1) 110(12) : Derivation of the Lorentz Angle Ω from the Cartan Torsion

Consider a Lorentz boost in the X axis :

$$t' = \gamma(t + vx/c^2)$$

$$x' = \gamma(x + vt), \quad y' = y, \quad t' = \gamma(t + vx/c^2), \quad x^0 = ct, \quad x^1 = x$$

$$\gamma = (1 - v^2/c^2)^{-1/2}, \quad \beta = v/c, \quad v/c = \tanh\phi \quad -(1)$$

$$\gamma = \cosh\phi, \quad \gamma\beta = \sinh\phi,$$

Then:

$$\begin{bmatrix} x^{0\prime} \\ x^{1\prime} \\ x^{2\prime} \\ x^{3\prime} \end{bmatrix} = \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}. \quad -(2)$$

The Lorentz boost matrix is:

$$B_X = \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad -(3)$$

Similarly (Enigmatic Photon, volume 4, p. 91 ff):

$$B_Y = \begin{bmatrix} \gamma & 0 & \gamma\beta & 0 \\ 0 & \gamma & 0 & 0 \\ \gamma\beta & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad -(4)$$

The boost infinitesimal generators are:

$$K_X = -i \left.\frac{\partial B_X}{\partial \phi}\right|_{\phi \to 0} = -i \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad -(5)$$

$$K_Y = -i \left.\frac{\partial B_Y}{\partial \phi}\right|_{\phi \to 0} = -i \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad -(6)$$

2)

The infinitesimal rotation generator about Z is:

$$J_Z = -i \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad -(7)$$

Therefore:

$$[K_x, K_y] = -i J_Z. \quad -(8)$$

It is found that:

$$[B_x, B_y] = i \gamma^2 \beta^2 J_Z \quad -(9)$$

$$\boxed{[B_x, B_y] = i \left(\frac{v}{c}\right)^2 \left(1 - \frac{v^2}{c^2}\right)^{-1} J_Z} \quad -(10)$$

This is a relativistic result, the commutator of two Lorentz boosts is a relativistically corrected infinitesimal rotation generator.

In notes 110 (11) the infinitesimal torsion generator T_Z was found to be:

$$T_Z = \frac{\omega}{c} J_Z \quad -(11)$$

where the angular velocity ω is:

$$\omega = \frac{d\theta}{dt} \quad -(12)$$

about the Z axis.

Now define the relativistically corrected

3) infinitesimal torsion generator as:

$$T_Z = \left(1 - \frac{v^2}{c^2}\right)^{-1} \frac{\omega}{c} J_Z \quad - (13)$$

$$\boxed{T_Z = \frac{\Omega}{c} J_Z} \quad - (14)$$

where:

$$\Omega = \omega \left(1 - \frac{v^2}{c^2}\right)^{-1} \quad - (15)$$

is the Thomas angle.

In notes 110(7) it was shown that the Thomas proper time is:

$$d\tau = \left(1 - \frac{v^2}{c^2}\right)^{1/2} dt \quad - (16)$$

For a rotation through 2π, the Thomas precession is:

$$\begin{aligned} \alpha &= \Omega \, d\tau - 2\pi \\ &= 2\pi \left(\left(1 - \frac{v^2}{c^2}\right)^{-1/2} - 1 \right) \end{aligned} \quad - (17)$$

where $\omega \, dt = 2\pi$ — (18)

THOMAS PRECESSION IS DUE TO THE CARTAN TORSION.