1) 108(12): Orbits for Various Force Laws

The basic orbital equation is eq. (7.21) of Marion and Thornton:

$$\frac{d^2u}{d\phi^2} + u = -\frac{\mu r^2}{l^2} F(r), \quad -(1)$$

$$u = 1/r.$$

If the orbit is the logarithmic spiral:

$$r = k e^{\alpha\phi}, \quad \frac{1}{r} = \frac{1}{k} e^{-\alpha\phi} \quad -(2)$$

then

$$\frac{d}{d\phi}\left(\frac{1}{r}\right) = -\frac{\alpha}{k} e^{-\alpha\phi} \quad -(3)$$

$$\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right) = \frac{\alpha^2}{k} e^{-\alpha\phi} \quad -(4)$$

i.e.

$$\frac{d^2u}{d\phi^2} = \frac{\alpha^2}{r} \quad -(5)$$

So:

$$F(r) = -\frac{l^2}{\mu r^3}\left(1+\alpha^2\right) \quad -(6)$$

The orbit of an inverse cubed force law is the logarithmic spiral of type (2).

If we now use the spiral of type:

$$\boxed{r^3 = k e^{\alpha\phi}, \quad e^{-\alpha\phi} = \frac{k}{r^3}} \quad -(7)$$

we find that:

2)

$$\frac{d^2u}{d\phi^2} = \frac{\alpha^2}{k} e^{-\alpha\phi} = \frac{\alpha^2}{r^3} \quad — (8)$$

So in eq (1):

$$\frac{1}{r} + \frac{\alpha^2}{r^3} = -\frac{\mu r^2}{l^2} F(r) \quad — (9)$$

$$\boxed{F(r) = -\frac{l^2}{\mu}\left(\frac{1}{r^3} + \frac{\alpha^2}{r^5}\right)} \quad — (10)$$

This is precisely the correct extra force law given by:

$$r_S = \frac{2GM}{c^2} + \frac{a}{r} \quad — (11)$$

in the relativistic Kepler problem. So eq (11) produces the extra attractive force $F(r)$ and results in the spiral orbit:

$$r^3 = k e^{\alpha\phi} \quad — (7)$$

i.e.

$$\boxed{r = k^{1/3} \exp\left(\frac{\alpha}{3}\phi\right)} \quad — (12)$$

This is another <u>logarithmic spiral.</u>

Finally, note that for a periodic $F(r)$, eq (1) is an <u>undamped</u> <u>resonator.</u>