III(6): Relativistically Corrected Scalar Potential of the Coulomb Law.

Start with:

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{2}{r}\frac{\partial \phi}{\partial r} = \frac{\rho_e}{4\pi\epsilon_0}\left(1 + \frac{2mG}{rc^2}\right) \quad -(1)$$

The solution is:

$$\boxed{\phi = \frac{e}{4\pi\epsilon_0 r}\left(1 + \frac{mG}{rc^2}\right)} \quad -(2)$$

where

$$\rho_e = \frac{e}{r^3}. \quad -(3)$$

Proof

Write $\phi = \phi_0 + \phi_1 \quad -(4)$.

where:

$$\phi_0 = \frac{e}{4\pi\epsilon_0 r}, \quad \phi_1 = \frac{C}{r}, \quad -(5)$$

$$C = \frac{e}{4\pi\epsilon_0}\cdot\frac{mG}{c^2}. \quad -(6)$$

Then:

$$\frac{\partial \phi_0}{\partial r} = -\frac{e}{4\pi\epsilon_0 r^2}, \quad \frac{\partial^2 \phi_0}{\partial r^2} = \frac{3e}{4\pi\epsilon_0 r^3}$$

and

$$\frac{\partial^2 \phi_0}{\partial r^2} + \frac{2}{r}\frac{\partial \phi_0}{\partial r} = \left(\frac{e}{r^3}\right)\cdot\frac{1}{4\pi\epsilon_0} = \frac{\rho_e}{4\pi\epsilon_0} \checkmark\checkmark$$

Similarly:

$$\frac{\partial \phi_1}{\partial r} = -\frac{2C}{r^3}, \quad \frac{\partial^2 \phi_1}{\partial r^2} = \frac{6C}{r^4} \quad -(6)$$

and

$$\frac{\partial^2 \phi_1}{\partial r^2} + \frac{2}{r}\frac{\partial \phi_1}{\partial r} = \frac{2C}{r^3}\cdot\frac{1}{r} \quad -(7)$$

$$= \frac{\rho_e}{4\pi\epsilon_0}\cdot\frac{2mG}{rc^2} \checkmark\checkmark$$