1) 110(11): Cartan Torsion for Rotation about the Z Axis

Rotation about the Z axis is defined by:

$$\begin{bmatrix} V_X' \\ V_Y' \\ V_Z' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_X \\ V_Y \\ V_Z \end{bmatrix} \quad -(1)$$

The rotation takes place at an angular velocity:

$$\omega = \frac{d\theta}{dt}. \quad -(2)$$

The tetrad is:

$$q^a_\mu = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad -(3)$$

and the spin connection is:

$$\omega^a{}_b = \kappa \epsilon^a{}_{bc} q^c \quad -(4)$$
$$= \frac{\omega}{c} \epsilon^a{}_{bc} q^c .$$

Therefore:

$$\left.\begin{aligned} q^1_1 &= \cos\theta, & q^1_2 &= \sin\theta, & q^1_3 &= 0, \\ q^2_1 &= -\sin\theta, & q^2_2 &= \cos\theta, & q^2_3 &= 0, \\ q^3_1 &= 0, & q^3_2 &= 0, & q^3_3 &= 1 \end{aligned}\right\} \quad -(5)$$

In the Cartesian coordinate system there are three tetrad vectors:

2)

$$\underline{q}^1 = \cos\theta \, \underline{i} + \sin\theta \, \underline{j} \quad -(6)$$

$$\underline{q}^2 = -\sin\theta \, \underline{i} + \cos\theta \, \underline{j} \quad -(7)$$

$$\underline{q}^3 = \underline{k} \quad -(8)$$

The spin connection and tetrad components are related by:

$$\omega^1{}_2 = \frac{\omega}{c} \epsilon^1{}_{23} q^3 = \kappa q^3 \quad -(9)$$

$$\omega^2{}_3 = \frac{\omega}{c} \epsilon^2{}_{31} q^1 = \kappa q^1 \quad -(10)$$

$$\omega^3{}_1 = \frac{\omega}{c} \epsilon^3{}_{12} q^2 = \kappa q^2 \quad -(11)$$

Therefore the non-zero spin connection components are:

$$\left.\begin{array}{ll} \omega^2{}_{13} = \kappa q^1{}_1, & \omega^3{}_{11} = \kappa q^2{}_1, \\ \omega^2{}_{23} = \kappa q^1{}_2, & \omega^3{}_{21} = \kappa q^2{}_2, \\ \omega^1{}_{32} = \kappa q^3{}_3 & \end{array}\right] \quad -(12)$$

The Cartan torsion is defined by:

$$T^a{}_{\mu\nu} = \partial_\mu q^a{}_\nu - \partial_\nu q^a{}_\mu + \omega^a{}_{\mu b} q^b{}_\nu - \omega^a{}_{\nu b} q^b{}_\mu \quad -(13)$$

It is assumed that θ depends only on t to produce the angular velocity (2). Therefore for example:

3)

$$T^{1}{}_{01} = \partial_0 q^1{}_1 - \partial_1 q^1{}_0 + \omega^1{}_{0b} q^b{}_1 - \omega^1{}_{1b} q^b{}_0 \quad -(14)$$

$$= \partial_0 q^1{}_1 = \frac{1}{c}\frac{\partial}{\partial t}\cos\theta(t)$$

$$= -\frac{1}{c}\left(\frac{\partial\theta}{\partial t}\right)\sin\theta(t) = -\frac{\omega}{c}\sin\theta(t) \quad -(15)$$

It is found that:

$$\left.\begin{aligned} T^1{}_{01} &= -\frac{\omega}{c}\sin\theta(t), & T^2{}_{02} &= -\frac{\omega}{c}\sin\theta(t) \\ T^1{}_{02} &= \frac{\omega}{c}\cos\theta(t), & T^2{}_{01} &= -\kappa\cos\theta(t) \end{aligned}\right\} -(16)$$

i.e. :

$$T^a{}_{0\mu} = \frac{1}{c}\frac{\partial q^a{}_\mu}{\partial t}$$

$$\boxed{T^a{}_{0\mu} = \frac{\omega}{c}\begin{bmatrix} -\sin\theta & \cos\theta & 0 \\ -\cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix}} \quad -(17)$$

The rotation generator about Z is defined by:

$$J^a{}_\mu = \frac{1}{i}\left.\frac{\partial q^a{}_\mu}{\partial t}\right|_{\theta=0} = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad -(18)$$

(Ryder, "Quantum Field Theory", p. 31)

4)

Therefore:

$$\boxed{J^a_\mu = \frac{1}{i} T^a_{0\mu}\Big|_{\theta=0}} \quad -(19)$$

Conclusion

The Cartan torsion is proportional to the rotation generator of the Lorentz group.

Eq. (19) can be written as:

$$J_Z = -i\, T_Z(\theta=0) \quad -(20)$$

In 4-D (Ryder page 37)

$$J_Z = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad -(21)$$

and:

$$[K_X, K_Y] = -i J_Z \quad -(22)$$

where K_X and K_Y are Lorentz boosts. Also:

$$[J_X, J_Y] = i J_Z \quad -(23)$$

et cyclicum.

Therefore the three rotation generators of the

5) Lorentz group or Poincaré group are three Cartan torsions.

The Thomas precession is explained by eq. (22), where K_X and K_Y are Lorentz boost generators.

Discussion

The Thomas precession cannot be described by the Christoffel symbol because the Christoffel torsion is always zero:

$$T^{\kappa}_{\mu\nu} = \Gamma^{\kappa}_{\mu\nu} - \Gamma^{\kappa}_{\nu\mu}$$

$$= 0 \quad -(24)$$

and so the Cartan torsion is zero:

$$T^{a}_{\mu\nu} = q^{a}_{\kappa} T^{\kappa}_{\mu\nu} = 0 \quad -(25)$$

The Thomas precession is a rotating Minkowski frame, and the simplest example of rotation in relativity theory. The Thomas precession is due to the Cartan torsion.