1) 100(13): Relation between Torsion and Canonical Angular Energy Momentum density

In previous notes and flow charts it has been shown that the two main equations of the Bianchi identity are:

$$\boxed{\begin{aligned} D_\mu \tilde{T}^{\kappa\mu\nu} &= -\tilde{R}^{\kappa}{}_{\mu}{}^{\mu\nu} \quad &(1)\\ D_\mu T^{\kappa\mu\nu} &= -R^{\kappa}{}_{\mu}{}^{\mu\nu} \quad &(2)\end{aligned}}$$

being the homogeneous and inhomogeneous field equations respectively. These are defined in a space-time with curvature and torsion. It is important to note that $\|g\|^{1/2}$ cancels out in both equations.

Electrodynamics

$$F^{\kappa\mu\nu} = A^{(0)} T^{\kappa\mu\nu} \quad (3)$$

Dynamics

$$T^{\kappa\mu\nu} = k J^{\kappa\mu\nu} \quad (4)$$

Eqs. (3) and (4) extend the Einsteinian postulate:

$$G_{\mu\nu} = k T_{\mu\nu} \quad (5)$$

to electrodynamics and to the Cartan torsion tensor. The latter is proportional to the tensor:

$$J^{\kappa\mu\nu} = -\frac{1}{2}\left(T^{\kappa\mu}x^{\nu} - T^{\kappa\nu}x^{\mu}\right), \quad (6)$$

the canonical angular energy / momentum density tensor.

2)

S.I. Units Check

$$k = 1.86595 \times 10^{-26}\ m\ kg^{-1}$$

$G_{\mu\nu} = m^{-2}$, so $T_{\mu\nu} = kgm\ m^{-3}$,

$J^{\kappa\mu\nu} = kg\ m^{-2}$, $T^{\kappa\mu\nu} = m^{-1}$

Therefore eq. (2) is:

$$\boxed{D_\mu F^{\kappa\mu\nu} = -A^{(0)} R^{\kappa}{}_{\mu}{}^{\mu\nu}} \quad -(7)$$

electrodynamics

and

$$\boxed{D_\mu J^{\kappa\mu\nu} = -\frac{1}{k} R^{\kappa}{}_{\mu}{}^{\mu\nu}} \quad -(8)$$

dynamics

The electromagnetic field density is directly proportional to the canonical angular energy momentum density

$$\boxed{F^{\kappa\mu\nu} = k A^{(0)} J^{\kappa\mu\nu}} \quad -(9)$$

It is now possible to proceed by expanding the left hand sides of eqs. (7) and (8) using the rule for the covariant derivative of a rank-three tensor.

3) The general rule for the covariant derivative for a tensor of any rank is given by Carroll eq (3.13):

$$D_\sigma T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} = \partial_\sigma T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} \qquad -(7)$$

$$+ \Gamma^{\mu_1}_{\sigma\lambda} T^{\lambda\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} + \Gamma^{\mu_2}_{\sigma\lambda} T^{\mu_1\lambda\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} + \cdots$$

$$- \Gamma^{\lambda}_{\sigma\nu_1} T^{\mu_1\mu_2\cdots\mu_k}{}_{\lambda\nu_2\cdots\nu_l} - \Gamma^{\lambda}_{\sigma\nu_2} T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\lambda\cdots\nu_l} - \cdots$$

So:

$$D_\mu F^{\kappa\mu\nu} = \partial_\mu F^{\kappa\mu\nu} + \Gamma^{\kappa}_{\mu\lambda} F^{\lambda\mu\nu} + \Gamma^{\mu}_{\mu\lambda} F^{\kappa\lambda\nu}$$

$$+ \Gamma^{\nu}_{\mu\lambda} F^{\kappa\mu\lambda} \qquad -(8)$$

$$D_\mu J^{\kappa\mu\nu} = \partial_\mu J^{\kappa\mu\nu} + \Gamma^{\kappa}_{\mu\lambda} J^{\lambda\mu\nu} + \Gamma^{\mu}_{\mu\lambda} J^{\kappa\lambda\nu}$$

$$+ \Gamma^{\nu}_{\mu\lambda} J^{\kappa\mu\lambda} \qquad -(9)$$

Approximation to Eq. (7)

$$\partial_\mu F^{\kappa\mu\nu} = -A^{(0)}\left(R^{\kappa}{}_{\mu}{}^{\mu\nu} + \Gamma^{\kappa}_{\mu\lambda} T^{\lambda\mu\nu} + \Gamma^{\mu}_{\mu\lambda} T^{\kappa\lambda\nu} + \Gamma^{\nu}_{\mu\lambda} T^{\kappa\mu\lambda}\right) \qquad -(10)$$

1) Assume that the electromagnetic part of the right hand side of eq (10) is pure rotational.

2) Assume that the gravitational torsion is small.

4)

Then:

$$\boxed{\partial_\mu F^{\kappa\mu\nu} \doteq -A^{(0)}\left(R^{\kappa}{}_{\mu}{}^{\mu\nu}\right)_{grav}} \quad \text{— (11)}$$

This is the equation used in paper 93.

Approximation to Eq (8)

$$\partial_\mu J^{\kappa\mu\nu} = -\frac{1}{k}\Big(R^{\kappa}{}_{\mu}{}^{\mu\nu} + \Gamma^{\kappa}{}_{\mu\lambda} T^{\lambda\mu\nu} + \Gamma^{\mu}{}_{\mu\lambda} T^{\kappa\lambda\nu} + \Gamma^{\nu}{}_{\mu\lambda} T^{\kappa\mu\lambda}\Big)$$

— (12)

If:

$$T^{\lambda\mu\nu} << R^{\kappa}{}_{\mu}{}^{\mu\nu} \quad \text{— (13)}$$

$$\boxed{\partial_\mu J^{\kappa\mu\nu} \doteq -\frac{1}{k}\left(R^{\kappa}{}_{\mu}{}^{\mu\nu}\right)_{grav}} \quad \text{— (14)}$$

In Ricci flat spacetimes:

$$\boxed{\partial_\mu F^{\kappa\mu\nu} = 0, \quad \partial_\mu J^{\kappa\mu\nu} = 0} \quad \text{— (15}$$

Note that these approximations are equivalent

5)

to assuming that the right hand side of eq. (10) or eq. (12) is torsion free, so that the gamma connection becomes the Christoffel connection. Then the left hand side of eq. (10) represents electromagnetism and the right hand side represents gravitation. The left hand side of eq. (14) however must be interpreted as the limit where the torsion is small compared with the curvature.

The Coulomb and Ampère Maxwell laws are given by eq. (11), as in earlier notes to paper 100, and the Newton inverse square law is generalized by eq. (14).

Alternatively it is possible to regard eqs. (11) and (14) as weak field limits where

$$D_\mu F^{\kappa\mu\nu} \doteq \partial_\mu F^{\kappa\mu\nu} \quad -(16)$$

$$D_\mu J^{\kappa\mu\nu} \doteq \partial_\mu J^{\kappa\mu\nu} \quad -(17)$$

Finally eq. (11) is interpreted as:

$$\boxed{A^{(0)}\left(\partial_\mu T^{\kappa\mu\nu}\right)_{\text{electromagnetic}} \doteq -A^{(0)}\left(R^{\kappa}{}_{\mu}{}^{\mu\nu}\right)_{\text{gravitation}}}$$