110(9) : Diagonalization of the Tetrad Matrix.

The definition of the tetrad in terms of the metric is:

$$g_{\mu\nu} = q^a_\mu q^b_\nu \eta_{ab} \quad -(1)$$

where:

$$\eta_{ab} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad -(2)$$

Now assume that the metric is diagonal:

$$g_{\mu\nu} = \begin{bmatrix} g_{00} & 0 & 0 & 0 \\ 0 & g_{11} & 0 & 0 \\ 0 & 0 & g_{22} & 0 \\ 0 & 0 & 0 & g_{33} \end{bmatrix} \quad -(3)$$

It follows from eq. (1) that:

$$g_{00} = q^0_0 q^0_0 \eta_{00} + q^1_0 q^1_0 \eta_{11} + q^2_0 q^2_0 \eta_{22} + q^3_0 q^3_0 \eta_{33}$$

$$g_{11} = q^0_1 q^0_1 \eta_{00} + q^1_1 q^1_1 \eta_{11} + q^2_1 q^2_1 \eta_{22} + q^3_1 q^3_1 \eta_{33}$$

$$g_{22} = q^0_2 q^0_2 \eta_{00} + q^1_2 q^1_2 \eta_{11} + q^2_2 q^2_2 \eta_{22} + q^3_2 q^3_2 \eta_{33}$$

$$g_{33} = q^0_3 q^0_3 \eta_{00} + q^1_3 q^1_3 \eta_{11} + q^2_3 q^2_3 \eta_{22} + q^3_3 q^3_3 \eta_{33}$$

$$-(4)$$

Eqs. (4) can be arranged as:

$$\begin{bmatrix} g_{00} & 0 & 0 & 0 \\ 0 & g_{11} & 0 & 0 \\ 0 & 0 & g_{22} & 0 \\ 0 & 0 & 0 & g_{33} \end{bmatrix} = \begin{bmatrix} q^{0\,2}_0 & 0 & 0 & 0 \\ 0 & q^{1\,2}_1 & 0 & 0 \\ 0 & 0 & q^{2\,2}_2 & 0 \\ 0 & 0 & 0 & q^{3\,2}_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$-(5)$$

2)

an arrangement that implies that off-diagonal elements of the tetrads vanish:

$$q^0_1 = \dots = q^3_2 = 0 \quad — (6)$$

Therefore:

$$g_{00} = q^{0\,2}_0, \quad g_{11} = q^{1\,2}_1, \quad g_{22} = q^{2\,2}_2, \quad g_{33} = q^{3\,2}_3.$$

$$— (7)$$

The tetrad matrix is defined by:

$$V'^a = q^a_\mu V^\mu. \quad — (8)$$

If the tetrad is diagonal:

$$\begin{bmatrix} V'^0 \\ V'^1 \\ V'^2 \\ V'^3 \end{bmatrix} = \begin{bmatrix} q^0_0 & 0 & 0 & 0 \\ 0 & q^1_1 & 0 & 0 \\ 0 & 0 & q^2_2 & 0 \\ 0 & 0 & 0 & q^3_3 \end{bmatrix} \begin{bmatrix} V^0 \\ V^1 \\ V^2 \\ V^3 \end{bmatrix} \quad — (9)$$

i.e.

$$\left.\begin{aligned} V'^0 &= q^0_0 V^0, \quad V'^1 = q^1_1 V^1, \\ V'^2 &= q^2_2 V^2, \quad V'^3 = q^3_3 V^3 \end{aligned}\right\} — (10)$$

An example of eq (10) is spacetime translation

$$x^{\mu\prime} = x^\mu + a^\mu \quad — (11)$$

Therefore a Lorentz transform of type (11) can be represented by a diagonal tetrad. However, 3)

in a Lorentz boost, a spacetime ~~E~~ rotation, there are off diagonal elements of the tetrad, because the tetrad elements are those of the Lorentz transfor matrix. The most general definition of the tetrad is used as in eq. (8), i.e. a matrix linking two vectors. A Lorentz transfor matrix is a matrix linking two vectors. In general, any two vectors ~~are~~ can be related by a matrix.

The Diagonal Tetrad Postulate

In general, the tetrad postulate is:

$$D_\mu q^a_\sigma = \partial_\mu q^a_\sigma + \omega^a_{\mu b} q^b_\sigma - \Gamma^\lambda_{\mu\sigma} q^a_\lambda \quad -(12)$$

When the tetrad is diagonal:

$$a = \sigma \quad -(13)$$

so, for example:

$$D_\mu q^0_0 = \partial_\mu q^0_0 + \omega^0_{\mu b} q^b_0 - \Gamma^\lambda_{\mu 0} q^0_\lambda = 0 \quad -(14)$$

Therefore:

$$b = \lambda = 0 \quad -(15)$$

and

$$D_\mu q^0_0 = \partial_\mu q^0_0 + \omega^0_{\mu 0} q^0_0 - \Gamma^0_{\mu 0} q^0_0 = 0 \quad -(16)$$

4) In Minkowski spacetime, and for diagonal tetrads:

$$q^0_0 = 1, \; q^1_1 = 1, \; q^2_2 = 1, \; q^3_3 = 1, \quad -(17)$$

So: $$\omega^0_{\mu 0} = \Gamma^0_{\mu 0} = 0 \quad -(18)$$

So-called Schwarzschild Metric

In this case:

$$q^0_0 = \left(1 - \frac{2GM}{c^2 r}\right)^{1/2}, \; q^1_1 = \left(1 - \frac{2GM}{c^2 r}\right)^{-1/2} \quad -(19)$$

From eq. (16):

$$D_1 q^0_0 = \partial_1 q^0_0 + \omega^0_{10} q^0_0 - \Gamma^0_{10} q^0_0 \quad -(20)$$
$$= 0$$

i.e. $$\frac{\partial q^0_0}{\partial r} = \left(\Gamma^0_{10} - \omega^0_{10}\right) q^0_0 \quad -(21)$$
$$= \frac{GM}{c^2 r^2}\left(1 - \frac{2GM}{c^2 r}\right)^{-1/2}$$

Therefore: $$\Gamma^0_{10} - \omega^0_{10} = \frac{GM}{c^2 r^2}\left(1 - \frac{2GM}{c^2 r}\right)^{-1} \quad -(22)$$

Now use Carroll, eq. (7.33), to find that:

$$\Gamma^0_{10} = \frac{GM}{c^2 r^2}\left(1 - \frac{2GM}{c^2 r}\right)^{-1} \quad -(23)$$

5) Therefore:

$$\omega^0{}_{10} = 0 \quad -(24)$$

<u>Result</u>

If $q^0_0 = g_{00}^{1/2}$, then $\omega^0{}_{10} = 0$, and:

$$\partial_1 q^0_0 = \Gamma^0{}_{10} q^0_0, \quad -(25)$$

$$\partial_0 q^0_1 = 0 \quad -(26)$$

The <u>Cartan torsion</u> element is therefore:

$$T^0{}_{10} = \partial_1 q^0_0 - \partial_0 q^0_1 + \omega^0{}_{10} q^0_0 - \omega^0{}_{01} q^1_1$$

$$\boxed{T^0{}_{10} = \partial_1 q^0_0 - \omega^0{}_{01} q^1_1} \quad -(27)$$

From eq. (12):

$$D_0 q^1_0 = \partial_0 q^1_0 + \omega^0{}_{01} q^1_0 - \Gamma^0{}_{01} q^1_0 - (28)$$

i.e. $\omega^0{}_{01} = \Gamma^0{}_{01} = \Gamma^0{}_{10}$ $-(29)$

Therefore: $$\boxed{T^0{}_{10} = \partial_1 q^0_0 = \frac{GM}{c^2 r^2}\left(1 - \frac{2GM}{c^2 r}\right)^{-1/2}}$$

$-(30)$

Finally the Christoffel torsion is:

$$\Gamma^{\kappa}_{\mu\nu} = q^{\kappa}_{\ a} T^{a}_{\ \mu\nu} \quad -(31)$$

$$= q^{0}_{\ 0} T^{0}_{\ \mu\nu} \quad -(32)$$

$$= 0$$

Therefore there is a contradiction, because:

$$\Gamma^{0}_{\ 01} = q^{0}_{\ 0} T^{0}_{\ 01} = 0 \quad -(33)$$

and $$q^{0}_{\ 0} \neq 0, \ T^{0}_{\ 01} \neq 0 \quad -(34).$$

<u>Conclusion</u>

The assumption of a diagonal tetrad matrix can only be used for the static Minkowski spacetime, whose metric is diag $(1, -1, -1, -1)$.