1) 100(16) Proof of the Hodge Dual Relation

Theorem

$$\boxed{\begin{array}{ll} \text{If} & R \wedge q := D \wedge T \\ \text{then} & \tilde{R} \wedge q := D \wedge \tilde{T}. \end{array}} \qquad \begin{array}{l} -(1) \\ -(2) \end{array}$$

Proof The Bianchi identity (1) is a cyclic sum of Riemann tensor definitions (GCUFT1, Chapter 7, Appendices):

$$R^{\lambda}{}_{\rho\mu\nu} = \partial_\mu \Gamma^{\lambda}{}_{\nu\rho} - \partial_\nu \Gamma^{\lambda}{}_{\mu\rho} + \Gamma^{\lambda}{}_{\mu\sigma}\Gamma^{\sigma}{}_{\nu\rho} - \Gamma^{\lambda}{}_{\nu\sigma}\Gamma^{\sigma}{}_{\mu\rho}. \quad -(3)$$

It is possible to define:

$$R^{\lambda}{}_{\rho}{}^{\mu\nu} := \partial^\mu \Gamma^{\lambda\nu}{}_{\rho} - \partial^\nu \Gamma^{\lambda\mu}{}_{\rho} + \Gamma^{\lambda\mu}{}_{\sigma}\Gamma^{\sigma\nu}{}_{\rho} - \Gamma^{\lambda\nu}{}_{\sigma}\Gamma^{\sigma\mu}{}_{\rho}. \quad -(4)$$

The Hodge dual of eqn (3) is:

$$\tilde{R}^{\lambda}{}_{\rho}{}^{\alpha\beta} = \frac{1}{2}\|g\|^{1/2}\epsilon^{\alpha\beta\mu\nu}R^{\lambda}{}_{\rho\mu\nu} \quad -(5)$$

and the Hodge dual of eqn (4) is:

$$\tilde{R}^{\lambda}{}_{\rho\alpha\beta} = \frac{1}{2}\|g\|^{1/2}\epsilon_{\alpha\beta\mu\nu}R^{\lambda}{}_{\rho}{}^{\mu\nu} \quad -(6)$$

The cyclic sum from eq. (3) is:

$$R^{\lambda}{}_{\rho\mu\nu} + R^{\lambda}{}_{\nu\rho\mu} + R^{\lambda}{}_{\mu\nu\rho} := \text{cyclic sum of definitions (3)} \quad -(7)$$

The cyclic sum from eq. (4) is:

$$R^{\lambda}{}_{\rho}{}^{\mu\nu} + R^{\lambda}{}_{\nu}{}^{\rho\mu} + R^{\lambda}{}_{\mu}{}^{\nu\rho} := \text{cyclic sum of definitions (4)} \quad -(8)$$

2)

Now define the inverse Hodge duals:

$$R_\rho{}^{\lambda\ \mu\nu} = \frac{1}{2}\|g\|^{1/2}\epsilon^{\mu\nu\alpha\beta}\tilde{R}^{\lambda}{}_{\rho\alpha\beta} \quad -(9)$$

$$\partial^\mu \Gamma^{\lambda\nu}{}_\rho = \frac{1}{2}\|g\|^{1/2}\epsilon^{\mu\nu\alpha\beta}\left(\widetilde{\partial_\alpha \Gamma^{\lambda}{}_{\beta\rho}}\right) \quad -(10)$$

$$\partial^\nu \Gamma^{\lambda\mu}{}_\rho = \frac{1}{2}\|g\|^{1/2}\epsilon^{\nu\mu\beta\alpha}\left(\widetilde{\partial_\beta \Gamma^{\lambda}{}_{\alpha\rho}}\right) \quad -(11)$$

$$\Gamma^{\lambda\mu}{}_\sigma \Gamma^{\sigma\nu}{}_\rho = \frac{1}{2}\|g\|^{1/2}\epsilon^{\mu\nu\alpha\beta}\left(\widetilde{\Gamma^{\lambda}{}_{\alpha\sigma}\Gamma^{\sigma}{}_{\beta\rho}}\right) \quad -(12)$$

$$\Gamma^{\lambda\nu}{}_\sigma \Gamma^{\sigma\mu}{}_\rho = \frac{1}{2}\|g\|^{1/2}\epsilon^{\nu\mu\beta\alpha}\left(\widetilde{\Gamma^{\lambda}{}_{\beta\sigma}\Gamma^{\sigma}{}_{\alpha\rho}}\right) \quad -(13).$$

then:

$$\tilde{R}^{\lambda}{}_{\rho\alpha\beta} = \left(\widetilde{\partial_\alpha \Gamma^{\lambda}{}_{\beta\rho}}\right) - \left(\widetilde{\partial_\beta \Gamma^{\lambda}{}_{\alpha\rho}}\right) + \left(\widetilde{\Gamma^{\lambda}{}_{\alpha\sigma}\Gamma^{\sigma}{}_{\beta\rho}}\right) - \left(\widetilde{\Gamma^{\lambda}{}_{\beta\sigma}\Gamma^{\sigma}{}_{\alpha\rho}}\right) \quad -(14)$$

and:

$$\tilde{R}^{\lambda}{}_{\rho\alpha\beta} + \tilde{R}^{\lambda}{}_{\beta\rho\alpha} + \tilde{R}^{\lambda}{}_{\alpha\beta\rho} := \text{cyclic sum of defns (14)} \quad -(15)$$

i.e. $\tilde{R} \wedge q := D \wedge \tilde{T}$ <u>Q.E.D.</u>

$-(16)$

3)

In general, if:

$$\tilde{R}^{\lambda}{}_{\rho\mu\nu} = \partial_\mu X^{\lambda}{}_{\nu\rho} - \partial_\nu X^{\lambda}{}_{\mu\rho} + X^{\lambda}{}_{\mu\sigma} X^{\sigma}{}_{\nu\rho} - X^{\lambda}{}_{\nu\sigma} X^{\sigma}{}_{\mu\rho} \quad -(17)$$

and if

$$\tilde{T}^{\lambda}{}_{\mu\nu} = X^{\lambda}{}_{\mu\nu} - X^{\lambda}{}_{\nu\mu} \quad -(18)$$

then:

$$D_\rho \tilde{T}^{\lambda}{}_{\mu\nu} + D_\nu \tilde{T}^{\lambda}{}_{\rho\mu} + D_\mu \tilde{T}^{\lambda}{}_{\nu\rho} := -\left(\tilde{R}^{\lambda}{}_{\rho\mu\nu} + \tilde{R}^{\lambda}{}_{\nu\rho\mu} + \tilde{R}^{\lambda}{}_{\mu\nu\rho}\right) \quad -(19)$$

Here $X^{\lambda}{}_{\nu\rho}$ is any three index connection. Eq. (19) is

$$\boxed{D_\mu T^{\lambda\mu\nu} := -R^{\lambda}{}_{\mu}{}^{\mu\nu}} \quad -(20)$$

Paradox in the Standard Model

For the Christoffel connection

$$\Gamma^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{\nu\mu} \quad -(21)$$

$$T^{\lambda\mu\nu} = 0, \quad R^{\lambda}{}_{\mu}{}^{\mu\nu} \neq 0 \quad -(22)$$

So the Christoffel connection is not compatible with the Bianchi identity (20) when torsion is properly considered.

4)

This is a major problem for the standard model, because the Bianchi identity used in EH neglects torsion.

Approximate Resolution of the Paradox in Paper 93

In paper 93 eqn (20) was used with two interacting fields, electromagnetic (e/m) and gravitational (grav.)

So:

$$T^{\lambda\mu\nu} = T^{\lambda\mu\nu}(e/m) + T^{\lambda\mu\nu}(\text{grav.}) \quad -(23)$$

$$R^{\lambda}{}_{\mu}{}^{\mu\nu} = R^{\lambda}{}_{\mu}{}^{\mu\nu}(e/m) + R^{\lambda}{}_{\mu}{}^{\mu\nu}(\text{grav}) \quad -(24)$$

The Christoffel connection was used for gravitation

so:

$$T^{\lambda\mu\nu}(\text{grav.}) = 0 \quad -(25)$$

For the e/m field it was assumed that:

$$\tilde{R} \wedge q = \omega \wedge \tilde{T} \quad -(26)$$

so eq. (20) is:

$$\boxed{\partial_\mu F^{\lambda\mu\nu}(e/m) \doteq -R^{\lambda}{}_{\mu}{}^{\mu\nu}(\text{grav.})} \quad -(27)$$

However, gravitation in isolation cannot be described with a Christoffel connection.