110(6): Tetrads of the Lorentz Transformation

In this note it is shown that the Lorentz transformation generates Cartan torsion. The latter is therefore present throughout physics, even in special relativity.

The Lorentz transformation is denoted (Carroll chapter 1)

$$x^{\mu'} = \Lambda^{\mu'}{}_{\nu} x^{\nu} \quad -(1)$$

The Lorentz matrix $\Lambda^{\mu'}{}_{\nu}$ is a tetrad matrix by definition of the tetrad:

$$V^a = q^a_\mu V^\mu \quad -(2)$$

Here:

$$a = \mu', \quad q^a_\mu = \Lambda^{\mu'}{}_{\nu}, \quad V^a = x^{\mu'}, \quad V^\mu = x^\nu \quad -(3)$$

The tetrad is thought of as the matrix that links two vectors. The usual application of the tetrad is to a tangential vector V^a and base manifold vector V^μ. In eq. (1) however both vectors $x^{\mu'}$ and x^ν are in the base manifold. The usual rules of differential geometry are then used to define the Cartan torsion from the tetrad. The spin connection is defined through the Lorentz transform matrix.

Rotations

$$\Lambda^{\mu'}{}_{\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad -(3)$$

2)

Lorentz Boost in x

$$\Lambda^{\mu'}{}_{\nu} = \begin{bmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad -(4)$$

where

$$\phi = \tanh^{-1}\frac{x}{t} \quad -(5)$$

$$\left.\begin{aligned} t' &= t\cosh\phi - x\sinh\phi, \\ x' &= -t\sinh\phi + x\cosh\phi, \\ v &= x/t, \\ t' &= \gamma(t - vx), \quad x' = \gamma(x - vt). \end{aligned}\right\} \quad -(6)$$

Tetrad Elements for Rotation

$$\left.\begin{aligned} q^0_0 &= 1, & q^0_1 &= 0, & q^0_2 &= 0, & q^0_3 &= 0, \\ q^1_0 &= 0, & q^1_1 &= \cos\theta, & q^1_2 &= \sin\theta, & q^1_3 &= 0, \\ q^2_0 &= 0, & q^2_1 &= -\sin\theta, & q^2_2 &= \cos\theta, & q^2_3 &= 0, \\ q^3_0 &= 0, & q^3_1 &= 0, & q^3_2 &= 0, & q^3_3 &= 1. \end{aligned}\right] \quad -(7)$$

The metric is defined by:

$$g_{\mu\nu} = q^a_\mu q^b_\nu \eta_{ab}, \quad -(8)$$

$$\eta_{ab} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad -(9)$$

3)

For example:

$$g_{00} = q^a_0 \, q^b_0 \, \eta_{ab}$$
$$= -q^0_0 q^0_0 + q^1_0 q^1_0 + q^2_0 q^2_0 + q^3_0 q^3_0$$
$$= -1 \qquad -(10)$$

Therefore:

$$g_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \qquad -(11)$$

This means that all elements of the conventionally defined Christoffel symbols and Riemann tensor elements are zero. There is therefore no conventionally defined curvature and no conventionally defined torsion.

There is however torsion from the Lorentz matrix.

Spacetime Torsion from the Lorentz Matrix

This is defined in the notation of differential geometry as:

$$T^a_{\mu\nu} = \partial_\mu q^a_\nu - \partial_\nu q^a_\mu + \omega^a_{\mu b} q^b_\nu - \omega^a_{\nu b} q^b_\mu . \qquad -(12)$$

From eqs. (7):

$$T^a_{\mu\nu} = \omega^a_{\mu b} q^b_\nu - \omega^a_{\nu b} q^b_\mu \qquad -(13)$$

because the tetrads have no space and time dependence, they depend only on θ, and we consider a constant

4) i.e. for a given angle of rotation θ.

The Spin Connection Elements

These are defined in Carroll's eq. (3.130) of his 1997 notes:

$$D_\mu X^a = \partial_\mu X^a + \omega^a_{\mu b} X^b \quad -(14)$$

The spin connections define the movement of the frame of reference. An example of a spin connection matrix is the one that defines a frame that has rotated with respect to another frame. Its units must be inverse metres. Therefore:

$$\boxed{\omega^a_{\mu b} = \kappa_\mu \Lambda^a{}_b} \quad -(15)$$

i.e.

$$\omega^a_{\mu b} = \kappa_\mu \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad -(16)$$

In previous work it was found that for pure rotation:

$$\omega^a_{\mu b}(A) = \kappa_0 \epsilon^a{}_{bc} q^c_\mu \quad -(17)$$

where $\epsilon^a{}_{bc}$ is the anti-symmetric unit tensor.

In order for eq. (17) to be true:

5)

$$a \neq b \neq c, \quad -(18)$$

for example:

$$a = 1, \ b = 2, \ c = 3, \quad -(19)$$

and

$$\epsilon^{a}{}_{bc} = -\epsilon^{b}{}_{ac}. \quad -(20)$$

Therefore:

$$\omega^{a}{}_{\mu b}(A) = -\omega^{b}{}_{\mu a}(A). \quad -(21)$$

From eq. (16) the antisymmetric part of $\omega^{a}{}_{\mu b}$ is:

$$\boxed{\omega^{a}{}_{\mu b}(A) = \kappa_\mu \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \sin\theta & 0 \\ 0 & -\sin\theta & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}} \quad -(22).$$

There are only two non-zero elements:

$$\omega^{1}{}_{\mu 2}(A) = -\omega^{2}{}_{\mu 1}(A) = \kappa_\mu \sin\theta \quad -(23)$$

From eqs. (7), (17) and (23), the only non-zero elements of the spin connection are:

$$\boxed{\omega^{1}{}_{32}(A) = -\omega^{2}{}_{31}(A) = \kappa_3 \sin\theta = \kappa_0} \quad -(24)$$

Thus:

$$\boxed{\sin\theta = \frac{\kappa_0}{\kappa_3}} \quad -(25)$$

6) The non-vanishing elements of torsion are found from eq. (13):

$$T^1{}_{3\nu} = \omega^1{}_{32}(A)\, q^2_\nu \qquad -(26)$$

$$T^2{}_{3\nu} = \omega^2{}_{31}(A)\, q^1_\nu \qquad -(27)$$

Therefore:

$$\left.\begin{aligned} T^1{}_{31} &= \omega^1{}_{32}(A)\, q^2_1 \\ T^1{}_{32} &= \omega^1{}_{32}\, q^2_2 \\ T^2{}_{31} &= \omega^2{}_{31}(A)\, q^1_1 \\ T^2{}_{32} &= \omega^2{}_{31}(A)\, q^1_2 \end{aligned}\right\} \qquad -(28)$$

i.e.

$$\boxed{\left.\begin{aligned} T^1{}_{31} &= -K_3 \sin^2\theta \\ T^1{}_{32} &= K_3 \sin\theta\cos\theta \\ T^2{}_{31} &= -K_3 \sin\theta\cos\theta \\ T^2{}_{32} &= -K_3 \sin^2\theta \end{aligned}\right\} \qquad -(29)}$$

Finally the antisymmetry of torsion is checked by:

$$T^1{}_{32} = \omega^1{}_{32}\, q^2_2 - \omega^1{}_{23}\, q^3_3 = \omega^1{}_{32}\, q^2_2$$

$$T^1{}_{23} = \omega^1{}_{23}\, q^3_3 - \omega^1{}_{32}\, q^2_2 = -\omega^1{}_{32}\, q^2_2$$

$$-(30)$$

7) The Gamma Connections

These are found from the tetrads and spin connections using the tetrad postulate:

$$D_\mu q^a_\sigma = \partial_\mu q^a_\sigma + \omega^a_{\mu b} q^b_\sigma - \Gamma^\lambda_{\mu\sigma} q^a_\lambda$$
$$= 0 \quad -(31)$$

In the case of rotation:

$$\omega^a_{\mu b} q^b_\sigma = \Gamma^\lambda_{\mu\sigma} q^a_\lambda, \quad -(32)$$

so for example:

$$T^1_{31} = \omega^1_{32}(A)\, q^2_1 \quad -(33)$$
$$= \Gamma^\lambda_{31} q^1_\lambda$$

i.e. $$T^1_{31} = \Gamma^1_{31}\cos\theta + \Gamma^2_{31}\sin\theta \quad -(34)$$

Finally use:

$$T^\kappa_{\mu\nu} = T^a_{\mu\nu} q^\kappa_a \quad -(35)$$

to find that:

$$T^\kappa_{\mu\nu} = \Gamma^\kappa_{\mu\nu} - \Gamma^\kappa_{\nu\mu} \quad -(36)$$

Note carefully that $\Gamma^\kappa_{\mu\nu}$ is <u>not</u> the Christoffel connection, which is zero.