100(19): Notes on Hodge Duality (Carroll, Chapter 1)

The Hodge dual of a differential form is defined in Carroll, chapt 1 as:

$$\tilde{A}_{\mu_1 \dots \mu_{n-p}} = \frac{1}{p!} \epsilon^{\nu_1 \dots \nu_p}{}_{\mu_1 \dots \mu_{n-p}} A_{\nu_1 \dots \nu_p} \quad -(1)$$

For example, in Minkowski spacetime:

$$\tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}. \quad -(2)$$

In ECE theory the convention is adopted that the inverse Hodge dual is:

$$F_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \tilde{F}^{\rho\sigma}, \quad -(3)$$

and that:

$$\epsilon^{\mu\nu\alpha\beta} = \epsilon_{\mu\nu\alpha\beta} = 1. \quad -(4)$$

So we obtain results such as:

$$\partial_1 \tilde{F}_{23} + \partial_3 \tilde{F}_{12} + \partial_2 \tilde{F}_{31} = \partial_1 F^{10} + \partial_3 F^{30} + \partial_2 F^{20} \quad -(5)$$

and:

$$\partial_1 F_{23} + \partial_3 F_{12} + \partial_2 F_{31} = \partial_1 \tilde{F}^{10} + \partial_3 \tilde{F}^{30} + \partial_2 \tilde{F}^{20}. \quad -(6)$$

In Minkowski spacetime the theory of classical electrodynamics is the Maxwell Heaviside theory

$$d \wedge F = 0 \quad -(7)$$

$$d \wedge \tilde{F} = \tilde{J} / \epsilon_0 \quad -(8)$$

$$F = d \wedge A \quad -(9)$$

2) In ~~the~~ special case:

$$\tilde{J} = 0 \quad -(10)$$

we obtain:

$$d \wedge F = 0 \quad -(11)$$

$$d \wedge \tilde{F} = 0 \quad -(12)$$

which are invariant under:

$$\boxed{\begin{array}{l} F \rightarrow \tilde{F} \\ \tilde{F} \rightarrow F \end{array}} \quad \begin{array}{l} -(13) \\ -(14) \end{array}$$

In this convention:

$$\boxed{\begin{array}{l} d \wedge F := \partial_\mu F_{\nu\rho} + \partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} \\ d \wedge \tilde{F} := \partial_\mu \tilde{F}_{\nu\rho} + \partial_\rho \tilde{F}_{\mu\nu} + \partial_\nu \tilde{F}_{\rho\mu} \end{array}} \quad -(15)$$

which is ~~the~~ index lowered convention that leads to:

$$\boxed{\begin{array}{l} d \wedge F := \partial_\mu \tilde{F}^{\mu\nu} \\ d \wedge \tilde{F} := \partial_\mu F^{\mu\nu} \end{array}} \quad -(16)$$

These equations are duality invariant, as Carroll points out on p. 24, chapt 1 of his 1997 notes.

In ECE ~~theory~~

$$\boxed{\begin{array}{l} d \wedge F^a = j^a / \epsilon_0 \\ d \wedge \tilde{F}^a = \tilde{j}^a / \epsilon_0 \end{array}} \quad -(17)$$

where:

$$F^a = d \wedge A^a + \omega^a{}_b \wedge A^b \quad -(18)$$

It is seen that the ECE field equations are duality invariant under:

$$\boxed{\begin{array}{ll} F^a \rightarrow \tilde{F}^a ; & j^a \rightarrow \tilde{j}^a \\ \tilde{F}^a \rightarrow F^a , & \tilde{j}^a \rightarrow j^a \end{array}} \quad -(19)$$

$$-(20)$$

In tensor notation:

$$\boxed{\partial_\mu \tilde{F}^{a\mu\nu} = \tilde{j}^{a\nu} / \epsilon_0} \longleftrightarrow \boxed{\partial_\mu F^{a\mu\nu} = j^{a\nu} / \epsilon_0}$$

where $\longleftrightarrow$ indicates the Hodge duality invariance.

Here:

$$j^a = \epsilon_0 A^{(0)} \left(R^a{}_b \wedge q^b - \omega^a{}_b \wedge T^b \right)$$

$$\tilde{j}^a = \epsilon_0 A^{(0)} \left(\tilde{R}^a{}_b \wedge q^b - \omega^a{}_b \wedge \tilde{T}^b \right)$$

$$\tilde{j}^{a\nu} = -\epsilon_0 A^{(0)} \left(\tilde{R}^a{}_\mu{}^{\mu\nu} + \omega^a{}_{\mu b} \tilde{T}^{b\mu\nu} \right)$$

$$j^{a\nu} = -\epsilon_0 A^{(0)} \left(R^a{}_\mu{}^{\mu\nu} + \omega^a{}_{\mu b} T^{b\mu\nu} \right)$$

$$-(21)$$

The duality invariance of the ECE field equations is an important property of the basic Cartan geometry.