110(8): Factorization of the Metric into Tetrads.

In this note the condition is defined under which the Cartan torsion may be non-zero when the Christoffel torsion is zero.

In general:

$$T^a_{\mu\nu} = T^\kappa_{\mu\nu} q^a_\kappa \quad -(1)$$

and:

$$T^\kappa_{\mu\nu} = q^\kappa_a T^a_{\mu\nu}. \quad -(2)$$

We consider the case of the Christoffel torsion:

$$T^\kappa_{\mu\nu} = \Gamma^\kappa_{\mu\nu} - \Gamma^\kappa_{\nu\mu} = 0. \quad -(3)$$

The Cartan torsion $T^a_{\mu\nu}$ in this case is defined by:

$$q^\kappa_a T^a_{\mu\nu} = 0 \quad -(4)$$

where

$$q^a_\kappa q^\kappa_a = 4. \quad -(5)$$

Therefore:

$$q^\kappa_0 T^0_{\mu\nu} + q^\kappa_1 T^1_{\mu\nu} + q^\kappa_2 T^2_{\mu\nu} + q^\kappa_3 T^3_{\mu\nu} = 0. \quad -(6)$$

In general the metric is defined by:

$$g_{\mu\nu} = q^a_\mu q^b_\nu \eta_{ab} \quad -(7)$$

where the Minkowski metric is:

$$\eta_{ab} = \text{diag}(1, -1, -1, -1). \quad -(8)$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad -(9)$$

2) Assume that $g_{\mu\nu}$ is diagonal:

$$g_{\mu\nu} = \begin{bmatrix} g_{00} & 0 & 0 & 0 \\ 0 & g_{11} & 0 & 0 \\ 0 & 0 & g_{22} & 0 \\ 0 & 0 & 0 & g_{33} \end{bmatrix} \quad -(10)$$

then:

$$\begin{bmatrix} g_{00} & 0 & 0 & 0 \\ 0 & g_{11} & 0 & 0 \\ 0 & 0 & g_{22} & 0 \\ 0 & 0 & 0 & g_{33} \end{bmatrix} = \begin{bmatrix} (q^0_0)^2 & (q^1_0)^2 & (q^2_0)^2 & (q^3_0)^2 \\ (q^0_1)^2 & (q^1_1)^2 & (q^2_1)^2 & (q^3_1)^2 \\ (q^0_2)^2 & (q^1_2)^2 & (q^3_3)^2 & (q^3_2)^2 \\ (q^0_3)^2 & (q^1_3)^2 & (q^2_3)^2 & (q^3_3)^2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad -(11)$$

Multiply both sides of eq. (11) by η_{ab} to obtain:

$$\begin{bmatrix} (q^0_0)^2 & (q^1_0)^2 & (q^2_0)^2 & (q^3_0)^2 \\ (q^0_1)^2 & (q^1_1)^2 & (q^2_1)^2 & (q^3_1)^2 \\ (q^0_2)^2 & (q^1_2)^2 & (q^3_3)^2 & (q^3_2)^2 \\ (q^0_3)^2 & (q^1_3)^2 & (q^2_3)^2 & (q^3_3)^2 \end{bmatrix} = \begin{bmatrix} g_{00} & 0 & 0 & 0 \\ 0 & g_{11} & 0 & 0 \\ 0 & 0 & g_{22} & 0 \\ 0 & 0 & 0 & g_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad -(12)$$

i.e.

$$\boxed{q^0_0 = g_{00}^{1/2},\ q^1_1 = g_{11}^{1/2},\ q^2_2 = g_{22}^{1/2},\ q^3_3 = g_{33}^{1/2}} \quad -(13)$$

and

$$q^0_1 = q^0_2 = \dots = q^3_2 = 0 \quad -(14)$$

Therefore when $g_{\mu\nu}$ is diagonal, the tetrads are the square roots in eq (13). All off-diagonal tetrad elements are zero.

Therefore in eq. (6):

$$q^0_0 T^0_{\mu\nu} + q^0_1 T^1_{\mu\nu} + q^0_2 T^2_{\mu\nu} + q^0_3 T^3_{\mu\nu} = 0 \quad -(15)$$

and so on. This means that

$$T^a_{\mu\nu} = 0 \quad -(16)$$

<u>Conclusion</u>

When $g_{\mu\nu}$ is diagonal, then $T^a_{\mu\nu}$ is zero if $T^\kappa_{\mu\nu}$ is zero.

However, when $g_{\mu\nu}$ contains off diagonal elements, then the Cartan torsion may be non-zero when the Christoffel torsion is zero. For example, if g_{01} is non-zero, eq. (12) becomes:

$$\begin{bmatrix} (q^0_0)^2 & (q^1_0)^2 & (q^2_0)^2 & (q^3_0)^2 \\ (q^0_1)^2 & (q^1_1)^2 & (q^2_1)^2 & (q^3_1)^2 \\ (q^0_2)^2 & (q^1_2)^2 & (q^3_3)^2 & (q^3_2)^2 \\ (q^0_3)^2 & (q^1_3)^2 & (q^2_3)^2 & (q^3_3)^2 \end{bmatrix} = \begin{bmatrix} g_{00} & g_{01} & 0 & 0 \\ g_{10} & g_{11} & 0 & 0 \\ 0 & 0 & g_{22} & 0 \\ 0 & 0 & 0 & g_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad -(17)$$

i.e.

$$\left.\begin{aligned} (q^1_0)^2 &= -g_{01} \\ (q^0_1)^2 &= -g_{01} \end{aligned}\right\} \quad -(18)$$

and in eq. (15):

$$q^0_0 T^0_{\mu\nu} + q^0_1 T^1_{\mu\nu} = 0 \quad -(19)$$

4) and:

$$q^1_0 T^0_{\mu\nu} + q^1_1 T^1_{\mu\nu} = 0 \quad - (20)$$

i.e. $$\left(q^1_1 q^0_0 - q^1_0 q^0_1\right) T^1_{\mu\nu} = 0 \quad - (21)$$

This means that if:

$$\boxed{q^1_1 q^0_0 = q^1_0 q^0_1} \quad - (22)$$

then $T^1_{\mu\nu}$ is not necessarily zero.

Similarly:

$$\left(q^1_0 q^0_1 - q^1_1 q^0_0\right) T^0_{\mu\nu} = 0 \quad - (23)$$

and if eq. (22) is true then $T^0_{\mu\nu}$ is not necessarily zero.

<u>Thomas Precession</u>

In this case:

$$\left.\begin{aligned} q^0_0 = g_{00}^{1/2} = \left(1 - \frac{v^2}{c^2}\right)^{1/2}, \quad q^1_1 = 1 \\ q^2_2 = r, \quad q^3_3 = 1 \end{aligned}\right\} \quad - (24)$$

and $$q^0_2 = \sqrt{2}\, r\omega^{1/2} = q^2_0. \quad - (25)$$

Eq (22) becomes:

$$q^2_0 q^0_2 = q^2_2 q^0_0 \quad - (26)$$

i.e. $$2r^2\omega = \left(1 - \frac{v^2}{c^2}\right)^{1/2} r$$