1)

110(10): Cartan and Christoffel Torsions in the so-called Schwarzschild Metric.

In this case:

$$T^{\kappa}_{\mu\nu} = q^{\kappa}_{a} T^{a}_{\mu\nu} = 0 \quad -(1)$$

where $T^{\kappa}_{\mu\nu}$ is the Christoffel torsion, q^{κ}_{a} is the inverse tetrad and $T^{a}_{\mu\nu}$ is the Cartan torsion. If the tetrad metric is diagonal, all torsions vanish. As shown in note 110(9) the use of a diagonal tetrad is inconsistent. Therefore assume that the q^{1}_{0} and q^{0}_{1} elements exist, i.e.

$$q^{a}_{\mu} = \begin{bmatrix} q^{0}_{0} & q^{0}_{1} & 0 & 0 \\ q^{1}_{0} & q^{1}_{1} & 0 & 0 \\ 0 & 0 & q^{2}_{2} & 0 \\ 0 & 0 & 0 & q^{3}_{3} \end{bmatrix} \quad -(2)$$

By definition: $$g_{\mu\nu} = q^{a}_{\mu} q^{b}_{\nu} \eta_{ab}, \quad -(3)$$

so $$g_{00} = q^{0\,2}_{0} - q^{1\,2}_{0} \quad -(4)$$

$$g_{11} = q^{0\,2}_{1} - q^{1\,2}_{1}. \quad -(5)$$

The tetrad postulate is:

$$D_{\mu} q^{a}_{\sigma} = \partial_{\mu} q^{a}_{\sigma} + \omega^{a}_{\mu b} q^{b}_{\sigma} - \Gamma^{\lambda}_{\mu\sigma} q^{a}_{\lambda} = 0 \quad -(6)$$

2) From eq. (1)

$$q^0_0 T^0_{\mu\nu} + q^0_1 T^1_{\mu\nu} = 0 \quad -(7)$$

and $$q^1_0 T^0_{\mu\nu} + q^1_1 T^1_{\mu\nu} = 0. \quad -(8)$$

However:

$$T^a_{\mu\nu} = q^a_\kappa T^\kappa_{\mu\nu} \quad -(9)$$

so if $T^\kappa_{\mu\nu} = 0$, $T^a_{\mu\nu}$ is also zero for all q^a_κ.

For the Christoffel connection, all elements of ~~of~~ the Christoffel and Cartan torsion are zero. This result is true for all tetrads and connections.

Checks a Self Consistency

In the tetrad postulate (6), set:

$$\mu = a = \sigma = 0 \quad -(10)$$

Therefore:

$$\partial_0 q^0_0 + \omega^0_{0b} q^b_0 - \Gamma^\lambda_{00} q^0_\lambda = 0 \quad -(11)$$

and:

$$\partial_0 q^0_0 + \omega^0_{00} q^0_0 + \omega^0_{01} q^1_0 - \Gamma^0_{00} q^0_0 - \Gamma^1_{00} q^0_1 = 0 \quad -(12)$$

In the so-called Schwarzschild metric there is no time dependence, so:

$$\partial_0 q^0_0 = 0 \quad -(13)$$

In the tetrad postulate (6), set:

$$\mu = 0, \quad a = 0, \quad \sigma = 1 \quad -(14)$$

so:

$$\partial_0 q^0_1 + \omega^0_{0b} q^b_1 - \Gamma^\lambda_{01} q^0_\lambda = 0 \quad -(15)$$

i.e.
$$\omega^0_{00} q^0_1 + \omega^0_{01} q^1_1 - \Gamma^0_{01} q^0_0 - \Gamma^1_{01} q^0_1 = 0 \quad -(16)$$

and from (12)

$$\omega^0_{00} q^0_0 + \omega^0_{01} q^1_0 - \Gamma^0_{00} q_1 - \Gamma^1_{00} q^0_1 = 0 \quad -(17)$$

In these equations:

$$\Gamma^0_{01} = \frac{GM}{r^2c^2}\left(1 - \frac{2GM}{rc^2}\right)^{-1} \quad -(18)$$

$$\Gamma^1_{00} = \frac{GM}{r^2c^2}\left(1 - \frac{2GM}{rc^2}\right) \quad -(19)$$

$$\Gamma^1_{01} = 0 \quad -(20)$$

$$\Gamma^0_{00} = 0 \quad -(21)$$

$-(22)$

Therefore:

$$\omega^0_{00} q^0_1 + \omega^0_{01} q^1_1 - \Gamma^0_{01} q^0_0 = 0$$

$$\omega^0_{00} q^0_0 + \omega^0_{01} q^1_0 - \Gamma^1_{00} q^0_1 = 0 \quad -(23)$$

4)

i.e.

$$\boxed{\left(q^0_{\ 1}+q^0_{\ 0}\right)\omega^0_{\ 00}+\left(q^1_{\ 0}+q^1_{\ 1}\right)\omega^0_{\ 01} = \Gamma^0_{\ 01}\,q^0_{\ 0}+\Gamma^1_{\ 00}\,q^0_{\ 1}} \quad -(24)$$

Therefore there is no self-inconsistency produced by the assumption (2) on the structure of the tetrad matrix.

From eq. (24) the off diagonal elements of the tetrad can be defined in terms of the diagonal elements and spin connection. The torsion is zero, so:

$$d\wedge q+\omega\wedge q=0 \quad -(25)$$

$$\partial_\mu q^a_{\ \nu}-\partial_\nu q^a_{\ \mu}+\omega^a_{\ \mu b}\,q^b_{\ \nu}-\omega^a_{\ \nu b}\,q^b_{\ \mu}=0 \quad -(26)$$

Set:

$$a=\mu=0,\ \nu=1 \quad -(27)$$

Then:

$$\partial_0 q^0_{\ 1}-\partial_1 q^0_{\ 0}+\omega^0_{\ 0b}\,q^b_{\ 1}-\omega^0_{\ 1b}\,q^b_{\ 0}=0$$

i.e.

$$\partial_1 q^0_{\ 0}=\omega^0_{\ 00}\,q^0_{\ 1}+\omega^0_{\ 01}\,q^1_{\ 1}-\omega^0_{\ 10}\,q^0_{\ 0}-\omega^0_{\ 11}\,q^1_{\ 0} \quad -(28)$$

5) In eq. (26), set:

$$a = 1, \quad \mu = 0, \quad \nu = 1 \qquad \text{— (29)}$$

then:

$$\partial_0 q^1_1 - \partial_1 q^1_0 + \omega^1{}_{0b} q^b_1 - \omega^1{}_{1b} q^b_0 = 0 \qquad \text{— (30)}$$

i.e.

$$\partial_1 q^1_0 = \omega^1{}_{0b} q^b_1 - \omega^1{}_{1b} q^b_0 \qquad \text{— (31)}$$

$$\partial_1 q^1_0 = \omega^1{}_{00} q^0_1 + \omega^1{}_{01} q^1_1 - \omega^1{}_{10} q^0_0 - \omega^1{}_{11} q^1_0 \qquad \text{— (32)}$$

and

$$\partial_1 q^0_0 = \omega^0{}_{00} q^0_1 + \omega^0{}_{01} q^1_1 - \omega^0{}_{10} q^0_0 - \omega^0{}_{11} q^1_0, \qquad \text{— (33)}$$

where

$$\omega^0{}_{00} q^0_1 + \omega^0{}_{01} q^1_1 = \Gamma^0{}_{01} q^0_0 \qquad \text{— (22)}$$

and

$$\omega^0{}_{00} q^0_0 + \omega^0{}_{01} q^1_0 = \Gamma^1{}_{00} q^0_1 \qquad \text{— (23)}$$

Fm eq (22) & (33):

$$(\partial_1 + \omega^0{}_{10}) q^0_0 = \Gamma^0{}_{01} q^0_0 - \omega^0{}_{11} q^1_0 \qquad \text{— (24)}$$

i.e.

$$\boxed{(\partial_1 + \omega^1{}_{10} - \Gamma^0{}_{01}) q^0_0 + \omega^0{}_{11} q^1_0 = 0} \qquad \text{— (25)}$$

6)

Complete Set of Equations

$$g_{00} = q^{0}_{0}{}^{2} - q^{1}_{0}{}^{2} \quad -(26)$$

$$g_{11} = q^{0}_{1}{}^{2} - q^{1}_{1}{}^{2} \quad -(27)$$

$$(q^{0}_{1} + q^{0}_{0})\,\omega^{0}{}_{00} + (q^{1}_{0} + q^{1}_{1})\,\omega^{0}{}_{01} \quad -(28)$$

$$= \Gamma^{0}{}_{01}\, q^{0}_{0} + \Gamma^{1}{}_{00}\, q^{0}_{1}$$

$$(\partial_1 + \omega^{1}{}_{10} - \Gamma^{0}{}_{01})\, q^{0}_{0} + \omega^{0}{}_{11}\, q^{1}_{0} = 0 \quad -(29)$$

If it is assumed that:

$$q^{1}_{0} = q^{0}_{1} \quad -(30)$$

Then:

$$g_{00} = q^{0}_{0}{}^{2} - g_{11} + q^{1}_{1}{}^{2} \quad -(32)$$

and

$$q^{0}_{0}{}^{2} + q^{1}_{1}{}^{2} = g_{00} + g_{11} \quad -(33)$$

$$= 1 - \frac{2GM}{rc^2} + \left(1 - \frac{2GM}{rc^2}\right)^{-1} \quad -(34)$$

A particular solution of eq. (34) is:

$$q^{0}_{0} = \left(1 - \frac{2GM}{rc^2}\right)^{1/2}, \quad -(35)$$

$$q^{1}_{1} = \left(1 - \frac{2GM}{rc^2}\right)^{-1/2}. \quad -(36)$$

$$q^{1}_{0} = q^{0}_{1} = 0 \quad -(37)$$

7) There is no self-inconsistency because:

$$q^0_0 \omega^0_{00} + q^1_1 \omega^0_{01} = \Gamma^0_{01} q^0_0 \quad -(38)$$

and:

$$(\partial_1 + \omega^1_{10}) q^0_0 = \Gamma^0_{01} q^0_0 \quad -(39)$$

Therefore with spin connection defined in this way, the diagonal tetrad approximation is valid.