110(1) : Cross Check on the Derivation of the ECE Lemma

Derivation 1

The Lemma is based on the tetrad postulate:

$$D_\mu q^a_{\ \sigma} = 0. \quad -(1)$$

Therefore:

$$D^\mu \left(D_\mu q^a_{\ \sigma} \right) = 0 \quad -(2)$$

By definition:

$$D^\mu \phi = \partial^\mu \phi \quad -(3)$$

where ϕ is scalar valued. Therefore:

$$D^\mu 0 = \partial^\mu 0 \quad -(4)$$

i.e all elements of $D_\mu q^a_{\ \sigma}$ are zero and all elements are scalar valued. Therefore eq (4) is true for all elements and therefore:

$$\partial^\mu \left(D_\mu q^a_{\ \sigma} \right) = 0 \quad -(5)$$

Using the rule for covariant derivative of a mixed index rank two tensor, $q^a_{\ \mu}$, then:

$$\partial^\mu \left(\partial_\mu q^a_{\ \sigma} + \omega^a_{\ \mu b} q^b_{\ \sigma} - \Gamma^\lambda_{\ \mu\sigma} q^a_{\ \lambda} \right) = 0 \quad -(6)$$

i.e.

$$\Box q^a_{\ \sigma} = \partial^\mu \left(\Gamma^\lambda_{\ \mu\sigma} q^a_{\ \lambda} - \omega^a_{\ \mu b} q^b_{\ \sigma} \right) \quad -(7)$$

where

$$\Box := \partial^\mu \partial_\mu . \quad -(8)$$

Now define a scalar curvature R by:

2)

$$R q^a_{\ \sigma} := \partial^\mu \left(\Gamma^\lambda_{\mu\sigma} q^a_{\ \lambda} - \omega^a_{\ \mu b} q^b_{\ \sigma} \right) \quad — (9)$$

Multiply both sides of eq. (9) by $q^\sigma_{\ a}$ and use:

$$q^a_{\ \sigma} q^\sigma_{\ a} = 4 \quad — (10)$$

Thus

$$\boxed{\square q^a_{\ \sigma} = R q^a_{\ \sigma}} \quad — (11)$$

where:

$$R := \frac{1}{4} q^\sigma_{\ a} \partial^\mu \left(\Gamma^\lambda_{\mu\sigma} q^a_{\ \lambda} - \omega^a_{\ \mu b} q^b_{\ \sigma} \right)$$

$$— (12)$$

The ECE Lemma (11) is a basic geometrical property of Cartan. It is derived by re-arranging the well known tetrad postulate. It has been shown that all the wave equations of physics can be derived from eq. (11).

Derivation 2

Regard $D_\mu q^a_{\ \sigma}$ as a rank three mixed index tensor. Its covariant derivative is therefore:

$$D^\mu \left(D_\mu q^a_{\ \sigma} \right) = \partial^\mu \left(D_\mu q^a_{\ \sigma} \right) + \omega^{a\mu}_{\ \ b} D_\mu q^b_{\ \sigma} - \Gamma^{\lambda\mu}_{\ \ \mu} D_\lambda q^a_{\ \sigma}$$

$$- \Gamma^{\lambda\mu}_{\ \ \sigma} D_\lambda q^a_{\ \mu}$$

$$= 0 \quad — (13)$$

3)

Now use the tetrad postulate:

$$D_\mu q^b_{\ \sigma} = D_\lambda q^a_{\ \sigma} = D_\lambda q^a_{\ \mu} = 0 \quad -(14)$$

to find eq. (5) again, Q.E.D.

Therefore the correctness of eq. (5) has been cross checked.

Finally the ECE wave equation is found by the postulate:

$$R = -kT \quad -(15)$$

where k is Einstein's constant and where T is an index reduced canonical energy-momentum. It is now known that eq. (15) is the correct version of Einstein's original postulate. The latter was derived from the so-called "second Bianchi identity" and the Noether Theorem. In so doing no account of torsion was made. The original derivation by Einstein of eq. (15) is:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = k T_{\mu\nu}. \quad -(16)$$

Multiply on both sides by $g^{\mu\nu}$ and use:

4)

$$g^{\mu\nu} g_{\mu\nu} = 4 \quad -(17)$$

$$R := g^{\mu\nu} R_{\mu\nu} \quad -(18)$$

$$T := g^{\mu\nu} T_{\mu\nu} \quad -(19)$$

to find:

$$R = -kT \quad -(20)$$

Note carefully that R in eq. (18) is defined in a different way from the R of ECE theory. This is why eq (12) is known as the ECE scalar curvature. The metric in ECE theory is defined by:

$$g_{\mu\nu} = q^a_\mu q^b_\nu \eta_{ab} \quad -(21)$$

where η_{ab} is the Minkowski metric. Eq. (21) is given for example in Carroll chapter 3.

The "second Bianchi identity" is:

$$D \wedge R = 0 \quad -(22)$$

whereas the correct identity (paper 88) is:

$$D \wedge (D \wedge T) := D \wedge (R \wedge q) \quad -(23)$$