), but no U(1) theory can produce a physical vortex field perpendicular to the plane of definition of U(1) without changing the relevant gauge group to O(3). If so, the photon can no longer be a two dimensional particle and must have mass.

Chapter 7. Photon Mass in Electromagnetic Theory

We have seen in Chaps. 1 to 6 that there exists a field, $\boldsymbol{B}^{(3)}$, in vacuo which shows that the photon, if particulate, cannot be two dimensional. If so, it must carry mass, which we denote m. In this chapter we explore in relatively simple terms the effect of non-zero photon mass on electrodynamics, classical and quantum mechanical. It is shown that our ideas about light are changed radically when m is not equal to zero, for example, there is no longer a simple relation between the classical angular frequency, ω, and the classical wave-vector, $\boldsymbol{\kappa}$. If the photon is massive, $\kappa = \omega/c$ is no longer true in vacuo. The photon of mass m has a rest frame, in which the de Broglie Guiding theorem holds. The latter defines a rest frequency ω_0, through

$$\hbar\omega_0 = mc^2, \tag{378}$$

but in any other frame, this theorem has to be modified relativistically using the Lorentz transformation, as for any other particle with mass. In Sec. 7.1 it is shown that the potential four-vector is a physical four-vector for the photon of mass m, meaning that the scalar potential cannot arbitrarily disappear, as in the transverse gauge [47]. This section is followed by an elementary development of some relativistic properties of A_μ in free space, with a view to emphasizing its physical nature. It is no longer a convenience for deriving fields from d'Alembert's equation [1,2]. The Proca equation takes over from the latter, and can be quantized with none of the difficulties associated with canonical quantization [23] of the photon with $m = 0$. Our novel energy density (373), which is gauge invariant, shows that photon mass is reconcilable straightforwardly with gauge invariance in O(3).

7.1 EINSTEIN EQUATION FOR A_μ

The Einstein equation is a cornerstone of classical special relativity and is present in any textbook on the subject. It introduces the rest energy, $En = mc^2$, and relates it to the Lorentz invariant product of p_μ with itself, where p_μ is the energy-momentum four-vector. In standard [23] covariant-contravariant notation

$$p^{\mu}p_{\mu} = m^2c^2, \tag{379}$$

vhere $p^{\mu} = (\boldsymbol{p}, En/c)$, $p_{\mu} = (-\boldsymbol{p}, En/c)$. Here $\boldsymbol{p}$ is the particle momentum and En he total particle energy in a given Lorentz frame of reference, with m denoting particle nass, a scalar Lorentz invariant. In the rest frame, the momentum is zero, $\boldsymbol{p} = 0$, because the particle is at rest (i.e., does not translate) with respect to the frame of observation being used. From these elements it should be borne in mind that the photon nass must also be a Lorentz invariant, its mass in any Lorentz frame is the *same* as its rest mass [57]. The translation of a particle at c with respect to the observer is often nisunderstood to mean that the particle's mass vanishes. This cannot be so, however, f mass is a Lorentz scalar, because a Lorentz scalar does not vary with frame ransformation. Therefore special relativity runs into fundamental difficulties when the elative speed between two Lorentz frames becomes c identically. It ceases to be physically meaningful, because mathematical divergence is present in this condition.

The obvious way out of this is to assert that there is no particle without mass, as shown for the photon by our Eqs. (357). This means that the photon never translates at c with respect to the observer, and that the range of electromagnetic radiation is inite. The constant c then becomes an axiom of special relativity rather than *the speed of light*, because the speed of light *varies* from Lorentz frame to Lorentz frame. Our Eq. (373) shows that gauge invariance in O(3) is compatible with a photon of mass m. The problems of accepting a massless photon are well known [1,2,23,93] and are so acute that the idea is untenable: some of the inconsistencies to which it leads have been isted at the end of the last chapter. Since the discovery of the cyclic relations (357), however, it has become glaringly flawed rather than tolerably inconsistent, and the heory of finite photon mass takes center stage in this chapter.

Before writing the Einstein equation for A_{μ}, it must be emphasized that there are no experimental data that can be used to show that m must be zero identically, despite the fact that it is confidently asserted to be so in so many textbooks. Equations (357) now out-date these texts and show conclusively that m *cannot* be zero identically. It s however, very small in magnitude, probably much less than 10^{-45} kgm [1,2]. Therefore the existence of $\boldsymbol{B}^{(3)}$ shows that m is identically non-zero, but does not put a number to it without much more work. The photon mass m is very small because light reaches us from sources that are far distant from the Earth, and in accord with the Liénard-Wiechert concept, we are seeing this source as it *was* in the far distant past, not as it is today. The experimental attempts to measure photon mass have been reviewed briefly in Vol. 1 [1], and indeed, limits on m appear in the standard tables. Such efforts are diametrically at odds with numerous texts in electrodynamics which assert that $m = 0$ identically, almost axiomatically and without thought. It is now clear that the assertion $m = 0$ contradicts Eq. (357) without justification, but it will, perhaps, be a long time before this fatal flaw in the massless photon achieves acceptance, so deeply

immersed in orthodoxy is the claim $m = 0$.

In the meantime, we develop in this chapter a more self-consistent view, based on special relativity applied in free space to the four-potential A_{μ}, which we take from the outset to be a physical four-vector, and not a mathematical convenience as in classical electrodynamics.

From Eq. (379) the Einstein equation can be written in the familiar form

$$En^2 = c^2\boldsymbol{p}\cdot\boldsymbol{p} + m^2c^4, \tag{380}$$

and we develop this form for a free photon considered as a particle of mass m. Gauge invariance shows [1,2,22-25] that in the presence of A_{μ}, the four-momentum p_{μ} of any particle becomes $p_{\mu} + eA_{\mu}$, where e is the elementary unit of electric charge. This is a fundamental statement of energy-momentum-charge conservation, a statement which can be interpreted to mean that the four-momentum of the free photon is eA_{μ}. Therefore the Einstein equation for a free photon with mass m is

$$A^{\mu}A_{\mu} = \left(\frac{mc}{e}\right)^2. \tag{381}$$

This equation shows that if the mass m were zero, the product $A^{\mu}A_{\mu}$ would vanish. This result is precisely the one derived in Vol. 1 [1] and identified recently by Roy and Evans [11] as a limiting form of the Dirac condition [48,49]. Equation (381) for finite m becomes identifiable with the Dirac condition itself. If m is of the order 10^{-45} kgm or less, then the product $A^{\mu}A_{\mu}$ is of the order 10^{-36} (kgm m s^{-1} $C^{-1})^2$ or less.

If the four-potential is written in terms of its scalar (ϕ) and vector ($\boldsymbol{A}$) components as

$$A^{\mu} := \left(\boldsymbol{A}, \frac{\phi}{c}\right), \tag{382}$$

Eq. (382) becomes

$$\phi^2 = c^2\boldsymbol{A}\cdot\boldsymbol{A} + \frac{m^2c^4}{e^2}. \tag{383}$$

This is an equation of electrodynamics for a particle of mass m which we wish to identify with the particulate photon after quantization. The latter can proceed with the axioms

$$p = eA = \hbar\kappa = -i\hbar\nabla, \qquad En = e\phi = \hbar\omega = i\hbar\frac{\partial}{\partial t}, \tag{384}$$

which convert [1,2] Eq. (383) into the Proca wave equation,

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)A_\mu = -\left(\frac{mc}{\hbar}\right)^2 A_\mu. \tag{385}$$

The well known axioms (384) can be combined to give

$$P^\mu = eA^\mu = \hbar\kappa^\mu, \tag{386}$$

where κ^μ is the wave four-vector,

$$\kappa^\mu = \left(\kappa, \frac{\omega}{c}\right). \tag{387}$$

Splitting Eq. (386) into vector and scalar components gives

$$eA = \hbar\kappa, \qquad e\phi = \hbar\omega, \tag{388}$$

and taking moduli in the first of these gives the equivalence condition (188). The latter is seen to be a restatement of the familiar axioms (384) and the Proca wave equation is seen to be a natural outcome of these axioms and the general physical law of conservation (Noether's theorem).

The second of Eqs. (388) shows that the scalar potential is the free photon energy divided by e, and this result is demanded by conservation of energy-momentum in special relativity, i.e., by type two gauge transformation. Therefore the scalar potential for the free photon cannot be set to zero arbitrarily, as is the custom inherited from the nineteenth century in classical electrodynamics [47]. This result was demonstrated in a different way in Chap. 1 by considering limits of the Dirac equation. It was found there that setting $\phi = 0$ violated conservation of energy in a field-fermion interaction. For the free photon without mass Eqs. (388) are equivalent if $\phi = cA^{(0)}$. This condition was used in Chaps. 3 and 4 as being valid, F.A.P.P., in the strong field limit, and the latter is therefore a result of the axioms (384) of quantum mechanics applied to the free photon.

The customary assertion $\phi = 0$ is, nonetheless, the one made in defining the transverse gauge [47]. In the Coulomb gauge, ϕ can be a non-zero constant, but is, again, often set to zero [47]. This procedure is, however, glaringly self-inconsistent even in the limit $m = 0$, which we henceforth take to be a mathematical artifice without physical meaning. For example, for $m = 0$, $A^\mu A_\mu = 0$ identically, meaning that

$$\phi^2 = c^2 A \cdot A, \tag{389}$$

so again ϕ is not zero if $A \cdot A$ is not zero. Simple considerations such as these lead to the more abstract but more general criticisms by Weinberg [93], who shows that A_μ cannot be quantized for the massless photon. In contrast, canonical quantization of the Proca equation (385) is straightforward [23], giving a physical wave-particle in three space dimensions.

For identically non-zero m, however, there exists a rest frame for the photon, defined by the condition $A \cdot A = 0$, in which we recover the de Broglie Guiding theorem [1,2],

$$e\phi = mc^2 = \hbar\omega_0. \tag{390}$$

If $\phi = 0$ then the de Broglie theorem is invalidated for finite m, showing again the self-inconsistency of this assertion. In a frame other than the rest frame, the de Broglie theorem becomes

$$m^2c^4 = e^2(\phi^2 - c^2 A \cdot A). \tag{391}$$

In the rest frame for identically non-zero m the de Broglie theorem implies, conversely, that $A \cdot A = 0$. This condition is not possible unless A is zero while ϕ is non-zero, the extreme opposite of the usual light-like condition associated with the photon with $m = 0$. The condition $A \cdot A = 0$ can never be attained in the transverse gauge, and corresponds to a photon which is at rest in the frame in which it is being observed. This condition is possible if and only if m is identically non-zero.

The orthodox view sets $m = 0$ identically, and this means Lorentz transformation cannot take place from one frame to another. Momentum can no longer be mass multiplied by velocity, and rest energy can no longer be mass multiplied by c^2. The only basis for these assertions is that the energy-momentum density of radiation transforms in the same way as the energy density of a particle from Lorentz frame to Lorentz frame [66]. There is, furthermore, no experimental justification possible for the assertion that there exists a massless particle. The recent emergence of Eqs. (357)

shows conclusively that the idea of a massless particle is self-inconsistent in special relativity because a massless particle can have only two physically meaningful dimensions. Equations (357) produce, experimentally [16-21], three physically meaningful fields in three physical dimensions.

Finally, the wave vector and angular frequency for a photon with mass are no longer related by the simple $\kappa = \omega / c$, but through the quantized version of the Einstein equation,

$$\hbar^2 \omega^2 = \hbar^2 c^2 \kappa^2 + m^2 c^4, \tag{392}$$

so that κ, whenever it occurs in electrodynamics, must be replaced in S.I. units by

$$\kappa = \frac{1}{\hbar c}\left(\hbar^2 \omega^2 - m^2 c^4\right)^{\frac{1}{2}}, \tag{393}$$

or in $c = \hbar = 1$ units by

$$\kappa = \left(\omega^2 - m^2\right)^{\frac{1}{2}}. \tag{394}$$

The theory can be understood in terms of a complex wavenumber, with real and imaginary part for identically non-zero m; a theory that then becomes directly analogous with that of absorption and dispersion [105] in media, for example molecular ensembles in which the dielectric permittivity and refractive index are complex. The overall effect of finite photon mass can therefore be thought of as a vacuum which makes the wavenumber complex, and introduces *vacuum friction* analogous to the friction coefficient in a theory of dielectric loss such as that of Debye [103—105]. These conclusions follow from Eq. (393),

$$\kappa^2 = \left(\frac{\omega}{c}\right)^2 - \left(\frac{mc}{\hbar}\right)^2, \tag{395}$$

if we define

$$\kappa' := \frac{\omega}{c}, \qquad \kappa'' := \frac{mc}{\hbar}. \tag{396}$$

This gives

$$\kappa^2 = \kappa\kappa^* = \kappa'^2 - \kappa''^2, \tag{397}$$

where

$$\kappa := \kappa' + i\kappa'' = \frac{\omega}{c} + i\frac{mc}{\hbar}, \tag{398a}$$

$$\kappa^* := \kappa' - i\kappa'' = \frac{\omega}{c} - i\frac{mc}{\hbar}. \tag{398b}$$

At the point $\kappa' = \kappa''$, we recover from Eq. (396) the rest frequency of the de Broglie Guiding theorem

$$\omega_0 = \frac{mc^2}{\hbar}, \tag{399}$$

and therefore in the photon rest frame

$$\kappa\kappa^* = 0. \tag{400}$$

The physical meaning of this result can be found by using an analogy with the classical theory of dielectric loss [96] in a medium in which the permittivity becomes complex,

$$\epsilon := \epsilon' + i\epsilon'', \tag{401}$$

and $\epsilon\epsilon^* = 0$ occurs at the point $\epsilon' = \epsilon''$. This does not mean that the physical observables ϵ' and ϵ'' have vanished. In Debye's theory for example, the condition $\epsilon' = \epsilon''$ occurs at the peak of the dielectric loss curve, and defines the Debye relaxation time and relaxation frequency.

7.2 THE DE BROGLIE POSTULATES AND FINITE PHOTON MASS

The Planck Einstein condition $En = \hbar\omega$ was augmented by de Broglie with his postulate $\boldsymbol{p} = \hbar\boldsymbol{\kappa}$ and these are two cornerstones of quantum mechanics. While $\boldsymbol{p} = \hbar\boldsymbol{\kappa}$ has become well accepted in orthodoxy, and has justifiably remained so, de Broglie's original path to this great discovery, and thereby to matter waves and quantum mechanics, has been obscured by time. The recent centennial volume [6], however, shows that there is a substantial fraction of contemporary physicists involved in devising experimental tests for the empty wave hypothesis, through which de Broglie arrived at $\boldsymbol{p} = \hbar\boldsymbol{\kappa}$ in his work published in 1923 [106]. In this section we develop the theory of the massive photon through use of the de Broglie postulates. In so doing, we make use of some excellent papers of the de Broglie centennial volume [6].

It is convenient to develop the empty wave hypothesis through an elementary consideration of the Lorentz transformations firstly of p_μ, the energy-momentum four-vector, and secondly of x_μ, the space-time four- vector. The former transforms as $p'_\mu = a_{\mu\nu}p_\nu$, where $a_{\mu\nu}$ is the Lorentz transformation matrix. In S.I. units [1,2]

$$\begin{bmatrix} cp'_X \\ cp'_Y \\ cp'_Z \\ iEn' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & i\gamma\beta \\ 0 & 0 & -i\gamma\beta & \gamma \end{bmatrix} \begin{bmatrix} cp_X \\ cp_Y \\ cp_Z \\ iEn \end{bmatrix}, \tag{402}$$

where $\beta = v/c$ and $\gamma = (1 - \beta^2)^{-1/2}$. This transforms the quantities in the column four-vectors from one Lorentz frame K to another, K', translating at v in Z with respect to K. According to Einstein's second hypothesis, physical equations are Lorentz covariant, and are valid in both frames. Gauge invariance of type two is derived from this principle [23].

From Eq. (402) the energy transforms as

$$En' = -\beta\gamma c p_Z + \gamma En. \tag{403}$$

The inverse transformation [47] is $p_\mu = p'_\nu a_{\nu\mu}$, and gives

$$En = \beta\gamma c p'_Z + \gamma En'. \tag{404}$$

If both particle and frame K' are moving at v in Z, then the particle is at rest in frame K', which is therefore called the rest frame, or proper frame. The particle has no momentum in its own rest frame, and so $p'_Z = 0$. Equation (404) becomes

$$En = \gamma En', \tag{405}$$

an equation which defines the energy of the particle as it appears to the stationary observer. This is therefore the experimentally measured energy.

Repeating this analysis for x_μ produces the equations

$$t' = -\gamma\beta Z + \gamma t, \qquad t = \gamma\beta Z' + \gamma t', \tag{406}$$

and since the particle is not translating in its own rest frame, $Z' = 0$, and

$$t = \gamma t', \tag{407}$$

which illustrates that time as it appears to the observer in the fixed frame is different from time in the moving frame. The units of frequency are the inverse of the units of time, so the frequency equation corresponding to Eq. (407), essentially the explanation of light aberration [47], is

$$f = \gamma^{-1} f'. \tag{408}$$

Therefore the experimentally measurable frequency is given by Eq. (408), while the experimentally measurable energy is given by Eq. (405). These two equations have been derived from the same Lorentz transformation, and therefore from the first principles of special relativity.

The first principle of quantum mechanics, the Planck-Einstein hypothesis, asserts that energy is proportional to frequency through the Planck constant. This principle is well supported experimentally, and so is special relativity in classical mechanics. However, Eqs. (405) and (408) show that if energy is made proportional to frequency in the rest frame,

$$En' = hf' = mc^2, \tag{409}$$

they do *not* remain so

$$En = h\gamma^2 f = hf\left(1 - \frac{v^2}{c^2}\right)^{-1}. \tag{410}$$

The $\boldsymbol{B}^{(3)}$ field now shows that if the photon is particulate in nature, there is a finite photon mass, so this problem is also present in the theory of electromagnetic radiation. Equation (409) is the de Broglie Guiding theorem in the rest frame of the photon with mass m, and in the observer frame it becomes Eq. (410). Stated in another way, following the interesting discussion by Awobode [107], there are two frequencies present in the observer frame. Equation (405) gives

$$f_1 = \frac{En}{h} = \frac{mc^2}{h}\left(1 - \frac{v^2}{c^2}\right)^{-1/2}, \tag{411}$$

and this is a different frequency from that given by Eq. (408). There is in contrast only one frequency present in the rest frame, that given by Eq. (409). Furthermore, if v is made identically equal to c, the frequency f_1 becomes infinite, while the frequency f becomes zero. Special relativity loses physical meaning unless the concept of Lorentz transformation is abandoned for $v = c$.

It is well known that de Broglie discovered this problem as a student, and suggested his wave hypothesis as a solution [106]. This line of thought predated the famous $\boldsymbol{p} = \hbar\boldsymbol{\kappa}$, which first appeared in print as a footnote to a short paper of 1924 [108]. As pointed out by Ferrero and Santos [109], the relation $\boldsymbol{p} = \hbar\boldsymbol{\kappa}$ was a generalization of the harmony of phases, or wave hypothesis. The generalization has become well known and well supported experimentally, while experimental evidence for the empty wave is still being sought contemporaneously [110]. The two frequency problem represented by Eqs. (408) and (411) therefore remains a fundamental challenge to relativistic quantum physics, because if de Broglie's own solution is not supported experimentally, another must be found. It is not known at the time of writing whether the empty wave exists experimentally, although several elegant experimental tests have been carried out [6]. Despite the passage of seventy years or more, this remains a fundamental challenge to physics, and in particular, to the theory of electromagnetic radiation. The appearance of the $\boldsymbol{B}^{(3)}$ field puts it center stage, because the photon, if it is a particle, can no longer be thought of as massless, and it is no longer possible to by-pass the problem by asserting that for the photon, Lorentz transformation is not applicable because there is no rest frame. In our opinion, this assertion was always dubious, and is shown to be erroneous by the cyclic relations (357) as argued throughout these volumes [1,2].

Having discovered the two frequency paradox, which is a paradox more severe than any other in relativistic quantum theory, de Broglie proposed a solution by using the fact that the phase of a travelling wave is a relativistically invariant quantity [47]. If there exists a plane wave that propagates in Z at phase velocity w with respect to the observer then the phase associated with the frequency f_1 at position R is

$$\phi = f_1\left(t - \frac{R}{w}\right). \tag{412}$$

If time, t, is now defined with respect to the observer frame, i.e., is the experimentally measured time in the observer's frame of reference, then the phase of the plane wave a time R/w earlier is relativistically invariant and equal to the phase (412) in the observer's frame or any other frame. The relativistic invariance of phase therefore makes it possible to assert that

$$ft = f_1\left(t - \frac{R}{w}\right), \tag{413}$$

an equation which is satisfied by the solutions

$$R = vt, \qquad w = \frac{c^2}{v}. \tag{414}$$

This means that the *phase* velocity w is faster than light, or superluminal, if $v<c$.

In a scholarly article, Muggur-Schächter [110] has translated de Broglie's original derivation of the wave phenomenon, which he described originally as a *periodic element*, a standing wave in the rest frame of the particle described by

$$\psi_0 = a_0 \exp(2\pi i \nu_0 t_0). \tag{415}$$

The inverse Lorentz transformation was then used by de Broglie to express this standing wave in the observer frame

$$\psi = a_0 \exp\left(2\pi i\nu\left(t - \frac{R}{w}\right)\right), \tag{416}$$

where

$$\nu = \gamma\nu_0, \qquad w = \frac{c^2}{v}. \tag{417}$$

Therefore the standing wave in the fixed frame becomes a travelling plane wave in the observer frame. The wave is a standing wave in the rest frame, and so is always centered on the particle itself, the essential reason being that mc^2 is the only non-zero energy in the particle rest frame, and is the only possible source of the frequency defined by de Broglie Guiding theorem (409).

The relevance of $\boldsymbol{B}^{(3)}$ is that the latter shows conclusively that the photon, if particulate, is massive, and so gives further support to the basic idea that matter waves and light waves are essentially manifestations of the same thing. This idea is the source of quantum mechanics, and was a direct result of the basic paradox just described between relativity and the light quantum hypothesis. The first explanation [110] given by de Broglie for the two frequency paradox has been developed into an elegant edifice of twentieth century thought [111—113]. The empty wave concept developed through the idea that all energy and momentum is carried by the particulate photon, so that the accompanying plane wave is bereft of these attributes. As we have just seen, the existence of this plane wave was postulated to account for the two frequency paradox. At the time of writing, evidence for the empty wave is being sought in several ways [6]. Closely related is the concept of the pilot wave, or guiding wave, and the following section looks at the connection between $\boldsymbol{B}^{(3)}$ and the pilot wave.

7.3 $\boldsymbol{B}^{(3)}$ AS A PILOT FIELD

It has been inferred in these volumes that the conventional view of free space electromagnetism is incomplete, because the classical theory produces a novel magnetic flux density in vacuo, $\boldsymbol{B}^{(3)}$. The imaginary axial vector quantity,

$$\boldsymbol{I}_A = \frac{c}{\mu_0}\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} = i\frac{c}{\mu_0}B^{(0)}\boldsymbol{B}^{(3)*}, \tag{418}$$

is the antisymmetric part of the free space light intensity tensor, and is therefore directly proportional to $\boldsymbol{B}^{(3)}$ in the vacuum. The question of how $\boldsymbol{B}^{(3)}$ acts as a pilot field [6] can be approached by firstly deriving the equation [7]

$$\frac{\boldsymbol{B}^{(3)}}{B^{(0)}} = \frac{\boldsymbol{J}^{(3)}}{\hbar} = \boldsymbol{e}^{(3)}, \tag{419}$$

as a straightforward consequence of the quantization of the electromagnetic field. In Eq. (419), $\hbar$ is the Dirac constant, and the real and physical

$$\boldsymbol{J}^{(3)} = \hbar\boldsymbol{e}^{(3)}, \tag{420}$$

is an angular momentum with magnitude $\hbar$ of the particle being considered, assumed to be the photon. Since $\boldsymbol{B}^{(3)}$ is produced from a cross product of vector plane wave functions $\boldsymbol{B}^{(1)}$ and $\boldsymbol{B}^{(2)}$, it satisfies the criteria originally proposed by de Broglie [6] for pilot waves, and is also a phase free magnetic flux density directly proportional to the angular momentum of the photon. It is therefore considered here as the pilot field of the angular momentum of the photon.

Equation (419) can be derived from fundamentals through a consideration of the electromagnetic torque density,

$$\boldsymbol{T}_V^{(3)*} = -\frac{i}{\mu_0}\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} = \frac{B^{(0)}}{\mu_0}\boldsymbol{B}^{(3)*}, \tag{421}$$

in which

$$\boldsymbol{m}^{(1)} = \boldsymbol{m}^{(2)*} = \frac{\boldsymbol{B}^{(1)}}{\mu_0}, \tag{422}$$

are oscillating magnetic dipole moments of the radiation itself. Thus

$$\boldsymbol{T}_V^{(3)*} = -i\boldsymbol{m}^{(1)} \times \boldsymbol{B}^{(2)}, \tag{423}$$

in formal analogy with the definition of magnetically generated torque in electrostatics and electrodynamics [114]. The real part of $\boldsymbol{T}_V^{(3)}$ is physical, and is proportional (see Sec. 7.4) to radiation angular momentum through $\boldsymbol{B}^{(3)}$. Thus

$$\boldsymbol{T}_v^{(3)*} = \omega\boldsymbol{J}_v^{(3)*}. \tag{424}$$

We now use in Eq. (424) one of the standard axioms of quantum mechanics, one based on the de Broglie relation, the axiom

$$\omega := \frac{\partial}{\partial t} = -i\frac{En}{\hbar} = \frac{|\boldsymbol{T}_V^{(3)}|}{\hbar}, \tag{425}$$

where En is energy. Since $\boldsymbol{J}_V^{(3)}$ is real, Eqs. (421), (424) and (425) give a real

$$T_V^{(3)*} = \frac{En}{\hbar}J_V^{(3)*} = \frac{En}{\hbar V}J^{(3)*}, \tag{426}$$

where V is the volume used to define the angular momentum density, $J_V^{(3)}$, and where $J^{(3)}$ has the units of angular momentum itself. Using the energy density

$$U = \frac{En}{V} = \frac{B^{(0)2}}{\mu_0}, \tag{427}$$

we obtain

$$T_V^{(3)*} = \frac{B^{(0)}}{\mu_0}B^{(3)*} = \frac{B^{(0)2}}{\mu_0\hbar}J^{(3)*}, \tag{428}$$

from which

$$B^{(3)} = B^{(0)}\frac{J^{(3)}}{\hbar}, \tag{429}$$

which is Eq. (419) with $J^{(3)} = \hbar e^{(3)}$. The result (429) was first derived in 1992 [7] using an independent method [1,2].

Equation (419) can be derived in a third way by using a straightforward adaptation of the standard Planck-Einstein equation,

$$\hbar\omega = \int U dV. \tag{430}$$

Instead of the usual $U = B^{(1)} \cdot B^{(2)}/\mu_0$ we use

$$U = \frac{1}{\mu_0}|B^{(1)} \times B^{(2)}| = \frac{B^{(0)2}}{\mu_0}, \tag{431}$$

and obtain

$$\hbar = \frac{1}{\mu_0\omega}\int |B^{(1)} \times B^{(2)}|dV. \tag{432}$$

In the basis ((1),(2),(3)), Eq. (432) becomes

$$iJ^{(3)*} := i\hbar e^{(3)*} = \frac{iB^{(0)}}{\mu_0\omega}\int B^{(3)*}dV, \tag{433}$$

and rearranging

$$B^{(3)} = \frac{\mu_0\omega}{B^{(0)}V}J^{(3)} = B^{(0)}\frac{J^{(3)}}{\hbar}, \tag{434}$$

which is again Eq. (419); the direct, self-consistent, result of two fundamental axioms of quantum mechanics, Eqs. (430) and (425), the Planck-Einstein and de Broglie relations.

To develop the idea of $B^{(3)}$ as a pilot field for photon spin we consider a Young experiment for $B^{(3)}$ carried out with a circularly polarized incident beam which is diffracted through the double aperture of the interferometer to form a diffraction pattern. In classical electromagnetism, this requires an exact solution of the Maxwell equations, as described recently by Jeffers *et al.* [115] for linearly polarized incident radiation. Using the free space relation,

$$I_0 = \frac{c}{\mu_0}B^{(0)2}, \tag{435}$$

it is seen that lines of diffraction due to $B^{(3)}$ will follow those due to $I_0^{1/2}$, and be similar to those due to I_0 computed by Jeffers *et al.* [115]. This is an interferogram [116] that shows considerable structure within a few wavelengths of the slits, a structure which is unobtainable with standard diffraction theory. There is, however, no such thing present as classical interference, i.e., no radiation actually crosses the symmetry axis, and no radiation passing through the top aperture arrives at a point below the axis of symmetry and vice-versa.

If $B^{(3)}$ is considered to be the pilot field of $\hbar$, then both quantities must be simultaneously measurable [6]. Lines of constant $B^{(3)}$ in a diffraction pattern would be lines of constant $\hbar e^{(3)}$. These ideas do not occur in conventional electrodynamics, in which $B^{(3)}$ is undeveloped. Its existence in vacuo, however, has been demonstrated self-consistently in these volumes, for example its magnetizing effect has been shown using the classical Hamilton-Jacobi equation of one electron (e) in the classical electromagnetic field represented by the four-potential (A_μ), a demonstration which shows that the trajectory of the electron in the beam is governed *entirely* by $B^{(3)}$ and by no other vacuum field. This can be understood through the fact that $T_v^{(3)}$ is a torque density of radiation in the vacuum, and to the fact that $\boldsymbol{B}^{(3)}$ is directly proportional to the radiation's real and physical angular momentum density (Eq. (434)). Prior to this understanding, $\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)}$ was an obscure experimental observable labelled by some nonlinear opticians as the *conjugate product*. (More precisely, it is one out of several [16-21] conjugate products.) The conjugate product $\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)}$ is the product of vacuum permeability and radiation torque density, a torque density which is expressible as $iB^{(0)}\boldsymbol{B}^{(3)*}/\mu_0$. The real and physical torque per unit volume of radiation is therefore proportional directly to the real and physical $\boldsymbol{B}^{(3)*}$ through the premultiplier $B^{(0)}$.

Our use [1] of the Hamilton-Jacobi equation of e in A_μ to demonstrate the existence of $\boldsymbol{B}^{(3)}$ from the principle of least action [1] is significant in at least two ways. Cushing [117] has pointed out that de Broglie originally saw the Hamilton-Jacobi equation as providing ".... an embryonic theory of the union of waves and particles, all in a manner consistent with a realist conception of matter". Equation (429) now shows that if $\hbar$ is the angular momentum of a particle, the photon, then $\hbar$ must be directly linked with $\boldsymbol{B}^{(3)}$, and in the realist view, be simultaneously observable with it. Rewriting Eq. (429)

$$\boldsymbol{J}^{(3)} = \hbar\left(\frac{\boldsymbol{B}^{(3)}}{B^{(0)}}\right), \tag{436}$$

we obtain an expression which is directly analogous with the Planck-Einstein and de Broglie relations, $En = \hbar\omega$ and $\boldsymbol{p} = \hbar\boldsymbol{\kappa}$ respectively.

Secondly, as shown by Bohm [85], the Schrödinger equation can be interpreted by developing it into a quantized Hamilton-Jacobi equation. This procedure requires the introduction of the quantum potential, and leads to non-locality and superluminal action at a distance [6]. In this context, Chap. 3 has suggested a connection between these concepts and an optically induced Aharonov-Bohm effect due to $\boldsymbol{B}^{(3)}$. These questions are addressed in the centennial volume [6] in many interesting ways, and the contemporary view [23,39] of the complicated topology of the vacuum may give us some answers. As described in Chap. 3, a particular vacuum topology is needed to sustain the Aharonov-Bohm effects. In the Copenhagen interpretation of quantum mechanics [45], the quantum equivalent of the classical $\boldsymbol{B}^{(3)}$ is an angular momentum operator, $\hat{\boldsymbol{B}}^{(3)}$, the photomagneton [4]. The latter is directly proportional [1,2] to $\hat{J}^{(3)}$, which is governed by angular momentum commutator equations, and operates on an angular momentum wavefunction. The latter has no physical reality until it is observed, when the wavefunction collapses.

Is it possible to use the $\boldsymbol{B}^{(3)}$ field to distinguish between the Copenhagen and Bohm-Vigier views of quantum mechanics? In order to begin to scratch the surface of

this question, it is convenient to look at Bohm's own discussion [85], and to try to adapt it to the line of argument that if $\boldsymbol{B}^{(3)}$ is the pilot field for $\hbar$, it is simultaneously measurable with it.

In the classical theory of electrodynamics, $\boldsymbol{B}^{(3)}$ is modified by diffraction as described accurately by Jeffers *et al.* [115] by numerically solving Maxwell's equations with well defined boundary conditions. The diffraction patterns caused by the two apertures are those of $\boldsymbol{B}^{(3)}$ itself, and so must be those of the angular momentum of classical radiation. The latter can be represented after quantization by $\hbar\boldsymbol{e}^{(3)}$, whose magnitude is $\hbar$. The particle (photon) concomitant with the diffracted wave therefore has angular momentum magnitude $\hbar$. If so, however, where is the particle after diffraction has occurred? If the incoming pilot field-particle ($\boldsymbol{B}^{(3)}$, $\hbar$) is equivalent to one photon of energy $\hbar\omega$, what happens to the particulate photon after diffraction? This question is answered entirely differently in the Bohm-Vigier and Copenhagen interpretations [6], thus giving scope for experimental investigation of these theoretical differences.

In the simpler case of light passing through a beam divider, the connection between $\boldsymbol{B}^{(3)}$ and $\hbar$ can be described as follows. In a naive interpretation the photon carries particulate information, and at random goes to one of two detectors, A or B, after the light has been split by the beam divider. However, radiation consisting of many photons and carrying the $\boldsymbol{B}^{(3)}$ field goes to both detectors simultaneously, detectors which measure split beam intensity. This is a statistical process. Since single photon (and neutron) generators are now available [6], these assumptions can be tested directly in principle and the need for statistical analysis by-passed entirely. Aspect *et al.* [118,119] appear to have shown that if the particulate photon goes one way in an interferometer, there is *nothing* present in the other arm, there is 100% anticorrelation. This suggests that the photon's angular momentum *and* the field $\boldsymbol{B}^{(3)}$ are both wholly present at A if the particulate photon has been so detected, and cannot therefore be present at B if there is nothing detected at B. The Aspect experiments, if interpreted in this way, show that $\hbar$ and $\boldsymbol{B}^{(3)}$ are at A, but this interpretation depends on the reliability and relevance of the available data [6,118]. We would expect that if $\boldsymbol{B}^{(3)}$ and $\hbar$ for one photon were both wholly present at A, there would be a heat effect (due to electromagnetic intensity) and a resonance effect (due to photon angular momentum) of the type described in Chaps. 1 to 3 in the weak field limit. Neither effect would be present at B in this naive point of view.

In the Copenhagen interpretation, described by Croca [120], the light incident on the beam splitter is divided into two wave packets, and when one of these hits a detector A, for example, the photon has chosen that path. The wave function in the Copenhagen interpretation is a wave of probability, and a detector is a measuring device that has the effect of bringing the photon into existence. Thus, causality is lost or obscured, reality is something that is asserted to follow measurement. The photon if detected at A has been brought into measurable existence by the process of detection at A, and is not simultaneously measurable at B. In the Copenhagen interpretation of quantum optics, if a photon is made to exist at A, so is $\boldsymbol{B}^{(3)}$, because the latter is an operator of quantum mechanics whose eigenvalue is the observable. The operation of $\hat{B}^{(3)}$ on a wavefunction brings $\boldsymbol{B}^{(3)}$ into experimentally measurable existence as an expectation value at the detector A, and the wavefunction collapses at A. This view appears to be in agreement with the results of Aspect *et al.* [118], but this is contested by several papers [6].

In the empty wave interpretation [6], the field $\boldsymbol{B}^{(3)}$, if it is the pilot field for $\hbar$, itself carries no angular momentum. The latter is asserted to be a purely particulate property of the photon. The empty wave can presumably be divided indefinitely at successive beam dividers, but the photon, if it is a particle, must always take one path or another. It is therefore possible that at the detector A there is a particulate photon and an empty field $\boldsymbol{B}^{(3)}$, with halved intensity, while at detector B there is an empty wave with halved intensity but no particulate photon. Several experiments to test this point of view have been proposed and carried out [1,2,6]. We have implicitly assumed that the empty wave has intensity in the presence of a photon, but we have also assumed that it has intensity in the absence of a photon, because the beam divider has been assumed to halve the incoming intensity. If so, we would expect heat effects at both A *and* B, but resonance effects at A only. In the naive and Copenhagen interpretations there would be no heat effect at B and both heat and resonance effects at A. If, on the other hand, an empty field carries no intensity, it would produce no measurable heat effect at B and we would still not be able to decide which of our interpretations is applicable. However, if there were empty waves at both A and B they could be made to recombine in principle to produce a measurable interference pattern. These questions are discussed in the contemporary literature [6], and are fundamental and central to the quantum theory of light and to physics in general, because they address the severe and fundamental paradox represented by the different frequencies in Eqs. (405) and (408). The paradox is that the elementary principles of special relativity are not compatible with those of the quantum theory, as first pointed out by Louis de Broglie [6]. The empty wave is a hypothesis put forward to account for this paradox, and in the Copenhagen interpretation there appears to be no known solution for it, a wholly unsatisfactory state of affairs. (It is essential to note that the de Broglie matter wave is not the empty wave, the former was proposed from a comparison [110] of the Hamilton and Fermat principles, the latter was proposed in an abstract [110] attempt to address the foregoing paradox. The matter wave was immediately accepted by the Copenhagen School, the empty wave evidently was not.) The emergence of $\boldsymbol{B}^{(3)}$ makes the photon look more than ever like a particle, because it must have mass, cannot be two dimensional, and can translate at less than c.

The description of $\boldsymbol{B}^{(3)}$ as a pilot field is conveniently developed [85] by assuming that there exists an entity, ψ, guiding the particle, an entity which can be written in terms of the real mechanical action S,

$$\psi = R\exp\left(i\frac{S}{\hbar}\right). \tag{437}$$

Here R^2 is the probability that a particle of mass m have a velocity $V = \nabla S/m$. For the photon, m is the photon mass suggested by the existence of $\boldsymbol{B}^{(3)}$ itself. In his paper of 1952 [85], Bohm showed that this view is plausible, and leads to all the major results of quantum mechanics. In so doing he met the objections of Pauli to de Broglie's original harmony of phases theorem and his later proposal of 1930 [121]. A slight extension of Bohm's original idea is all that is need to describe $\boldsymbol{B}^{(3)}$ as a pilot field,

$$\boldsymbol{\psi}^{(1)} = \boldsymbol{R}^{(1)} e^{iS/\hbar} = \boldsymbol{\psi}^{(2)*}, \tag{438}$$

where S is electromagnetic action [1,2] defined by

$$S = \hbar(\omega t - \boldsymbol{\kappa}\cdot\boldsymbol{r}), \tag{439}$$

and where $\boldsymbol{R}^{(1)}$ and $\boldsymbol{R}^{(2)}$ are

$$\boldsymbol{R}^{(1)} = \boldsymbol{e}^{(1)}, \qquad \boldsymbol{R}^{(2)} = \boldsymbol{e}^{(3)}. \tag{440}$$

In this picture

$$\psi^{(0)2} = |\boldsymbol{\psi}^{(1)} \times \boldsymbol{\psi}^{(2)}|, \tag{441}$$

is the probability of finding a particle with an angular momentum given by

$$|\boldsymbol{J}^{(3)}| = \frac{\partial S}{\partial \phi}, \tag{442}$$

where [85] ϕ is the azimuthal angle. Therefore $\boldsymbol{B}^{(3)}$, which is directly proportional to $\boldsymbol{J}^{(3)}$, as we have shown in several ways, is a pilot field of the particulate angular momentum $\hbar$ which we ascribe to the photon.

7.4 A NEW APPROACH TO THE TWO FREQUENCY PARADOX

It is possible to devise a new approach to the paradox which led de Broglie to his proposal of the empty wave, a proposal which has not yet been verified experimentally, however. Our new developments, described in this section, are meant to augment de Broglie's harmony of phases theorem, which we regard as a valid answer to the fundamental paradox represented by Eqs. (405) and (408). Recall that these two equations show that special relativity is not compatible with the Planck-Einstein hypothesis without some additional hypothesis such as that of the empty wave. So the paradox remains a central issue in contemporary physics, since it appears not to be addressed at all in the Copenhagen agreement [1,2] or standard texts.

Our suggestion is that the quantum of electromagnetic energy can be expressed in terms of wavelength as

$$En = \hbar_1 \lambda, \tag{443}$$

where $\hbar_1$ is a Lorentz invariant force,

$$\hbar_1 = \frac{\omega^2}{2\pi c}\hbar. \tag{444}$$

Thus, Lorentz transformation of the energy proceeds through Lorentz transformation of the wavelength, i.e.,

$$\lambda = \gamma \lambda_0, \tag{445}$$

while Lorentz transformation of frequency occurs always as in Eq. (408). Thus frequency always transforms as frequency, and wavelength always transforms as wavelength. This self-consistently produces the result

$$\lambda f = \lambda_0 f_0 = c, \tag{446}$$

in all Lorentz frames.

These conclusions are derived as follows. We accept c as a constant of special relativity from Einstein's first hypothesis, and if so then the product of wavelength and frequency must be invariant under Lorentz transformation. This is automatic in the conventional theory, because there the photon is massless and propagates at c, and it is asserted that Lorentz transformation has no meaning since there is no rest frame. If the photon is massive, and it is assumed that its wavelength in its rest frame is λ_0, then its counterpart in the observer frame is $\gamma \lambda_0$. The counterpart of the rest frequency f_0 is, however, $\gamma^{-1} f_0$, and so equation (446) is recovered. This is consistent with the fact that c is frame invariant by Einstein's first hypothesis. Therefore the wavelength transforms as the energy in Eq. (445), while frequency transforms as in Eq. (408). Louis de Broglie considered that frequency can also transform as energy, through our Eq. (411), but this is not clear from the foregoing fundamentals. This assumption by de Broglie causes two frequencies to appear in the observer frame, and as described earlier in this chapter, this led to the harmony of phases theorem.

Our new proposal, Eq. (443), is based on the assumption that energy can be proportional to wavelength, because both transform in the same way from Lorentz frame to Lorentz frame. The Planck-Einstein relation is therefore augmented by Eq. (443) and the quantum of energy is proportional to wavelength. This result can be deduced by first linking the wavenumber to the Thompson radius [122], $\lambda/2\pi$,

$$\lambda = \frac{2\pi}{\kappa}. \tag{447}$$

We express the unit of time as $2\pi/\omega$. This means that the quantum hypothesis becomes

$$p^\mu = \hbar\left(\frac{2\pi}{\lambda}e^{(3)}, \frac{2\pi}{ct}\right), \tag{448}$$

giving the result

$$p^\mu p_\mu = m^2c^2 = h^2\left(\frac{1}{c^2t^2} - \frac{1}{\lambda^2}\right). \tag{449}$$

In the limit $m \to 0$ we recover $\lambda \to ct$, which is the counterpart of the usual $\omega \to c\kappa$. Thus,

$$\frac{\lambda}{t} = \frac{\omega}{\kappa} = c, \qquad \lambda\kappa = \omega t = 2\pi, \tag{450}$$

showing that our assumptions assign a value 2π to the phase ωt. More generally,

$$\lambda\kappa = \omega t = 2\pi + \alpha, \tag{451}$$

where α is an arbitrary phase variable.

The unit of action in the $m \to 0$ limit is

$$S = \hbar\omega t = \hbar\lambda\kappa = Ent, \tag{452}$$

where $En = \hbar\omega$ is the quantum of electromagnetic energy, the photon. The photon energy En is thus

$$En = \hbar\frac{\kappa}{t}\lambda = \frac{\hbar\kappa\omega}{2\pi}\lambda = \left(\frac{\hbar\omega^2}{2\pi c}\right)\lambda, \tag{453}$$

and becomes proportional to the wavelength of radiation,

$$En = \hbar_1\lambda = \hbar\omega, \tag{454}$$

where

$$\hbar_1 = \frac{\omega^2}{2\pi c}\hbar, \tag{455}$$

is a Lorentz invariant because for $m \to 0$, the difference $\omega^2 - \frac{\kappa^2}{c^2} = 0$ is a Lorentz invariant. If a difference of two quantities is invariant, and one quantity is equal to the other, then both quantities individually are Lorentz invariant. Thus our new constant $\hbar_1$ is an invariant under Lorentz transformation. Unlike $\hbar$ it is not a universal constant because of the presence of ω^2 in its definition. The product of $En = \hbar\omega$ with $En = \hbar_1\lambda$ gives

$$En^2 = hf\hbar_1\lambda = \hbar^2\omega^2, \quad \lambda f = c, \tag{456}$$

showing that the product of wavelength and frequency is c as required. Since c is a universal constant, this result is also Lorentz invariant. In the limit $m \to 0$ therefore, En^2 is also invariant.

Our assertion that energy transforms as wavelength is based on the fact that x_μ and p_μ are both four-vectors of special relativity. Since mc^2 is an energy, the de Broglie Guiding theorem becomes, with the use of Eq. (454),

$$\hbar_1\lambda_0 = mc^2, \tag{457}$$

which in the observer frame becomes

$$\hbar_1\lambda = \hbar_1\gamma\lambda_0 = mc^2. \tag{458}$$

Frequency transforms through Eq. (408), i.e., as a frequency. Therefore we arrive at

$$\lambda = \frac{\lambda_0}{\left(1 - \frac{v^2}{c^2}\right)^{1/2}} = \frac{mc^2}{\hbar_1}, \tag{459}$$

i.e., *wavelength transforms as wavelength*, and,

$$f = \left(1 - \frac{v^2}{c^2}\right)^{1/2} f_0 = \frac{mc^2}{\hbar}, \tag{460}$$

frequency transforms as frequency.

In all Lorentz frames,

$$\lambda f = \lambda_0 f_0 = \frac{m^2c^4}{\hbar\hbar_1} = \left(\frac{m^2c^4}{En^2}\right)c, \tag{461}$$

and if $En = mc^2$, we recover

$$\lambda f = \lambda_0 f_0 = c. \tag{462}$$

Since En^2 and m^2c^4 are both frame invariant if $m \to 0$, and since mc^2 is the only energy available in the rest frame, Eq. (462) follows from Eq. (461).

Therefore we have arrived at the conclusion that the oscillating phenomenon postulated by de Broglie [6] can be understood without the use of the harmony of phases theorem, provided that energy transforms as wavelength, and not as frequency.

Accompanying the particle, therefore, is a wavelength, λ_0 and through the relation $f_0\lambda_0 = c$ in the rest frame, a frequency. These are the wavelength and frequency of the empty wave, and f_0 and λ_0 are linked in the same way as for an electromagnetic wave. The empty wave moves forward at the velocity of the particle. There appears to be no need, however, to postulate a superluminal phase velocity for the empty wave. These conclusions are important for experiments which try to detect the empty wave through its interference with an electromagnetic wave [6]. We note that the energy assigned in Eqs. (454) is that of the particulate photon, and if this photon is removed by a beam divider [6] for example, the remaining empty wave has no energy. This is the standard interpretation.

7.5 ROLE OF $\boldsymbol{B}^{(3)}$

The definition of $\hbar_1$ in Eq. (455) suggests that it can be related to $\boldsymbol{B}^{(3)}$ through Eq. (280), which shows that $\hbar_1$ is Lorentz invariant,

$$\hbar\omega^2 = ec^2|\boldsymbol{B}^{(3)*}|, \tag{463}$$

because $\boldsymbol{B}^{(3)}$ is so. This equation gives a simple proportionality

$$\hbar_1 = ec|\boldsymbol{B}^{(3)*}| = eE^{(0)}, \tag{464}$$

showing that $\hbar_1$ has the units of torque per unit length. These are also the units of force (newtons), but torque per unit length is a pseudo, or axial, vector, and force is a polar vector. The units of torque (J) are also those of energy, so the modulus of torque is energy. From Eq. (240) we can write

$$\boldsymbol{B}^{(3)*} = -\frac{i}{c}\boldsymbol{v}^{(1)} \times \boldsymbol{B}^{(2)}, \tag{465}$$

so that the quantum of torque modulus from Eq. (454) is

$$En = |\boldsymbol{Tq}^{(3)*}| = ec\lambda|\boldsymbol{B}^{(3)*}|. \tag{466}$$

Therefore the electromagnetic torque per unit length, taken to be the wavelength, λ,

is directly proportional to $\boldsymbol{B}^{(3)}$, and is the new radiation constant $\hbar_1$,

$$\hbar_1 = ec|\boldsymbol{B}^{(3)*}|. \tag{467}$$

The electromagnetic torque itself can be written as

$$\boldsymbol{Tq}^{(3)} = ec\lambda\boldsymbol{B}^{(3)} = \hbar_1\lambda\boldsymbol{e}^{(3)}, \tag{468}$$

and is directly proportional to $\boldsymbol{B}^{(3)}$, a result which was also obtained for the torque per unit volume in Eq. (184) or (421). Comparison of Eqs. (468) and (184) or (421) gives

$$B^{(0)} = \frac{\mu_0 ec\lambda}{V}, \tag{469}$$

and if $r^{(0)} = \lambda/2\pi$, this is Eq. (192). Equations (184) and (468) both show that $\boldsymbol{B}^{(3)}$ is observable directly in the Beth effect [123], which measures the electromagnetic torque experimentally through the action of circularly polarized radiation on a crystal suspended from a torsion wire.

Therefore we conclude that $\boldsymbol{B}^{(3)}$ is measurable directly in the Beth effect, and conversely, is responsible for the Beth effect.

The Planck-Einstein hypothesis becomes

$$En = \hbar\omega = |\boldsymbol{Tq}^{(3)}| = \hbar_1\lambda, \tag{470}$$

which shows that $\hbar\omega$ can be thought of as an energy quantum which is also a quantum of electromagnetic torque. The torque quantum $\hbar_1\lambda$ is conversely also an energy quantum. Both quanta are mc^2 according to de Broglie's rest frame hypothesis developed in the preceding section. The de Broglie Guiding theorem therefore becomes

$$\hbar\omega_0 = \hbar_1\lambda_0 = mc^2, \tag{471}$$

where λ_0 and f_0 are the rest wavelength and frequency respectively. The theorem gives the photon mass directly in terms of $\boldsymbol{B}^{(3)}$

$$m = \frac{e}{c}\lambda_0|\boldsymbol{B}^{(3)}|, \tag{472}$$

and we see $\boldsymbol{B}^{(3)}$ appearing frequently in fundamental equations, a clear sign that $\boldsymbol{B}^{(3)}$ itself is of fundamental significance.

We therefore propose that the quantum of energy $\hbar\omega$ is also a torque quantum, $\hbar_1\lambda$, where $\hbar_1$ is Lorentz invariant and proportional to the modulus of $\boldsymbol{B}^{(3)}$. The quantum of energy and the quantum of torque are equal to each other in Eq. (471), but *transform differently*. The energy quantum transforms as frequency, the torque quantum as wavelength. If they are equal to each other in the rest frame they are no longer so in any other Lorentz frame. The equality of the torque and energy quanta of electromagnetic radiation is a Lorentz invariant equality if and only if the mass m of the photon is zero identically. The two frequency paradox therefore melts away for this reason.

7.6 NEW FUNDAMENTAL EQUATIONS OF ELECTRODYNAMICS

We suggest the following hypotheses of electrodynamics.

1) In Euclidean, or three dimensional, space, the equations,

$$\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} = iB^{(0)}\boldsymbol{B}^{(3)*}, \text{ et cyclicum,} \tag{473}$$

are fundamental equations of motion of electrodynamics in vacuo or in matter. Here,

$$\boldsymbol{B} = \boldsymbol{B}^{(1)} + \boldsymbol{B}^{(2)} + \boldsymbol{B}^{(3)}, \tag{474}$$

is magnetic flux density.

2) In flat space-time (special relativity), these equations retain their structure, but the $\boldsymbol{B}$ fields become four dimensional generators [1].

3) In Riemannian or curved space-time (general relativity), there is an equivalent set of equations of motion for electromagnetism in the presence of gravitation, equations which must be written with Riemann's geometry.

4) These equations are also equations of quantum mechanics if the $\boldsymbol{B}$ fields become angular momentum operators of quantum mechanics. They then become Heisenberg commutator relations [1].

5) They are more general than Maxwell's equation because of the fundamental role of $\boldsymbol{B}^{(3)}$, which in quantum mechanics becomes the photomagneton [4].

Chapter 8. Primordial $\boldsymbol{B}^{(3)}$ in Relativistic Cosmology

The origin of cosmic magnetic fields has raised considerable interest among astrophysicists over the past few decades. Cosmic magnetic fields may owe their present strength to dynamo amplification, but to initiate the process a seed field is required [124]. The latter may be primordial or else a consequence of a battery mechanism in protogalaxies or in the first stars. In a recent review, Coles [125] has discussed elaborately the role of primordial magnetic fields on the formation of large scale structure. He found a limit $B \le 3 \times 10^{-10}$ tesla. He considered the constraints imposed by nucleosynthesis, which could provide the extra fluctuation needed to reconcile theories of galaxy formation with observations of large scale structure. He deduced a more stringent limit of $B \le 2.4 \times 10^{-11}$ tesla if the time scale for dissipation is short compared with the expansion limit. However, so far there are no satisfactory physical mechanisms for the generation of $\boldsymbol{B}$.

In this chapter, we shall consider the energy loss of the photon when it passes through a Maxwell vacuum with non-zero torsion, which together with the spin density of the background space-time gives rise to a non-zero conductivity coefficient and hence to a non-zero photon mass on the cosmological scale. Again, a magnetic field can be shown to be associated with non-zero torsion, a field which has been identified with $\boldsymbol{B}^{(3)}$ in vacuo. So $\boldsymbol{B}^{(3)}$ should be considered as the primordial relict magnetic field in relativistic cosmology.

8.1 TORSION, SPACE-TIME DEFECT AND GAUGE PRINCIPLE

While formulating the general theory of relativity Einstein was not aware of intrinsic spin, and this was not initially incorporated into the structure of space-time. Cartan [126] showed that the interaction of intrinsic spin with geometry must lead to torsion, the antisymmetric part of the connection coefficient now being connected to metric spin density. In 1929, Fock and Ivanenko [127] introduced the concept of local frames to fit the Dirac electron into Einstein's general relativity. The action of the Lorentz or Poincaré group on the local frame becomes the real prototype of local gauge symmetry. For convenience we discuss the Fock-Ivanenko coefficients [127].

The vierbein or 16 component tetrad fields $t_a^\alpha(x)$ introduced by Weyl [128] bear to the metric tensor the same relation as Dirac's γ - matrices bear to the unit matrix, i.e., choose $t_a^\alpha (a = 1, ..., 4)$ such that

$$t_a^\alpha t_b^\beta g_{\alpha\beta} = \delta_{ab}, \tag{475a}$$

where

$$\delta_{ab} = \begin{bmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{bmatrix}. \tag{475b}$$

Also, $t_\alpha^a t_\beta^b \delta_{ab} = g_{\alpha\beta}$, t_α^a being the inverse matrix of t_a^α. If L_a^b is a space-time dependent Lorentz matrix, then the transformation law for tetrads can be written as

$$t_a^\alpha = L_a^b L_b^\alpha, \tag{476}$$

and

$$t = \det\left(t_a^\alpha\right) = \sqrt{-g}. \tag{477}$$

Thus we can interpret the Lorentz group as the group of rotations of the tetrad. If γ^a are the space-time dependent Dirac matrices, we have

$$\gamma^\alpha = t_a^\alpha \gamma^a, \tag{478}$$

where the γ^a are the usual constant Dirac matrices satisfying $1/2\left(\gamma^a\gamma^b + \gamma^b\gamma^a\right) = \delta^{ab}$. In contrast,

$$\frac{1}{2}\left(\gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha\right) = g^{\alpha\beta}. \tag{479}$$

Consider now a spinor $\psi(x)$ which has the transformation law $\psi = S\psi$ where S is the space-time dependent spinor representations of the Lorentz matrix L, given as $S^{-1}\gamma^a S = L_b^a \gamma^b$, when the tetrad is rotated as $t_a^\alpha = L_a^b t_b^\alpha$. While dealing with the gravitational interaction of spinor fields, the fundamental field is no longer the metric

tensor but the vierbein fields t_a^α, the metric tensor being defined in terms of a quadratic product $t_\alpha^a t_\beta^b \delta_{ab} = g_{\alpha\beta}$ of these fields. In the quantum theory the expectation value of this product represents the metric. The Dirac matrices have *themselves* become fields and are no longer constant as in a flat space-time background. With this conceptual structure we can identify the components

$$\Gamma_\mu = -\frac{1}{4}\omega_{\alpha\beta\mu}\gamma^\alpha\gamma^\beta, \tag{480}$$

as the Fock-Ivanenko coefficients [127]. Here, $\omega_{\alpha\beta\mu}$ is the generalized Christoffel-like asymmetric connection.

This definition is unique up to the addition of a vector multiple of the unit matrix, i.e., the Γ_μ are arbitrary up to a gauge,

$$\Gamma_\mu \rightarrow \Gamma_\mu + A_\mu I, \tag{481}$$

where I is the identity matrix. This arbitrariness enables the introduction of the electromagnetic four-potential. Owing to gauge invariance, we can define a new derivative,

$$\gamma'_{a;\lambda} = \gamma_{a;\lambda} + \left[\Gamma'_\mu, \gamma_a(x)\right], \tag{482}$$

where $\Gamma'_\lambda = \Gamma_\lambda + B_\lambda$. Under space-time transformations the fields $\gamma^a(x)$ and Γ_α transform as standard contravariant and covariant quantities. The commutator of two covariant derivatives of a spinor is defined as

$$\psi_{;\mu\nu} = \psi_{;\nu\mu} = \Gamma_{\mu\nu}\psi - Q_{\mu\nu}^\lambda \psi_{,\lambda}. \tag{483}$$

Here

$$\Gamma_{\mu\nu} = \delta_\mu\Gamma_\nu - \delta_\nu\Gamma_\mu + \left[\Gamma_\nu, \Gamma_\mu\right], \tag{484a}$$

and

$$Q'_{\mu\nu} = \Gamma_{\mu\nu}^\lambda - \Gamma_{\nu\mu}^\lambda \tag{484b}$$

define the torsion tensor. The curvature spinor $\Gamma_{\mu\nu}$ transforms under tetrad rotations as follows

$$\Gamma_{\mu\nu} = S\Gamma_{\mu\nu}S^{-1}, \tag{485}$$

the fields γ^{α} transforming as

$$\gamma^{\alpha} = S\gamma^{\alpha}S^{-1}. \tag{486}$$

This gives rise to a relation between the curvature tensor and curvature spinor,

$$\Gamma_{\nu\lambda} = -\frac{1}{4}R_{\mu\rho\nu\lambda}\gamma^{\mu}\gamma^{\rho}. \tag{487}$$

For a torsion free space we recover the usual expression for curvature. These concepts give rise to a possible unified description of electroweak, strong, and gravitational interactions in the framework of the curved space Dirac equation with background torsion and with an energy dependent fundamental length L_0 scaling as $G^{1/2}$ or E^{-1}.

Torsion in space-time, from which we can define a fundamental length, can also be constructed from another point of view. According to Sakharov [129], space-time can be considered as a deformable medium with elastic properties. From the geometrical description of crystalline dislocations or defects, it is known that torsion plays the role of defect density in the limit of dislocation having a continuous distribution. If we consider a small closed circuit and write

$$l^{\alpha} = \oint Q^{\alpha}_{\beta\gamma}dA^{\beta\gamma}, \tag{488}$$

where $dA^{\beta\gamma} = dx^{\beta} \times dx^{\gamma}$ is the area element enclosed by the loop and $Q^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{[\beta\gamma]}$ is the torsion associated with the connection $\Gamma^{\alpha}_{\beta\gamma}$; l^{α} representing the closure failure. So torsion has intrinsic geometrical meaning, i.e., it represents the failure of the loop to close in analogy with a crystal, l^{α} having the dimensions of length.

In 1978, Hojman *et al.* [130] discussed in detail the generalized gauge principle, minimal coupling, and torsion. Gauge invariance and minimal coupling have drawn much attention in discussing the Einstein-Cartan theory with non-symmetric connection coefficients. This theory has non-zero torsion tensor,

$$\Gamma^{\mu}_{\nu\alpha} = \Gamma^{\mu}_{\alpha\nu} - \Gamma^{\mu}_{\nu\alpha}, \tag{489}$$

and the usual definition of the electromagnetic field tensor $F_{\mu\nu}$ in general relativity is

$$F_{\mu\nu} := A_{\mu;\nu} - A_{\nu;\mu}. \tag{490}$$

Here, the semi-colon denotes covariant differentiation involving the connection coefficients $\Gamma^{\alpha}_{\mu\nu}$. According to Hehl *et al.* [131] this definition is incompatible with the Einstein-Cartan theory if coupling of electromagnetic field to torsion is kept invariant under the usual gauge transformation $A^{\prime}_{\mu} \to A_{\mu} + \Lambda_{,\mu}$, where Λ is a scalar function. The solution suggested by Hehl *et al.* is to dispense with minimal coupling by defining $F_{\mu\nu}$ as

$$F_{\mu\nu} := A_{\nu|\mu} - A_{\mu|\nu}. \tag{491}$$

The bar symbol denotes a covariant derivative using the Christophel symbol of the metric. By definition now, photons are decoupled from torsion. However, if we accept the general principle that spinning particles both generate and react to torsion, it is quite reasonable to expect that the photon should be coupled to torsion. Hehl *et al.* used Eq. (490) to define the field tensor of the massive vector (Proca) field. Hojman *et al.* [130] rightly pointed out that the definition can be used for the Proca field because the massive vector field is not invariant under gauge transformation. Evans and Vigier [1] clearly established that the Proca field will be invariant under the usual gauge transformation if we take

$$A_{\mu}A_{\mu} \to 0, \tag{492}$$

for $A_{\mu} \neq 0$, $m_{\gamma} \neq 0$. They found condition (492) to be consistent with the Lorentz gauge, but not with the Coulomb gauge with zero scalar potential. If A_{μ} be considered as the complex solution of d'Alembert's equation, then the Proca field has also been shown [1] to be consistent with the Dirac condition [133].

Again, Hojman *et al.* have shown that Eq. (490) may be used in the pure electromagnetic case in a theory involving torsion. They proposed a modified gauge invariance with minimal coupling between electromagnetism and the torsion, which is allowed to propagate and to be non-zero in vacuo. The usual form of gauge invariance is recovered when the torsion vanishes. Within this framework a gauge transformation of the electromagnetic field can be written as

$$\psi \to \psi' = \exp(iq\Lambda)\psi\,, \qquad A_\mu \to A'_\mu = \exp(\phi\Lambda_{,\mu})A_\mu\,, \tag{493}$$

and the minimal electromagnetic coupling is defined by the prescription

$$\psi_{,\mu} \to \psi'_{,\mu} = \psi_{,\mu} - iee^{-\phi}A_\mu\,. \tag{494}$$

The *heplon* field ϕ serves as a potential for torsion. In this way it is also possible to define a theory with propagating torsion, in contrast to the original Einstein-Cardan theory. According to the prescription by Hojman *et al.* the torsion can be written in terms of the scalar function ϕ as

$$\Gamma^{\alpha}_{\mu\nu} = \delta^{\alpha}_{\mu}\phi_{,\nu} - \delta^{\alpha}_{\mu}\phi_{,\nu}\,. \tag{495}$$

8.2 MASS OF PHOTON, SPACE-TIME DEFECT AND NON-ZERO CONDUCTIVITY COEFFICIENT

If we endow the vacuum with non-zero conductivity, i.e., $\sigma \neq 0$ in vacuo, Maxwell's equations should be written in the form

$$\nabla\cdot\boldsymbol{E} = 0, \quad \nabla\cdot\boldsymbol{B} = 0, \quad \nabla\times\boldsymbol{E} = \sigma\boldsymbol{E} + \epsilon_0\chi_e\frac{\partial\boldsymbol{E}}{\partial t}, \quad \nabla\times\boldsymbol{H} = -\mu_0\chi_m\frac{\partial\boldsymbol{H}}{\partial t}, \tag{496}$$

where ϵ_0 and μ_0 are vacuum permittivity and permeability respectively, and where χ_e and χ_m are the relative dielectric and permeability constants. Again,

$$\nabla\times\nabla\times\boldsymbol{E} = -\nabla^2\boldsymbol{E}, \tag{497}$$

which, together with Maxwell's equations, give

$$\nabla^2\boldsymbol{E} = -\frac{1}{c^2}\epsilon_0\chi_e\chi_m\mu_0\frac{\partial^2\boldsymbol{E}}{\partial t^2} + \sigma\mu_0\chi_m\frac{\partial\boldsymbol{E}}{\partial t}\,. \tag{498}$$

This equation is not invariant under $\hat{T}$, the motion reversal operator. The second term on the right hand side indicates that there will be dissipation of energy during the propagation of the photon. If we consider plane waves in the Z direction,

$$E_X = b\exp\left(i\omega\left(t - \frac{Z}{v}\right)\right)H_Y = b\left(\frac{\epsilon_0\chi_e}{\mu_0\chi_m}\right)^{1/2}e^{i\omega(t-Z/v)}, \tag{499}$$

After simple calculation [133] the group velocity v_g, and the phase velocity v_p of the photon can be written as

$$v_p = n\left(1 - \frac{1}{8}\frac{\sigma^2}{(\epsilon_0\chi_e)^2 n^4\omega^2}\right)^{1/2}, \tag{500}$$

and

$$v_g = n\left(1 + \frac{1}{4}\frac{\sigma^2}{(\epsilon_0\chi_e)^2 n^4\omega^2}\right)^{1/2}, \tag{501}$$

for $\sigma/\omega \to 0$.

For small refractive index, i.e., for $n < 1$, the group velocity will be less than the speed of light c in vacuo. Hence no superluminal transmission is permitted in this type of vacuum. So, for $\sigma = 0$ and $n = 1$, $v_p = v_g = c$. Now, taking the imaginary refractive index, we can obtain the mass formula of photons as

$$m_\gamma^2 = \hbar^2\omega^2(1 - n^2) - \frac{\sigma^2\hbar^2}{4n^2(\epsilon_0\chi_e)^2}\,. \tag{502}$$

Several authors [134] have calculated the effective photon mass from

$$m_\gamma = \frac{\hbar H}{\sqrt{2}n}, \tag{503}$$

where H is Hubble's constant. For small ω (but $\omega > \sigma$), Eq. (502) reduces to

$$m_\gamma \sim \frac{\sigma h}{2n(\epsilon_0 \chi_e)}, \qquad (504)$$

/hen the refractive index is complex. Comparing Eqs. (502) and (504) we get

$$\sigma \sim \frac{H}{\sqrt{2}}, \qquad (505)$$

/hich is a direct relation between the conductivity coefficient and Hubble's constant. 'herefore by measuring the displacement current in vacuo in the laboratory, Hubble's arameter can be estimated, allowing a new test of cosmological theory in the iboratory.

Using plane wave solutions we get the following dispersion relation in a covariant orm

$$\left(|\boldsymbol{k}|^2 - n^2 k_0^2\right) A^\mu(k) = \chi_m \mu_0 \left(g^{\mu\nu} + \frac{(n^2-1)}{n^2} \eta^\mu \eta^\nu \right) J_\nu(x), \qquad (506)$$

/here $J_\mu = (\sigma \boldsymbol{E}, 0)$, $A^\mu = (\boldsymbol{A}, i\phi/c)$, $K_\mu = (\boldsymbol{k}, k_0)$. Here η^μ is the unit time- ke vector, which is $(\boldsymbol{0}, 1)$ for a medium at rest. It is evident from Eq. (492) that

$$|\boldsymbol{A}| \neq 0 \text{ but } \phi = 0 \text{ for } \sigma \neq 0, \qquad (507)$$

/hich is the usual Coulomb gauge. Again, the condition $A_\mu A_\mu \to 0$ is not consistent /ith the usual Coulomb gauge if the scalar potential is zero. So it seems that the gauge rinciple has to be reinterpreted for $m_j \neq 0$, $\sigma \neq 0$ in vacuo.

Within the framework of the Einstein-de Broglie-Proca (EBP) theory, the ondition $A_\mu A_\mu \to 0$ is consistent with gauge theory [135] if we write

$$J_\mu = (\sigma \boldsymbol{E}, J_0) \text{ with } J_0 = \boldsymbol{B}^{(0)}. \qquad (508)$$

Iere $\boldsymbol{B}^{(0)}$ is the magnetic flux density related to the photon as mentioned in earlier hapters. As no electrostatic field can be generated [1,2] out of $\boldsymbol{B}^{(3)}$, $\boldsymbol{E}$ must be roduced by magnetic induction. In the conventional framework of Maxwell's equation

$$\boldsymbol{J} = \sigma(\boldsymbol{E} + \boldsymbol{E}'), \qquad (509)$$

where $\boldsymbol{E}$ is the field derivable from a potential and where $\boldsymbol{E}'$ consists of all non-electrostatic fields. In the EBP framework $\boldsymbol{E} = \boldsymbol{0}$, $\boldsymbol{J} = \sigma\boldsymbol{E}$. This indicates that the current and field distributions are entirely defined by the medium. The presence of this type of non-conservative field as produced by $\boldsymbol{B}^{(3)}$ is responsible for the loss of energy of the photon when it propagates in this type of vacuum. In regard to gauge invariance, the Lagrangian [50],

$$L = -\frac{1}{4}\left(F_{\mu\nu}F_{\mu\nu} + m_\gamma^2 A_\mu A_\mu\right), \qquad (510)$$

with $m_\gamma \neq 0$, and $F_{\mu\nu} = \partial A_\mu/\partial x_\nu - \partial A_\nu/\partial x_\mu$ will be invariant to

$$A_\mu \to A_\mu + \frac{1}{c}\frac{\partial \Lambda}{\partial x_\mu} \qquad (511)$$

if and only if

$$m_\gamma^2 A_\mu A_\mu = 0. \qquad (512)$$

If $m_\gamma \neq 0$ then

$$A_\mu A_\mu \to 0, \quad A_\mu \neq 0, \qquad (513)$$

is the only alternative solution. Conventionally, it is asserted that the invariance of L under Eq. (511) means that $m_\gamma = 0$, and EBP theory is not consistent with the massless photon.

However, in our extended framework of EBP theory, with $\sigma \neq 0$ we can write the four-current $J_\mu = (\sigma \boldsymbol{E}, J_0)$ as discussed above, the dispersion relation (506) clearly indicating

$$\boldsymbol{A} \neq \boldsymbol{0}, \quad A_0 \sim J_0, \qquad (514)$$

for $\sigma \neq 0$. Roy and Evans [50] have recently suggested a novel gauge theory which reconciles non-zero photon mass with

$$A_0 \xrightarrow[FAPP]{} |A|, \tag{515}$$

o that

$$A_\mu A_\mu \to 0. \tag{516}$$

Iere FAPP denotes for all practical purposes as mentioned in earlier chapters. Equation 515) is a limiting form of and an excellent approximation to the condition introduced y Dirac [132], $A_\mu A_\mu$ = constant, Eq. (516). In extended EBP theory, with $\sigma \neq 0$, it s evident from the dispersion relation (506) that

$$|A_0| \to |A| \text{ for } |n^2| \sim \frac{1}{3}. \tag{517}$$

'or $|n^2| = 1/3$, the metric tensor $(g^{\mu\nu} + \frac{n^2-1}{n^2}\eta^\mu\eta^\nu)$ becomes positive definite ince $m_\gamma \neq 0$ for complex refractive index.

If torsion is considered to be non-zero, the modified Maxwell equations can be vritten as

$$F^{\mu\nu}_{\|\nu} = e^{-\phi}J^\mu - F^{\mu\nu}\phi_{,\nu}, \tag{518}$$

vhere $F^{\mu\nu}_{\|\nu}$ is covariant differentiation without torsion. This equation can be written as

$$\nabla\times \boldsymbol{H} + \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} = e^{-\phi}\boldsymbol{J} - \left(\boldsymbol{H}\times\nabla\phi + \frac{\boldsymbol{E}}{c}\frac{\partial\phi}{\partial t}\right). \tag{519}$$

et us consider the plane wave in the Z direction as taken previously, the Maxwell quation (498) takes the form

$$\frac{\partial^2 E_X}{\partial t^2} = \frac{1}{c^2}\frac{\partial^2 E_X}{\partial t^2} + \frac{1}{c}\frac{\partial\phi}{\partial t} + \frac{1}{c^2}\frac{\partial E_X}{\partial t}\frac{\partial\phi}{\partial t} + \frac{1}{c}\frac{\partial H_Y}{\partial t}\frac{\partial}{\partial t}\left(\frac{\partial\phi}{\partial t}\right) + \frac{1}{c^2}E_X\frac{\partial^2\phi}{\partial t^2}, \tag{520}$$

vith $\nabla\times(\nabla\times \boldsymbol{E}) = -\nabla^2\boldsymbol{E}$. Considering E_X and H_Y as in Eq. (499), and comparing t with Eq. (498) we get

$$\sigma \sim \frac{\partial\phi}{\partial Z} + \frac{1}{c}\frac{\partial\phi}{\partial t} + \frac{\partial}{\partial t}\left(\frac{\partial\phi}{\partial t}\right), \tag{521}$$

where $\partial^2\phi/\partial t^2 = 0$ is assumed, i.e., $\partial\phi/\partial t = L$, a constant. Again, as σ is constant,

$$\frac{\partial\phi}{\partial Z} = K, \text{ a constant}, \tag{522}$$

so that $\sigma = K + L$ (constant). Let $K = L$, i.e.,

$$\frac{\partial\phi}{\partial Z} = \frac{1}{c}\frac{\partial\phi}{\partial t}, \tag{523}$$

then the conductivity coefficient σ can be written as

$$\sigma \sim \frac{\partial\phi}{\partial Z} = \text{constant}, \tag{524}$$

i.e.,

$$\sigma_{vac} = \frac{\epsilon_0\chi_e}{\mu_0\chi_m}\frac{\partial\phi}{\partial Z}. \tag{525}$$

Taking the average over the surface,

$$\sigma^{av}_{vac} = \frac{\epsilon_0\chi_e}{\mu_0\chi_m}\frac{\oint\frac{\partial\phi}{\partial Z}dx\times dy}{\oint dx\times dy}. \tag{526}$$

As a result, the vacuum conductivity concomitant with photon mass may arise from space-time defect or from the torsion generated by defect in space-time. In the general case it is given by

$$\sigma_{vac}^{av} = \frac{\epsilon_0\chi_e}{\mu_0\chi_m} \frac{\Sigma_\alpha \oint \left(\delta_\nu^\alpha \phi_{,\mu} - \delta_\mu^\alpha \phi_{,\nu}\right) dx^\mu dx^\nu}{\sum_{\mu\nu} \oint dx^\mu dx^\nu}, \tag{527}$$

or

$$\sigma_{vac}^{av} = \frac{\epsilon_0\chi_e}{\mu_0\chi_m} \frac{\Sigma_\alpha l^\alpha}{\Sigma \int dx^\mu dx^\nu}. \tag{528}$$

8.3 NON-ZERO PHOTON MASS AND THE PRIMORDIAL MAGNETIC FIELD

From the previous section it is clear that if we consider the propagation of a photon through a vacuum with non-zero torsion, we obtain a relation between the conductivity coefficient and torsion

$$\sigma_c = Q = \frac{4\pi}{c^2} G \sigma_s \epsilon_0, \tag{529}$$

where the torsion $Q = \partial_\mu \phi$; ϕ being the torsion potential, and σ_s is the background spin density. G is the gravitational constant. De Sabbata and Gasparini [136] have found a relation between the torsion vector Q and the magnetic field B which is generated through the spin density σ_s,

$$B = \left(\frac{8\pi}{3c}\right)(2\alpha G)^{1/2} \sigma_s. \tag{530}$$

Comparing the following relations

$$m_\gamma \sim \frac{\sigma_c \hbar}{4\pi\epsilon_0 c^2}, \tag{531}$$

$$\sigma_c = \frac{4\pi}{c^2} G \sigma_s \epsilon_0, \tag{532}$$

we obtain

$$m_\gamma \sim \frac{hG\sigma_s}{c^4}. \tag{533}$$

Therefore the spin density for the Hubble universe can be estimated [136] as

$$\sigma_s \sim 3 \times 10^7. \tag{534}$$

Using this in Eq. (533) we find a limit on photon mass,

$$m_\gamma \lesssim 8 \times 10^{-64} \text{ kgm}. \tag{535}$$

Data from the Pioneer 10 flyby of Jupiter gave a limit [138], in comparison, of

$$m_\gamma \lesssim 8 \times 10^{-52} \text{ kgm}. \tag{536}$$

Using the value of σ_s in Eq. (530) we obtain

$$B \lesssim 2.9 \times 10^{-11} \text{ T}. \tag{537}$$

This magnetic field can be identified with the relict $B^{(3)}$ field for the following reasons. The photon loses its energy during its propagation through the vacuum with non-zero conductivity coefficient of the form $\sigma_s \sim \partial\phi / \partial Z$, the gradient of the torsion potential. So, the magnetic field B_Z can be written as $\sim \partial\phi / \partial Z$. For magnetic or electric fields, it is known that E^2 or B^2 is proportional to the energy density of the field. The analogy between Q and B already established suggests that we can interpret Q as the energy density of the torsion field. In the case of propagating torsion

$$Q = \partial_\mu \phi, \tag{538a}$$

and

$$Q^2 = \partial_\mu \phi \partial^\mu \phi , \tag{538b}$$

indicating that

$$|\boldsymbol{B}|^2 \sim |\frac{\partial \phi}{\partial X}|^2 + |\frac{\partial \phi}{\partial Y}|^2 + |\frac{\partial \phi}{\partial Z}|^2 - \frac{1}{c^2}|\frac{\partial \phi}{\partial t}|^2 = |\frac{\partial \phi}{\partial X}|^2 + |\frac{\partial \phi}{\partial Y}|^2, \tag{539}$$

since $\partial\phi/\partial Z = (1/c)(\partial\phi/\partial t)$.

As a result, we can write $|\boldsymbol{B_Z}| = B^{(0)}$ to be independent of phase and of time, since $\partial\phi/\partial t$ = constant. Here, the magnetic four-vector has been defined in terms of a torsion four-vector. In the circular basis,

$$B_\mu = \left(\boldsymbol{B}^{(1)}, \boldsymbol{B}^{(2)}, \boldsymbol{B}^{(3)}, iB^{(0)}\right), \tag{540}$$

and

$$E_\mu = \left(\boldsymbol{E}^{(1)}, \boldsymbol{E}^{(2)}, \boldsymbol{E}^{(3)}, iE^{(0)}\right), \tag{541}$$

are the magnetic and electric components of the plane wave in vacuo. It can be shown that $B^{(0)}$ and $E^{(0)}$ are the time like components of B_μ and E_μ respectively [4]. In this view, $E_\mu E_\mu$ and $B_\mu B_\mu$ are Lorentz invariants and contribute to the electromagnetic energy density in vacuo. The energy density of the field can be written as

$$U = \frac{1}{2}\left(\epsilon_0 E_\mu E_\mu + \frac{1}{\mu_0} B_\mu B_\mu\right). \tag{542}$$

Again,

$$E_\mu E_\mu = E^{(1)2} + E^{(2)2} + E^{(3)2} - E^{(0)2}, \tag{543}$$

and

$$B_\mu B_\mu = B^{(1)2} + B^{(2)2} + B^{(3)2} - B^{(0)2}. \tag{544}$$

Using the relation

$$\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} = iB^{(0)}\boldsymbol{B}^{(3)*}, \tag{545}$$

we obtain in the Maxwellian limit,

$$\boldsymbol{B}^{(3)} = \frac{1}{iB^{(0)}}\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} = B^{(0)}\boldsymbol{k}, \tag{546}$$

where $\boldsymbol{k}$ is a unit axial vector in the propagation axis of the plane wave in vacuo. Similarly it can be shown that

$$i\boldsymbol{E}^{(3)} = iE^{(0)}\boldsymbol{k}. \tag{547}$$

Using these expressions for $\boldsymbol{B}^{(3)}$ and $i\boldsymbol{E}^{(3)}$ we obtain [4]

$$\left(i\boldsymbol{E}^{(3)}\right)\cdot\left(i\boldsymbol{E}^{(3)}\right)^* - E^{(0)2} = 0 = \boldsymbol{B}^{(3)}\cdot\boldsymbol{B}^{(3)*} - B^{(0)2}. \tag{548}$$

Therefore,

$$B_\mu B_\mu = B^{(1)2} + B^{(2)2}, \qquad E_\mu E_\mu = E^{(1)2} + E^{(2)2}, \tag{549}$$

which is precisely the result indicated by the conventionally interpreted Planck law, where there are only two transverse degrees of freedom. Finite photon mass, however, implies three degrees of space polarization, two transverse and one longitudinal. In the above four-vector representation the longitudinal contribution is cancelled precisely by the time like-contribution. This occurs both for the real $\boldsymbol{B}^{(3)}$ and for the imaginary $-i\boldsymbol{E}^{(3)}$. The latter is unphysical at first order, but its square modulus is real. The compatibility of the four-vector representation with the field four-tensor $F_{\mu\nu}$ has been discussed in Chap. 11 of Vol. 1 [1].

The magnetic components $\boldsymbol{B}^{(3)}$ and $B^{(0)}$, the longitudinal and time-like components of B_μ [1], are time and phase independent, which means that

$$\frac{\partial \boldsymbol{B}^{(3)}}{\partial t} = 0, \qquad \frac{\partial B^{(0)}}{\partial t} = 0. \tag{550}$$

The magnetic field associated with vacuum conductivity and generated by torsion has similar properties to those of $\boldsymbol{B}^{(3)}$ and $B^{(0)}$,

$$B_Z \sim \frac{\partial\phi}{\partial Z} = \frac{1}{c}\frac{\partial\phi}{\partial t}, \qquad \frac{\partial}{\partial t}\left(\frac{\partial\phi}{\partial Z}\right) = 0, \tag{551}$$

$$\frac{\partial^2\phi}{\partial t^2} = 0. \tag{552}$$

o B_Z and $B^{(0)} \sim (1/c)(\partial\phi/\partial t)$ do not contribute to the field energy density, because $)^2 = \partial_\mu\phi\partial^\mu\phi \sim B^2$. This implies clearly that we can identify the field $B_Z \sim \partial\phi/\partial Z$ ith $B^{(3)}$ and $(1/c)(\partial\phi/\partial t)$ with $B^{(0)}$. Therefore the relict magnetic field $B^{(3)}$ can be ›nsidered as a candidate for the seed field in large scale structure formation. onsidering the photon mass limit $m_\gamma \sim 10^{-52}$ kgm we have,

$$B^{(3)} = B_Z \sim 10^{-7}\ \text{Gauss} = 10^{-11}\ \text{T}, \tag{553}$$

hich agrees well with the observational limit $\sim 10^{-10}$ T on the primordial magnetic eld.

4 COSMOGONIC IMPLICATIONS OF THE SEED MAGNETIC FIELD

Cosmic magnetic fields are ubiquitous within our own and other galaxies. They e essential for the process of synchrotron radiation and may be strong enough for agnetic stresses to be dynamically important; even a very weak field can inhibit ermal conductivity and associated diffusion processes. The issue of how magnetic elds can originate and evolve is therefore of interest in many astrophysical aspects, ırticularly for cosmogony. A field, even if very weak, can be amplified by a dynamo echanism in a medium with large scale internal motions. This can always happen ithin galactic discs, inside individual stars, and in other contexts. All dynamo echanisms rely, however, on a seed field: a non-zero field must initiate the process, herwise the dynamo has nothing to feed on. Several alternative theories [124] have en proposed to explain the presence of the seed field. In 1970, Harrison [139] oposed an interesting mechanism to create a weak field. A non-zero vorticity in the imordial perturbation was considered in this framework. The photons of the ckground thermal radiation (2.8 K) would be strongly coupled to the electrons via ıompson scattering, but less strongly so to the ions. Angular momentum conservation the photon-electron components implies that the angular velocity of a co-moving eddy lls off as R^{-1} during the expansion, whereas the angular velocities in the ionic mponent (where rest mass dominates and $\rho \propto R^{-3}$) would go as R^{-2}, having moved

independently of the other components. This difference tends to build up a circuit in each eddy. On galactic scales, the build up is so slow that the resultant fields would be only about 10^{-22} tesla. These might never be adequate seeds for a subsequent dynamo.

This idea of *primordial vorticity*, or *whirls* has lost favor with most cosmologists, largely because they decay during cosmic expansion, whereas irrotational density perturbations (arising from initial curvature fluctuations) would grow. The initial conditions required in order that rotational perturbations be still significant in the post-recombination universe seem rather implausible. While Harrison [139] proposed a mechanism for primordial magnetic fields in the pre-recombination plasma, Mishustin and Ruzmaikin [140] investigated the possible occurrence of weak seed magnetic fields during the period after the recombination of the residual plasma. However, the field generated through their mechanism is too weak, and of the order 10^{-22} T. The physical explanation for the generation of such a weak field is not at all clear in the framework of the theory of Mishustin *et al.* Moreover, a still weaker field, that affected neither the behavior of pregalactic gas nor the local physics at early epochs (i.e., nucleosynthesis) might nonetheless suffice to initiate a dynamo. A seed field of about 10^{-22} T could grow to the presently observed galactic strength (of the order 10^{-10} T) provided that the e-folding time were no more than a few times 10^8 years, comparable with the galactic rotation period. In a young disc galaxy, however, the magnetic fields would not have had enough time to grow to a dynamically significant level.

In contrast, the magnetic field generated due to non-zero torsion can be shown to be of the order 10^{-11} T, and we have shown that this is the $B^{(3)}$ field. Therefore a primordial $B^{(3)}$ field can be envisaged in the relict microwave background radiation. The forthcoming COBRA/SAMBA project of the European Space Mission [141] will measure the $B^{(3)}$ field through the ratio of longitudinal and transverse components of the microwave background radiation.

Chapter 9. $\boldsymbol{B}^{(3)}$, Experimental Status and Prospects

Chapter 7 of Vol. 1 [1] discussed the experimental evidence for the existence of the $\boldsymbol{B}^{(3)}$ field, and a revised version of the experiment of Deschamps *et al.* [17c] to provide evidence for its existence through the expected square root intensity profile of inverse Faraday induction. Currently, the strongest evidence for the existence of the $\boldsymbol{B}^{(3)}$ field remains the inverse Faraday effect (IFE), and this has already been discussed in previous volumes [1,2]. Conclusive evidence for the existence of the $\boldsymbol{B}^{(3)}$ field at first order would follow from the predicted dependence of magnetization on the square root of the power density of the incident circularly polarized radiation. In this chapter we propose a related experiment which appears to be easier to carry out. Chapter 2 shows that the effective magnetic flux density of the $\boldsymbol{B}^{(3)}$ field when interacting with one proton through the Dirac equation should range from nano to kilo tesla. To date, most attempts to detect effects due to $\boldsymbol{B}^{(3)}$ have been conducted in the optical region, and have provided conclusive evidence for its existence through the inverse Faraday effect. However, this is a second order effect in $iB^{(0)}\boldsymbol{B}^{(3)*}$. Conclusive evidence for $\boldsymbol{B}^{(3)}$ acting at first order is still to be obtained. Optical NMR, discussed in Chap. 2, shows that nuclear resonance lines can be shifted by circularly polarized visible radiation, and that the direction of the shift is very probably reversed with the sense of polarization. This phenomenon led to the discovery of the $\boldsymbol{B}^{(3)}$ field [7], and Chap. 2 describes how it reproduces the available ONMR data through its interaction with the third Pauli spinor. At visible frequencies this is a small second order effect as seen by Warren *et al.* [20] in an exhaustive and careful series of measurements.

The phenomenon of magnetization by circularly polarized light in non-absorbing media [17—21] was first demonstrated experimentally by van der Ziel *et al.* [17a] in a classic paper, following a theoretical prediction by Pershan [17a]. In the received view, the physical mechanism which gives rise to such magnetization is essentially different from magnetization produced in media with optical absorption. The latter is produced by polarized electrons in the medium by the transfer of angular momentum of the electromagnetic radiation to electrons via spin-orbit coupling. In the case of a non-absorbing media, this mechanism is absent. Some physical mechanisms which can produce this type of magnetization include the non-linear interaction of the light with the medium or the possibility that electromagnetic radiation itself has a longitudinal agnetic field. The $B^{(3)}$ field is now recognized to be a phase free example of a novel ass of longitudinal solutions in vacuo of the field equations of electrodynamics [2]. hese equations are in general non-Abelian, but when linearized become Maxwell's juations.

Some particular experimental problems arise in the study of optomagnetic effects, hich for diamagnetics and paramagnetics are small in magnitude and difficult to detect cperimentally [17—21]. The received view can lead, furthermore, to gross nderestimations such as that of Buckingham and Parlett [142], who claim, contrary to cperimental evidence [20], that nuclei in a fluid irradiated by a circularly polarized ght beam produces NMR shifts no less than nine orders of magnitude smaller than ose actually observed. The reason for this discrepancy is clearly the failure of these ithors to accept that the conjugate product $A^{(1)} \times A^{(2)}$ interacts with the third Pauli inor as described in Chaps. 1 and 2. This is a dramatic illustration of the fact that ceptance of $B^{(3)}$ leads to adequate agreement between data and theory as described Chap. 2, and that the received view does not. Surprisingly, Buckingham and Parlett 42] ignore the available data of Warren *et al.* [20] in an attempt to maintain a onservative point of view which in the last analysis is subjective assertion.

Nonetheless, improvement in the sensitivity of measurement of very weak magnetic elds is always to be sought, and the various available methods are reviewed by Lenz 43]. Currently, NMR quantum magnetometers are accurate typically to within 002% with a sensitivity limit of about 10^{-15}T in material with appropriate nuclear spin. ptical pumping in quantum magnetometry [144] is sensitive to 10^{-12}T/(Hz)$^{1/2}$ but limited ecause it requires spectral absorption. The superconducting quantum interference evice (SQUID) [145,146] is well known to be sensitive to about 10^{-14}T/(Hz)$^{1/2}$ at liquid elium temperatures. The sensitivity can be improved by modulation using elliptically olarized light, decreasing the temperature to about 10^{-4}K, or using multiple SQUIDS 47]. Another sensitive device is the flux superconducting differential transformer.

Induction magnetometers (IM) are based on Faraday induction. Various methods an be used to change the magnetic flux, for example modulating the polarization or the itensity of the light and using a synchronized lock-in amplifier. The sensitivity of IM limited by thermal noise in the resistance of the coil, and the minimum detectible agnetic field is about 10^{-12}T/(Hz)$^{1/2}$ at room temperature. The superconducting mtovolt preamplifier [148] can be used to minimize preamplifier noise. Noise is irther minimized by using a superconducting preamplifier circuit [149], which is apable of reducing ohmic noise by up to a factor of 10^5. The use of resonance reduces e incidence of preamplifier noise. Using the optimal coil and modulating frequency roduces a sensitivity of better than 10^{-16}T/(Hz)$^{1/2}$ at liquid helium temperatures.

Braginsky [150] has shown that the sensitivity of measurement can be improved ith a lightly damped mechanical oscillator such as a torque or mechanical pendulum, ith a quality factor of greater than 10^8. We are not aware of the use of such scillators for the measurement of magnetic fields. Earlier torque balances were applied the measurement of magnetic anisotropy with great sensitivity [151].

The torque resonance magnetometer (TRM) is a device wherein the sample forms one part of a mechanical oscillator. The sample is placed in an intense external field of for example 100 T so that the magnetic moment of the sample is perpendicular to the external magnetic field. If circularly polarized light creates a magnetic moment in the sample the latter will experience an alternating mechanical torque equal to the vector product of magnetic moment and external magnetic field. This method has a sensitivity of less than 10^{-16}T/(Hz)$^{1/2}$ and can also be adapted for use with an electrical superconductor magnetometer (SCM). In analogy with SCM it is very useful to apply the modulation principle of measurement in TRM. An optical interferometric method, for example, can be used to overcome the problem of sensitivity in the torque oscillator amplitude.

The Faraday effect is an indirect method of measuring the optomagnetic effect (OME) in fiber-optic magnetometers [152]. The induced magnetic field in OME can be detected through the rotation of the plane of polarization of a probe laser, and this method has been applied successfully for the highly sensitive optical registration of an EPR spectrum [153]. This method has been used for the detection of plane rotations as small as 0.001 arc seconds, and has also been used for the study of the IFE by reflection from a semiconductor [154]. A high intensity femtosecond laser used in this work leads to a useful surface scanning technique. For the investigation of OME however the method suffers from the drawback that the effective value of polarization-plane rotation also involves third order nonlinearities in the form of a four-wave mixing phenomenon. This effect has great value in the resonance condition when the probe and pump frequencies and also their sum or difference are the same as the energy levels of the sample [24].

For nonabsorbing media there is an unambiguous interrelation between magneto-optical and optomagnetic effects, and the constants are equal [155] in microscopic theory. The first example to be observed of such a relation is that between the Faraday effect and the IFE: the $\hat{T}$ and $\hat{P}$ symmetries of the circularly polarized field and $B^{(3)}$ are identical, and their effect on media is identical in the absence of absorption. In the presence of absorption [42] the IFE can still be described in terms of the conjugate product of vacuum electromagnetism, and therefore in terms of $B^{(3)}$. However, an ordinary magnetic field (as distinct from the radiated $B^{(3)}$) is obviously not absorbed by the sample at any frequency.

An ordinary magnetic field also produces magnetochiral birefringence (MB) [156], or nonreciprocal magneto-optic linear birefringence. The corresponding optomagnetic effect is inverse magnetochiral birefringence (IMB) [42], which is magnetization by a laser in a chiral medium. IMB may occur theoretically with linearly polarized light and its sign changes with the direction of light propagation [42]. It is further reviewed by Stedman [5b] and by Evans [5b]. Using $B^{(3)}$ theory, it becomes the optically produced equivalent of magnetic circular dichroism or optical rotatory dispersion, with the static magnetic field replaced by circularly polarized electromagnetic radiation, and relies essentially on the fact that in a chiral medium the parity difference between a magnetic and electric field becomes irrelevant [5b]. In chiral liquids the magnetization from IMB does not depend on the light polarization, and its direction is parallel to that of beam propagation, reversing sign for enantiomers. In crystals the effect depends on space group symmetry [157] and in general is about a hundred to a thousand times weaker than the IFE.

These volumes have shown that electromagnetic radiation includes the $\boldsymbol{B}^{(3)}$ field, now known to be a phase free component of a class of longitudinal solutions [158,159] in vacuo of the electromagnetic field equations. The $\boldsymbol{B}^{(3)}$ component propagates in vacuo and is a radiated field as described in earlier chapters. Its interaction with one electron is governed by the conjugate product $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$, which can be used either in the Dirac or Hamilton-Jacobi equations of motion, leading to expressions such as Eq. (403) of Vol. 1 and in Appendix F of this volume. The I dependence at visible frequencies gives way to an $I^{1/2}$ half dependence at radio frequencies under the right conditions.

The IFE has been studied in plasmas [17c] and in the liquid and solid states [17—21]. In the original experiment of Deschamps *et al.* [17c] microsecond, megawatt pulses of microwave radiation at 3 GHz were used to ionize a low pressure gas. In the received view [17a] the electrons so produced are driven into circular orbits by the electric field of the circularly polarized radiation and in consequence produce an axial magnetic field. In the new $\boldsymbol{B}^{(3)}$ theory the effect is described in Appendix F. In both cases the induced magnetic field is detected through induction in a pick-up coil surrounding the ionized gas. The intensity of the induced field and its dependence on the polarization of the incoming radiation were measured using two polarizers [17c]. Changing the angle between them produced a cosinal intensity variation. Over the range of intensity investigated, the inverse Faraday induction was found to be linear in the intensity I of the electromagnetic radiation.

The theoretically expected square root intensity dependence due to $\boldsymbol{B}^{(3)}$ acting at first order will start to dominate under the conditions discussed in Chap. 12 of Vol. 1, and these can be achieved by reducing the frequency of the radiation to the MHz range, for example 0.3 MHz, at which frequency the linear dependence curves off. In such an experiment, care should be taken to eliminate any effects which could produce an artifactual non-linear I profile. Chiang [100] has discussed the fact that ions would be accelerated in the opposite direction in experiments of this type, and in consequence the sample must be an electron beam. In plasma, artifactual ionic effects interfere to the order of 10% for an incident field amplitude $\boldsymbol{E}$ of the order of 2.89 x 10^5 volts per meter. This corresponds to an intensity of about 2.5 x 10^8 W/m^2, of the order of that needed to see the real, artifact free, square root intensity dependence. If an electron beam is used these artifacts are eliminated and furthermore, for a given incident field and induction coil, the induced voltage will be increased since many more electrons will be contributing to it. For example, if an electron gun were used to deliver say one amp cm^{-2}, and if the beam cross sectional area were 1 cm^2, then the total number of electrons exposed in a given pulse would be of the order 10^{13}, a hundred times greater than the electron density of the Deschamps experiment.

An alternative scheme would be to use continuous wave (cw) circularly polarized microwave or radio frequency radiation with a pulsed electron beam. Such an arrangement is commonly used in an undergraduate experiment to determine the charge to mass ratio of the electron. Helmholtz coils surround a glass bulb containing an electron gun and a low pressure gas. The electrons are accelerated perpendicular to the field produced by the Helmholtz coils and under the right conditions move in circular orbits of a few centimeters radius. Such apparatus can be used to detect and measure $\boldsymbol{B}^{(3)}$ by exposing it to circularly polarized cw radiation, pulsing the electron gun and using the Helmholtz coils as pick up coils to measure the induced voltage. The radio frequency radiation is circularly polarized using a log spiral or helical antenna. In this experiment, constant beam power is maintained and the frequency varied using a device such as that produced by Antenna Research. A vacuum system containing a high current electron gun is used (a few amps per square centimeter). The electron gun is pulsed and the pick up coil measures $\boldsymbol{B}^{(3)}$ induced in the sample as given in Appendix F. Theoretically, modest beam powers of the order of 40 W m^{-2} would be sufficient to see an induced voltage provided the electron density N were adequate.

9.1 CONCLUSIONS

A survey has been given of a variety of techniques for investigating the magnetizing effects of circularly polarized electromagnetic radiation, focussing on the experimental prospects for detecting and measuring $\boldsymbol{B}^{(3)}$ through its expected square root I profile.

Appendix A. Circular Basis for Pauli Spinors

In the text the vector potential, $\boldsymbol{A}^{(1)} = \boldsymbol{A}^{(2)*}$, of the electromagnetic wave has been set up in the circular basis defined by Eq. (32). In this appendix, the Pauli spinors are defined in the same basis in order to introduce $\boldsymbol{A}^{(1)}$ and $\boldsymbol{A}^{(2)}$ consistently into the Dirac equation consistently. In the basis ((1), (2), (3)) the Pauli spinors, by analogy with Eq. (32), are

$$\boldsymbol{\sigma}^{(1)} = \boldsymbol{\sigma}^{(2)*} = \frac{1}{\sqrt{2}}\left(\boldsymbol{\sigma}_X - i\boldsymbol{\sigma}_Y\right), \qquad \boldsymbol{\sigma}^{(3)} = \boldsymbol{\sigma}_Z \cdot \tag{A1}$$

In the usual Cartesian basis, the spinors are well known [1,2,22—25] to be

$$\boldsymbol{\sigma}_X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \boldsymbol{\sigma}_Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \boldsymbol{\sigma}_Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{A2}$$

and Eqs. (A1) and (A2) lead directly to Eq. (39) in the text. It can be checked by direct evaluation that the Pauli spinors in ((1), (2), (3)) have SU(2) symmetry, and are therefore representations of three dimensional space. They are real, and therefore obey the following rules for Hermitian transposition [2,22—25],

$$\begin{gathered}\boldsymbol{\sigma}^{(1)} = \boldsymbol{\sigma}^{(2)+} = \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{pmatrix}, \qquad \boldsymbol{\sigma}^{(2)} = \boldsymbol{\sigma}^{(1)+} = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}, \\ \boldsymbol{\sigma}^{(3)} = \boldsymbol{\sigma}^{(3)+} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{gathered} \tag{A3}$$

The $\boldsymbol{\sigma}^{(3)}$ spinor is the same as the $\boldsymbol{\sigma}_Z$ spinor in the usual Cartesian representation, which also has SU(2) symmetry [2,22—25]. The representations of the Pauli spinors in the Cartesian and circular bases are therefore equivalent, but the latter basis is convenient for the description of circular polarization [1,2].

Appendix B. Product Algebra of Spinors and Unit Vectors in the Circular Basis

In order to evaluate products such as $\boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{A}^{(1)}$ appearing in the Dirac equation it is necessary to define the product algebra of Pauli spinors and unit vectors in the circular basis ((1), (2), (3)).

The dot product in this basis is equivalent to the Cartesian dot product, because the two bases are equivalent representations of three dimensional space, and so,

$$\boldsymbol{\sigma} \cdot \boldsymbol{e} = \boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{e}^{(2)} + \boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{e}^{(1)} + \boldsymbol{\sigma}^{(3)} \cdot \boldsymbol{e}^{(3)} = \boldsymbol{\sigma}_X \cdot \boldsymbol{i} + \boldsymbol{\sigma}_Y \cdot \boldsymbol{j} + \boldsymbol{\sigma}_Z \cdot \boldsymbol{k} = \begin{pmatrix} 1 & 1-i \\ 1+i & -1 \end{pmatrix}, \tag{B1}$$

giving the sum

$$\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{e}^{(2)} + \boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{e}^{(1)} = \begin{pmatrix} 0 & 1-i \\ 1+i & 0 \end{pmatrix}. \tag{B2}$$

The individual terms in this sum are

$$\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{e}^{(2)} = \frac{1}{\sqrt{2}} \left(\boldsymbol{\sigma}_X - i\boldsymbol{\sigma}_Y \right) \cdot \frac{1}{\sqrt{2}} (\boldsymbol{i} + i\boldsymbol{j}) = \begin{pmatrix} 0 & 0 \\ 1+i & 0 \end{pmatrix}, \tag{B3}$$

and

$$\boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{e}^{(1)} = \begin{pmatrix} 0 & 1-i \\ 0 & 0 \end{pmatrix}, \tag{B4}$$

and are complex. The Hermitian transpose (the matrix transpose with simultaneous complex conjugation of each element) is, for each term,

$$(\sigma^{(1)} \cdot e^{(2)})^{+} = \sigma^{(2)} \cdot e^{(1)}, \quad (\sigma^{(2)} \cdot e^{(1)})^{+} = \sigma^{(1)} \cdot e^{(2)}. \tag{B5}$$

The product of terms is

$$(\sigma^{(1)} \cdot e^{(2)})(\sigma^{(2)} \cdot e^{(1)}) = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}. \tag{B6}$$

Using these rules, it can be checked directly that the product is

$$(\sigma^{(1)} \cdot e^{(2)})(\sigma^{(2)} \cdot e^{(1)}) = e^{(1)} \cdot e^{(2)} + i\sigma^{(3)} \cdot e^{(1)} \times e^{(2)}, \tag{B7}$$

which is Eq. (25a) of the text, and is the correct spinor algebra in the basis ((1), (2), (3)). It is of key importance because it introduces the conjugate product $e^{(1)} \times e^{(2)}$ multiplied by the spinor $\sigma^{(3)}$. This term represents mathematically the interaction of the conjugate product of a circularly polarized electromagnetic field with the spinor $\sigma^{(3)}$, and thus with the half integral spin of a fermion such as a proton or electron.

Finally in this appendix, it is shown that

$$(\sigma^{(1)} \cdot e^{(2)} v)^{+} = v^{+}(\sigma^{(1)} \cdot e^{(2)})^{+} = v^{+}\sigma^{(2)} \cdot e^{(1)}, \tag{B8}$$

a relation which must be used to construct the Hermitian transpose of Eq. (26a), i.e., to construct Eq. (26b). We write the two-spinor as the column vector,

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \tag{B9}$$

whose components are in general complex. The Hermitian transpose of this column vector is the row vector,

$$v^{+} = \begin{pmatrix} v_1^{*} & v_2^{*} \end{pmatrix}. \tag{B10}$$

Evaluating directly, it follows that

$$v^{+}(\sigma^{(1)} \cdot e^{(2)})^{+} = \begin{pmatrix} v_1^{*} v_2^{*} \end{pmatrix} \begin{pmatrix} 0 & 1-i \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & (1-i)\, v_1^{*} \end{pmatrix}, \tag{B11}$$

and

$$(\sigma^{(1)} \cdot e^{(2)} v)^{+} = \left(\begin{pmatrix} 0 & 0 \\ 1+i & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right)^{+} = \begin{pmatrix} 0 \\ v_1(1+i) \end{pmatrix}^{+} = \begin{pmatrix} 0 & (1-i)\, v_1^{*} \end{pmatrix}, \tag{B12}$$

and so we arrive at Eq. (B8). Hermitian transposition algebra of this kind is important in the Dirac equation [11,18] because it is used to show that the equation produces a rigorously non-negative probability, unlike the Klein-Gordon equation.

Appendix C. Hermitian Transposition of the Dirac Equation

We wish to prove that the Hermitian transpose of the Dirac equation,

$$\left(i\gamma^{\mu}\partial_{\mu} - e\gamma^{\mu}A_{\mu} - m\right)\psi = 0, \tag{C1}$$

is

$$\bar{\psi}\left(i\gamma^{\mu}\overleftarrow{\partial}_{\mu} - e\gamma^{\mu}A_{\mu}^{*} + m\right) = 0. \tag{C2}$$

In order to do so, it is necessary only to consider the term $\gamma^{\mu}A_{\mu}$ **when** A_{μ} is complex in general, because the Hermitian transposition of the other terms is entirely standard [22—25]. The relevant term is

$$\gamma^{\mu}A_{\mu} = \gamma^{o}A_{o} + \gamma^{i}A_{i}, \tag{C3}$$

and its Hermitian transposition implies the algebraic transposition of matrices elements with simultaneous complex conjugation of those elements. So Hermitian transposition of term (C3) results in

$$\gamma^{\mu+}A_{\mu}^{*} = \gamma^{o}A_{o}^{*} - \gamma^{i}A_{i}^{*}, \tag{C4}$$

where A_{μ}^{*} is the complex conjugate of A_{μ}. We have used [2,22—25],

$$\gamma^{o+} = \gamma^{o}, \qquad \gamma^{i+} = -\gamma^{i}, \tag{C5}$$

and multiplying by γ^{o}, using $\gamma^{i}\gamma^{o} = -\gamma^{o}\gamma^{i}$, results in

$$-\bar{\psi} e \gamma^{\mu} A_{\mu}^{*} := -e \psi^{+} \gamma^{o} \left(\gamma^{o} A_{o}^{*} + \gamma^{i} A_{i}^{*} \right), \tag{C6}$$

as in the text. This means that the operator $-e\gamma^{\mu}A_{\mu}^{*}$ multiplies the spinor $\bar{\psi} = \psi^{+}\gamma^{o}$ from the right, the operation in this case being matrix multiplication of the row vector $\bar{\psi}$, with four elements. In the original Dirac equation (C1) the operator $-e\gamma^{\mu}A_{\mu}$ multiplies the four-spinor ψ, a column vector, from the left. In the textbooks [22—25] the use of a complex A_{μ} in the Dirac equation is not standard, because the phenomenon of the anomalous Zeeman effect is being described in a static magnetic field whose vector potential is real. The standard treatment of the Dirac equation of a fermion in a plane wave is almost always restricted to linear polarization, and the key resonance term is missed.

Appendix D. Theory of Electrodynamics

	Fundamental Concept	Standard Theory	New Theory
1	Structure of Theory	linear, Abelian	**non-linear, Non-Abelian**
2	Maxwell's Equations in Vacuo	$\frac{\partial F_{\mu\nu}}{\partial x_{\nu}} = 0$, $\boldsymbol{B}^{(3)} = \boldsymbol{0}$	$\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} = i B^{(0)} \boldsymbol{B}^{(3)*}$, **et cyclicum**
3	Electromagnetic Sector Symmetry	$U(1) = O(2)$	$O(3)$
4	Poincaré Group Symmetry	$\hat{J}^{(3)}$ generator missing, unphysical little group	**magnetic fields are rotation generators**
5	Wigner Little Group	$E(2)$, unphysical planar, Euclidean	$O(3)$, **physical space rotation**
6	Source of $\boldsymbol{B}^{(3)}$ at Observer Point R and Instant t	not considered	**circling** e **at time** $t - \frac{R}{c}$ **earlier**
7	Propagation of $\boldsymbol{B}^{(3)}$	not considered	**through the Liénard-Wiechert potentials** $\boldsymbol{A}^{(1)} = \boldsymbol{A}^{(2)*}$

	Fundamental Concept	Standard Theory	New Theory
8	Gauge Group Definition of $\boldsymbol{B}^{(3)}$	not considered	$\boldsymbol{B}^{(3)*} = -i\frac{e}{\hbar}\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$
9	Free Space Four-Tensor	$F_{\mu\nu}$, Abelian in space ((1), (2), (3))	$G_{\mu\nu}$, Non-Abelian in space ((1), (2), (3))
10	Photon Helicities	-1 and 1	-1, 0, 1
11	Translational Poynting Theorem	$\nabla \cdot \boldsymbol{N} = -\frac{\partial U}{\partial t}$	same
12	Rotational Poynting Theorem	not considered	$\nabla \cdot \boldsymbol{J}^{(3)} = -\frac{\partial U^{(3)}}{\partial t}$
13	Magnetic Fields	$\boldsymbol{B}^{(1)} = \boldsymbol{B}^{(2)*}$	$\boldsymbol{B}^{(1)} = \boldsymbol{B}^{(2)*}$, $\boldsymbol{B}^{(3)}$
14	Electric Fields	$\boldsymbol{E}^{(1)} = \boldsymbol{E}^{(2)*}$	$\boldsymbol{E}^{(1)} = \boldsymbol{E}^{(2)*}$, $i\boldsymbol{E}^{(3)}$
15	Vector Potentials	$\boldsymbol{A}^{(1)} = \boldsymbol{A}^{(2)*}$	$\boldsymbol{A}^{(1)} = \boldsymbol{A}^{(2)*}$, $i\boldsymbol{A}^{(3)}$, A_0
16	Planck-Einstein Relation	$En = \hbar\omega$	$En = \hbar\omega = h_1\lambda$, $h_1 = ec\lvert\boldsymbol{B}^{(3)}\rvert$
17	de Broglie Relation	$p = \hbar\kappa$	$p = \hbar\kappa = \frac{h_1\lambda}{c}$

	Fundamental Concept	Standard Theory	New Theory
18	Quantum of Energy	$\hbar\omega = \frac{1}{\mu_0}\int(\boldsymbol{B}^{(1)} \cdot \boldsymbol{B}^{(1)*} + \boldsymbol{B}^{(2)} \cdot \boldsymbol{B}^{(2)*})dV$	$\hbar\omega = \frac{1}{\mu_0}\int(\boldsymbol{B}^{(1)} \cdot \boldsymbol{B}^{(1)*} + \boldsymbol{B}^{(2)} \cdot \boldsymbol{B}^{(2)*} + \boldsymbol{B}^{(3)} \cdot \boldsymbol{B}^{(3)*})dV_1$
19	Quantum of Angular Momentum	$\hbar$	$\hbar = h_1\frac{\lambda}{\omega}$
20	Quantum of Torque	$\hbar\omega$	$\hbar\omega = h_1\lambda$
21	Momentum Equivalence Condition	not considered	cyclic relations imply in vacuo $eA^{(0)} = \hbar\kappa$, the quantum of linear momentum
22	Mass of Photon	identically zero	$m = \frac{e\lambda_0\lvert\boldsymbol{B}^{(3)}\rvert}{c}$
23	Gauge Invariant Lagrangian Mass Term	not considered	$En = \frac{V}{\mu_0}\left(\frac{Mc^2}{\hbar\omega}\right)^2 \boldsymbol{B}^{(3)} \cdot \boldsymbol{B}^{(3)*}$ in $O(3)$ gauge group
24	Gauge Conditions on Four-Potential	1) transverse 2) Coulomb	1) not allowed 2) scalar, non-zero 3) $A_\mu A_\mu \rightarrow 0$

	Fundamental Concept	Standard Theory	New Theory
25	**Field Quantization**	canonical, beset with difficulties because A_μ is not covariant	direct quantization of cyclic field relations
26	**de Broglie Theorem**	not considered	$\hbar\omega_0 = h_1\lambda_0 = mc^2$
27	**Interaction with Fermion**	via $\boldsymbol{A}^{(1)} = \boldsymbol{A}^{(2)*}$ in minimal prescription, $A_0 = \phi = 0$	same as Standard Theory but finite scalar potential $A_0 = \phi \neq 0$
28	**Magneto-optics**	I dependence from $A_0 = 0$	$I^{1/2}$ dependence observable under the right conditions, $A_0 \neq 0$
29	Q.E.D.	no mass term	finite mass term
30	**Observation of $\boldsymbol{B}^{(3)}$ Field**	not considered	routinely observable through $i\boldsymbol{B}^{(0)}\boldsymbol{B}^{(3)*}$ in magneto-optics

Notes:

$\boldsymbol{N}$:= Poynting vector;
$\boldsymbol{J}^{(3)}$ = radiation angular momentum;
λ := wavelength;
V = volume of radiation;
$\phi = A_0$ = scalar potential.

U = energy density;
$U^{(3)} = \frac{1}{\mu_0}\boldsymbol{B}^{(3)}\cdot\boldsymbol{B}^{(3)*}$;
λ_0 = rest wavelength;
M = mass of radiation;

Appendix E. Dynamical Analogies of the Maxwell Equations

The existence of the cyclical field equations,

$$\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} = i B^{(0)} \boldsymbol{B}^{(3)*}, \text{ et cyclicum}, \tag{E1}$$

implies that there is a precise dynamical analogy to the Maxwell equations in free space. Equations (E1) can be used to derive the Maxwell equations because the former are fundamental relations between fields in free space, relations to which there is a precise dynamical analogy,

$$\boldsymbol{J}^{(1)} \times \boldsymbol{J}^{(2)} = i J^{(0)} \boldsymbol{J}^{(3)*}, \text{ et cyclicum}, \tag{E2}$$

where $\boldsymbol{J}$ denotes angular momentum. The close similarity between Eqs. (E1) and (E2) stems from the fact that both equations describe O(3) rotation generators, suggesting a profound analogy between the structure of space-time and electromagnetism. This can be extended to general relativity. In particular, the rotation generators obey a cyclically symmetric, compact, group algebra, that of O(3), whereas linear momentum generators do not. In consequence, there is a $\boldsymbol{B}^{(3)}$ field, corresponding to $\boldsymbol{J}^{(3)}$, but no $\boldsymbol{E}^{(3)}$, because the polar $\boldsymbol{p}^{(3)}$ cannot be generated from the axial product $\boldsymbol{p}^{(1)} \times \boldsymbol{p}^{(2)}$ of transverse momentum generators. This is true in any system of coordinates, both in special and in general relativity. At the most fundamental level, electromagnetism in free space, like gravity, is a consequence of the structure of space-time itself, be this Euclidean or Riemannian. Furthermore, the existence of cyclic relations between space rotation generators implies that $\boldsymbol{B}^{(1)}$ and $\boldsymbol{J}^{(1)}$ must be complex in nature, and since they are rotation generators, time dependent in some way. The Maxwell equations are invariant under the duality transformation [1,2] (Minkowski notation),

$$\left(\frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}\right) \longrightarrow \epsilon_{\mu\nu\rho\sigma}\left(\frac{\partial A_\sigma}{\partial x_\rho} - \frac{\partial A_\rho}{\partial x_\sigma}\right), \tag{E3}$$

where $\epsilon_{\mu\nu\rho\sigma}$ is the four-dimensional Levi-Civita symbol, and from the analogy between Eqs. (E1) and (E2), it is expected that there is a direct dynamical analogy to this result,

$$\left(\frac{\partial p_\nu}{\partial x_\mu} - \frac{\partial p_\mu}{\partial x_\nu}\right) \longrightarrow \epsilon_{\mu\nu\rho\sigma}\left(\frac{\partial p_\sigma}{\partial x_\rho} - \frac{\partial p_\rho}{\partial x_\sigma}\right), \tag{E4}$$

where p_μ is the energy-momentum four-vector. If $F_{\mu\nu}$ is the four-tensor of electric and magnetic fields [1,2], and $J_{\mu\nu}$ that of rotation and boost generators, then the above-mentioned analogy between electromagnetism and dynamics can be summarized through the statement that $F_{\mu\nu}$ is proportional to $J_{\mu\nu}$. Thus [1,2], magnetic fields are rotation generators, electric fields are boost generators, and the free space Maxwell equations become those of classical dynamics.

In free space, the Maxwell equations in S.I. are,

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \qquad \nabla \cdot \boldsymbol{B} = 0, \tag{E5a}$$

$$\nabla \times \boldsymbol{B} = \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t}, \qquad \nabla \cdot \boldsymbol{E} = 0. \tag{E5b}$$

The pair of equations (E5a) and (E5b) can be written respectively as

$$\frac{\partial F_{\mu\nu}}{\partial x_\nu} = 0, \qquad \frac{\partial}{\partial x_\nu}\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma} = 0, \tag{E6}$$

where $F^D_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}$ is the dual tensor [1,2] of $F_{\mu\nu}$. The duality transform (E3) can therefore be expressed through the fact that Maxwell's free space equations are invariant under

$$F_{\mu\nu} \longrightarrow F^D_{\mu\nu}. \tag{E7}$$

The duality transform offers considerable insight to the nature of electromagnetism and space-time at the most fundamental level. It can be expressed in terms of space-time as

$$\nabla\times \longrightarrow -\frac{i}{c}\frac{\partial}{\partial t}, \qquad \frac{\partial}{\partial t} \longrightarrow ic\nabla\times, \tag{E8}$$

and in terms of electromagnetism as

$$\boldsymbol{B}^{(0)} \longrightarrow i\frac{\boldsymbol{E}^{(0)}}{c}, \qquad \boldsymbol{E}^{(0)} \longrightarrow -ic\boldsymbol{B}^{(0)}. \tag{E9}$$

It is easily checked that both (E8) and (E9) leave the Maxwell equations (E5) unchanged. Therefore Eqs. (E5) can be written as

$$\boldsymbol{B} \longrightarrow i\frac{\boldsymbol{E}}{c}, \qquad \boldsymbol{E} \longrightarrow -ic\boldsymbol{B}. \tag{E10}$$

On the most fundamental level, therefore, Maxwell's equations indicate that the duality transform

$$\sqrt{+1}\,\boldsymbol{B} \longrightarrow \sqrt{-1}\,\frac{\boldsymbol{E}}{c}, \tag{E11}$$

is equivalent to the duality transform

$$(\nabla\times)\boldsymbol{B} \longrightarrow \left(-\frac{i}{c}\frac{\partial}{\partial t}\right)\frac{i\boldsymbol{E}}{c}. \tag{E12}$$

This can be seen clearly through the vector potential $\boldsymbol{A}$, because

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \longrightarrow -\frac{i}{c}\frac{\partial}{\partial t}\boldsymbol{A} = \frac{i\boldsymbol{E}}{c}, \tag{E13}$$

thus combining Eqs. (E8) and (E9). Ultimately, therefore, the generation of an electric from a magnetic field is a matter of replacing the operator $\nabla\times$ by the operator $-\frac{i}{c}\frac{\partial}{\partial t}$, both acting on $\boldsymbol{A}$, and this is a consequence of the structure of space-time itself. In the minimal prescription, the vector potential $\boldsymbol{A}$ is $\boldsymbol{p}/e$, where $\boldsymbol{p}$ is linear momentum, and from the Lorentz force equation, $\boldsymbol{F}$ is expressible dimensionally as $e\boldsymbol{E}$. Relations such as these provide an opportunity, therefore, of writing the Maxwell equations as equations of pure dynamics. Using $\boldsymbol{E} = -1/e(\partial\boldsymbol{p}/\partial t) = -\boldsymbol{F}/e$, it is easily checked that

$$\boxed{\nabla\times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}} \quad \longleftrightarrow \quad \boxed{\nabla\times \boldsymbol{F} = -\frac{\partial \boldsymbol{J}_A}{\partial t}} \qquad \text{(E14)}$$

where $\boldsymbol{J}_A$ is angular momentum per unit area. So in classical dynamics, the curl of force is the negative of torque per unit area, $-\partial \boldsymbol{J}_A/\partial t$. This is a precise analogy to the free space Maxwell equation on the left hand side of Eq. (E14), $\boldsymbol{F}$ being analogous to $\boldsymbol{E}$, $\boldsymbol{J}_A$ to $\boldsymbol{B}$. Similarly,

$$\boxed{\nabla\times \boldsymbol{B} = \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t}} \quad \longleftrightarrow \quad \boxed{\nabla\times \boldsymbol{J}_A = -\frac{1}{c^2}\frac{\partial \boldsymbol{F}}{\partial t}} \qquad \text{(E15)}$$

It follows, also, that

$$\boxed{\begin{aligned}\nabla\cdot \boldsymbol{B} &= 0\\ \nabla\cdot \boldsymbol{E} &= 0\end{aligned}} \quad \longleftrightarrow \quad \boxed{\begin{aligned}\nabla\cdot \boldsymbol{J}_A &= 0\\ \nabla\cdot \boldsymbol{F} &= 0\end{aligned}} \qquad \text{(E16)}$$

showing that if the rotational motion is taking place around the Z axis, there is no phase dependence of the type $\exp(i(\omega t - \kappa Z))$ in the Z components of $\boldsymbol{J}_A$ and $\boldsymbol{F}$.

In order to demonstrate more clearly that electric fields are boost generators and magnetic fields rotation generators [1,2] it is necessary to introduce linear momentum, $\boldsymbol{p}$, into the above relations, rather than force $\boldsymbol{F}$, whereupon

$$\boxed{\nabla\times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}} \quad \longleftrightarrow \quad \boxed{\frac{v^2}{\omega}\nabla\times \boldsymbol{p} = -\frac{\partial \boldsymbol{J}}{\partial t}} \qquad \text{(E17)}$$

with, as in classical dynamics,

$$\boldsymbol{v} = \boldsymbol{\omega}\times \boldsymbol{r}, \qquad \boldsymbol{J} = \boldsymbol{r}\times \boldsymbol{p}. \qquad \text{(E18)}$$

Here $\boldsymbol{\omega}$ and $\boldsymbol{J}$ denote angular velocity and momentum respectively, and $\boldsymbol{r}$ is the radius vector of the angular motion [1,2]. The free space Maxwell equation in analogy (E17) is therefore equivalent to the fundamental relation between linear and angular momentum in classical dynamics.

From Eq. (E14),

$$\nabla\times \boldsymbol{F} = \frac{1}{\pi r^2}\frac{\partial \boldsymbol{J}}{\partial t}, \qquad \text{(E19)}$$

where, in general, and in analogy with the general relation between $\boldsymbol{E}$ and $\boldsymbol{A}$,

$$\boxed{\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} - \nabla\phi} \quad \longleftrightarrow \quad \boxed{\boldsymbol{F} = -\frac{\partial \boldsymbol{p}}{\partial t} - \nabla U} \qquad \text{(E20)}$$

where U is potential energy. The curl of $\boldsymbol{F}$ is independent of U, and the curl of $\boldsymbol{E}$ is independent of ϕ. Continuing the analogy,

$$\boxed{\boldsymbol{B} = \nabla\times \boldsymbol{A}} \quad \longleftrightarrow \quad \boxed{\boldsymbol{J} = \pi r^2\nabla\times \boldsymbol{p}} \qquad \text{(E21)}$$

showing that $\boldsymbol{J}$ is independent of U just as $\boldsymbol{B}$ is independent of ϕ. Comparing Eqs. (E18) and (E21) shows that $\boldsymbol{r}\times = \pi r^2\nabla\times$. The right hand side of Eq. (E21) can be obtained by direct integration of Eq. (E19),

$$\boldsymbol{J} = -\pi r^2\int \nabla\times \boldsymbol{F}\,dt = -\pi r^2\nabla\times\int \boldsymbol{F}\,dt = \pi r^2\nabla\times \boldsymbol{p}, \qquad \text{(E22)}$$

using Eq. (E20). Therefore, the fundamentally analogous quantities are

$$\boldsymbol{B} \longleftrightarrow \boldsymbol{J}, \quad \boldsymbol{A} \longleftrightarrow \pi r^2 \boldsymbol{p}, \quad \boldsymbol{E} \longleftrightarrow \frac{v^2}{\omega}\boldsymbol{p}, \tag{E23}$$

showing that $\boldsymbol{B}$, $\boldsymbol{E}$, and $\boldsymbol{A}$ in electrodynamics are as fundamental as $\boldsymbol{J}$, $\boldsymbol{F}$ and $\boldsymbol{p}$ in dynamics. The dynamical analogy shows clearly that both fields and potentials are physically meaningful.

The duality transformation in dynamics can be developed in its clearest form by accepting the validity of the de Broglie guiding theorem,

$$En = mc^2 = \hbar\omega_0 . \tag{E24}$$

If v is identified with c in Eq. (E17), then,

$$\frac{c^2}{\omega_0} = \frac{\hbar}{m}, \tag{E25}$$

so that in the photon rest frame,

$$\nabla \times \boldsymbol{p} = -\frac{m}{\hbar}\frac{\partial \boldsymbol{J}}{\partial t}, \tag{E26}$$

where m is the mass of the photon. The duality transform therefore becomes

$$\boxed{\sqrt{+1}\,\boldsymbol{B} \longrightarrow \sqrt{-1}\,\frac{\boldsymbol{E}}{c}} \quad \longleftrightarrow \quad \boxed{\sqrt{+1}\,\boldsymbol{J} \longrightarrow \sqrt{-1}\,\frac{\hbar}{m}\boldsymbol{p}} \tag{E27}$$

which, with $\boldsymbol{v} = \boldsymbol{p}/m$, is equivalent to

$$\boldsymbol{v} \longrightarrow -\frac{i\boldsymbol{J}}{\hbar}, \quad \frac{\boldsymbol{J}}{\hbar} \longrightarrow i\boldsymbol{v} . \tag{E28}$$

Equation (E28) means that $\boldsymbol{J}/\hbar$ plays the role of $\boldsymbol{B}$ in the duality transform, and $\boldsymbol{v}$ that of $\boldsymbol{E}/c$. Thus, the magnetic field is a rotation generator and the electric field a boost generator [1,2], which is the momentum-energy within a factor. If there is photon

mass, $\boldsymbol{B}^{(3)}$ and the $\boldsymbol{J}^{(3)}$ are missing, or at best, ill-defined.

Equations of motion can be constructed directly from relations with group symmetry O(3) because this is the group of infinitesimal rotation generators and angular momentum operators [1]. Space itself is described by a set of relations between complex unit vectors [1,2], $\boldsymbol{e}^{(1)}$, $\boldsymbol{e}^{(2)}$, and $\boldsymbol{e}^{(3)}$ in the basis ((1), (2), (3)),

$$\boldsymbol{e}^{(1)} \times \boldsymbol{e}^{(2)} = i\boldsymbol{e}^{(3)*}, \text{ et cyclicum} . \tag{E29}$$

In order to convert this into an equation of motion, it is written as

$$\left(\boldsymbol{e}^{(1)}e^{i\phi}\right) \times \left(\boldsymbol{e}^{(2)}e^{-i\phi}\right) = i\boldsymbol{e}^{(3)*}, \text{ et cyclicum}, \tag{E30}$$

where $e^{i\phi}$, $\phi = \omega t - \boldsymbol{\kappa}\cdot\boldsymbol{r}$, is the de Broglie wave function. This is a generator of wave motion with relativistically invariant phase. Equations (E29) represent the framework of space, while Eqs. (E30) represent wave propagation with simultaneous rotational motion. If the propagation velocity of the wave is v, then $\boldsymbol{e}^{(3)}$ also propagates at this velocity, because at instant t, the frame ((1), (2), (3)) is centered on an origin at point $\boldsymbol{R}$. The complete frame must move forward to point $\boldsymbol{R}'$ at instant t' because if $\boldsymbol{e}^{(1)}$ and $\boldsymbol{e}^{(2)}$ propagate, so does $\boldsymbol{e}^{(3)}$, being always defined by $\boldsymbol{e}^{(1)} \times \boldsymbol{e}^{(2)}$.

Equation (E30) is therefore an equation which describes the characteristics of wave motion superimposed on the structure of space. It is a frame of reference spiralling forward at the propagation velocity of the wave. Equations (E30) become Eqs. (E1) provided

$$\boldsymbol{B}^{(1)} = iB^{(0)}\boldsymbol{e}^{(1)}e^{i\phi}, \quad \boldsymbol{B}^{(2)} = -iB^{(0)}\boldsymbol{e}^{(2)}e^{-i\phi}, \quad \boldsymbol{B}^{(3)} = B^{(0)}\boldsymbol{e}^{(3)}, \tag{E31}$$

showing that $\boldsymbol{B}^{(3)}$ propagates at the same speed as $\boldsymbol{B}^{(1)}$ and $\boldsymbol{B}^{(2)}$. Since Eq. (E1) is analogous with Eq. (E2), cyclically symmetric relations between angular momenta, and since angular and linear momentum and angular momentum and force are related fundamentally through equations identical in structure to the free space Maxwell equations, the cyclic equations (E1) imply the existence of the Maxwell equations themselves. *Additionally*, they imply the existence of $\boldsymbol{B}^{(3)}$ in vacuo and in matter, and are therefore field equations, relations between fields, which are more fundamental than the Maxwell equations. Equations (E1) show that all types of angular momentum theory can be applied directly to field theory, both in classical and quantum mechanics, in special and in general relativity.

It *is* possible to modify the Maxwell equations to account for $\boldsymbol{B}^{(3)}$ from equations (E1), but this procedure is self-consistent only in the O(3) gauge group, which is non-

Abelian [2]. Equations (E1) show that Maxwell's equations must be generalized, and are therefore more fundamental as we have argued. In the O(3) gauge group, covariant derivatives are used, borrowing techniques of gauge theory taken from general relativity [2,23]. This leads to the possibility of developing $\boldsymbol{B}^{(3)}$ in Riemannian geometry, with an eye to proving that space-time becomes curved in the presence of electromagnetism as well as gravitation. This would be a step towards a unified description of electroweak and gravitational fields. A glance at equations (E1) shows them to be non-Abelian and non-linear. The much older Maxwellian point of view is linear and Abelian, making unification with gravitational theory difficult.

The dynamical operation,

$$\boldsymbol{e}^{(1)} \longrightarrow \boldsymbol{e}^{(1)} \exp\left(i\omega\left(t - \frac{r}{c}\right)\right), \tag{E32}$$

is a boost of $\boldsymbol{e}^{(1)}$ [1], and is alone sufficient to produce a generalized version of the Maxwell equations in free space. It is an operation on $\boldsymbol{e}^{(1)}$ with the well known exponent [21-25] $\exp(iS/\hbar)$ where S is the electromagnetic action $\hbar(\omega t - \boldsymbol{\kappa}\cdot\boldsymbol{r})$. We have introduced $\hbar$ into the analysis, which has therefore become quantum mechanical in nature. In space-time the action in Minkowski notation is $S = \hbar\kappa_\mu x_\mu$. Therefore the generalized Maxwell equations are produced by the boost,

$$\boldsymbol{e}^{(1)} \longrightarrow \exp\left(i\hbar\kappa_\mu x_\mu\right)\boldsymbol{e}^{(1)} = \exp\left(ieA_\mu x_\mu\right)\boldsymbol{e}^{(1)} \tag{E33}$$

which is a rotation in space-time in special relativity. Defining the operator by the symbol $\hat{\mathfrak{L}} = e^{ieA_\mu x_\mu}$, transition to fields is achieved, finally, through

$$\frac{\boldsymbol{B}^{(1)}}{B^{(0)}} = i\,\hat{\mathfrak{L}}^{(1)}\boldsymbol{e}^{(1)}, \quad \frac{\boldsymbol{B}^{(2)}}{B^{(0)}} = -i\,\hat{\mathfrak{L}}^{(2)}\boldsymbol{e}^{(2)}, \quad \frac{\boldsymbol{B}^{(3)}}{B^{(0)}} = \boldsymbol{e}^{(3)}, \tag{E34}$$

and the duality transformation,

$$\sqrt{+1}\,B^{(0)} \longrightarrow \sqrt{-1}\,\frac{E^{(0)}}{c}, \tag{E35}$$

takes Eqs. (E1) to their equivalents for electric fields

$$\boldsymbol{E}^{(1)} \times \boldsymbol{E}^{(2)} = -E^{(0)}\left(i\boldsymbol{E}^{(3)}\right)^{*} \text{ et cyclicum}, \tag{E36}$$

where $i\boldsymbol{E}^{(3)}$ is pure imaginary and unphysical [1,2]. The electric field equations (E36) are as fundamental as the magnetic field equations (E1), being transformed into (E1) by fundamental tensorial duality.

Appendix F. $\boldsymbol{B}^{(3)}$ in an Electron Gas

For purposes of experimental investigation, we list in this Appendix equations relating $\boldsymbol{B}^{(3)}$ in free space to its value in a sample made up of N electrons in a volume V. In general, using the text's notation,

$$\boldsymbol{B}^{(3)}_{\text{in sample}} = \frac{N}{V}\cdot\frac{\mu_0 e^3 c^2}{2m\omega^2}\left(\frac{B^{(0)}}{(m^2\omega^2 + e^2 B^{(0)2})^{1/2}}\right)\boldsymbol{B}^{(3)}_{\text{free space}}. \tag{F1}$$

In *the low field (visible frequency) limit*, $m\omega \gg eB^{(0)}$, Eq. (F1) reduces to

$$\boldsymbol{B}^{(3)}_{\text{in sample}} \longrightarrow \frac{N}{V}\left(\frac{\mu_0 e^3 c^2 B^{(0)}}{2m^2\omega^3}\right)\boldsymbol{B}^{(3)}_{\text{free space}}, \tag{F2}$$

and in *the high field (radio frequency) limit*, $m\omega \ll eB^{(0)}$, Eq. (F1) becomes

$$\boldsymbol{B}^{(3)}_{\text{in sample}} \longrightarrow \frac{N}{V}\left(\frac{\mu_0 e^2 c^2}{2m\omega^2}\right)\boldsymbol{B}^{(3)}_{\text{free space}}. \tag{F3}$$

The *free space* value of $\boldsymbol{B}^{(3)}$ is [1,2],

$$\boldsymbol{B}^{(3)}_{\text{free space}} = \left(\frac{\mu_0}{c}I\right)^{1/2}\boldsymbol{e}^{(3)} = \left(\frac{I}{\epsilon_0 c^3}\right)^{1/2}\boldsymbol{e}^{(3)}. \tag{F4}$$

In terms of beam intensity, the low field limit is

$$B^{(3)}_{\text{in sample}} \longrightarrow \frac{N}{V}\left(\frac{\mu_0^2 e^3 c}{2m^2}\right)\frac{I}{\omega^3} e^{(3)}, \tag{F5}$$

and under the conditions used by Rikken [74], i.e., $I = 5.5 \times 10^{12}$ Wm^{-2}, $\omega = 1.77 \times 10^{16}$ rad s^{-1}, we obtain from Eq. (F5),

$$B^{(3)}_{\text{in sample}} = 1.06 \times 10^{-35} \frac{N}{V} e^{(3)} \sim 10^{-9} \text{ tesla}, \tag{F6}$$

$$\text{or } \sim 10^{-5} \text{ gauss,}$$

which, for $N/V = 10^{26} m^{-3}$, is about the same order of magnitude as reported experimentally by van der Ziel *et al.* [17] in the first inverse Faraday effect experiment. *This magnitude is about four or five orders below the limit of sensitivity reported by Rikken* [74]. It is important to note that the interaction of $\boldsymbol{B}^{(3)}$ with matter is relativistic, because $\boldsymbol{B}^{(3)}$ propagates, F.A.P.P., at c in vacuo. It is important not to confuse the free space magnitude of $\boldsymbol{B}^{(3)}$, i.e., Eq. (F4), with its magnitude within a sample. Finally, it is important to test $\boldsymbol{B}^{(3)}$ under conditions appropriate to the theory, i.e., ideally, in a free electron gas or electron beam. Liquid benzene, used by Rikken, has no free fermion spin, and is inappropriate for comparison with Eq. (F1), for *N electrons* in a sample V.

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