1)

113(2): Birkhoff / Jebsen Theorem

As developed in papers 111 and notes 112(1) the simplest form of the Birkhoff / Jebsen Theorem is:

$$n r = \frac{r}{m} = \int dr \quad -(1)$$

This seemingly simple and obvious equation must have a profound meaning in physics because it describes all known orbits without the Einstein equation. The Birkhoff / Jebsen theorem states that any spherically symmetric solution of the vacuum field equation of Einstein must be stationary and asymptotically flat. Given the Frobenius theorem, these properties are described in equation (1). The theorem was first stated by Jörg Tofte Jebsen in about 1920, and two years later by Birkhoff. The importance of eq. (1) is that the theorem is purely geometrical. To prove it, there is no need to assume the existence of a mass M. The latter actually enters into consideration following the data. From the Birkhoff theorem itself there would be no indication that mass is present. The Birkhoff theorem alone does not account for the interaction of a mass M with a mass m. This is as pointed out many times by Stephen Crothers.

The symmetries of a metric are characterized by the existence of Killing vectors. On the sphere S^2 there are three Killing vectors:

$$[V^{(1)}, V^{(2)}] = V^{(3)} \quad -(1)$$

et cyclicum

In three dimensional Euclidean space R^3 is spherically symmetric with respect to rotations around the origin. These

2) rotations are described as the flow of Killing vector fields. Points move into each other, but each point stays on an S^2 at a fixed distance from the origin. These are said to foliate R^3. The only exception is the point at the origin.

The metric on the entire manifold is of the form:

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = g_{IJ}(v)\, dv^I dv^J + f(v)\, \gamma_{ij}(u)\, du^i du^j \qquad -(2)$$

where $\gamma_{ij}(u)$ is the metric on the sub-manifold. It is to be noted that there are no cross terms such as $dv^I du^j$. Both $g_{IJ}(v)$ and $f(v)$ are functions of v^I alone. Here is a set of m coordinate functions u^i on the sub-manifold and a set of $n-m$ coordinate functions v^I defining the submanifold being considered.

Eq (2) implies that the metric on a spherically symmetric spacetime is:

$$ds^2 = g_{aa}(a,b)\, da^2 + g_{ab}(a,b)\,(da\, db + db\, da) + g_{bb}\, db^2 + r^2(a,b)\, d\Omega^2 \qquad -(3)$$

A particular solution of eq. (3) in spherical polar coordinates is:

$$ds^2 = g_{aa}(a,r)\, da^2 + g_{ar}(a,r)\,(da\, dr + dr\, da) + g_{rr}(a,r)\, dr^2 + r^2\, d\Omega^2 \qquad -(4)$$

Finally the procedure as described in Carroll's chapter seven is to eliminate the cross-term in eq. (4) by replacing the first three terms by

$$n\, dt^2 c^2 + n\, dr^2. \qquad -(5)$$

3)

This procedure uses:

$$dt = \frac{\partial t}{\partial a} da + \frac{\partial t}{\partial r} dr \quad -(6)$$

so:

$$dt^2 = \left(\frac{\partial t}{\partial a}\right)^2 da^2 + \left(\frac{\partial t}{\partial a}\right)\left(\frac{\partial t}{\partial r}\right)(da\,dr + dr\,da) + \left(\frac{\partial t}{\partial r}\right)^2 dr^2. \quad -(7)$$

Therefore:

$$ds^2 = m(t,r)\, dt^2 c^2 + n(t,r)\, dr^2 + r^2 d\Omega^2 \quad -(8)$$

if:

$$g_{aa} = m\left(\frac{\partial t}{\partial a}\right)^2, \quad -(9)$$

$$n + m\left(\frac{\partial t}{\partial r}\right)^2 = g_{rr}, \quad -(10)$$

and

$$m\left(\frac{\partial t}{\partial a}\right)\left(\frac{\partial t}{\partial r}\right) = g_{ar}. \quad -(11)$$

This procedure by Carroll is designed a priori to recover the Minkowski metric:

$$ds^2 = -c^2 dt^2 + dr^2 + r^2 d\Omega^2 \quad -(12)$$

The spacetime under consideration is Lorentzian, so either m or n ~~will~~ is negative. The m(t, r) function is chosen to be negative because the Minkowski metric (12) is itself spherically symmetric. Therefore:

$$ds^2 = -m(t,r)\, c^2 dt^2 + n(t,r)\, dr^2 + r^2 d\Omega^2 \quad -(13)$$

At this point Carroll starts to use the Einstein field equation with the Ricci flat assumption:

$$R_{\mu\nu} = 0 \quad -(14)$$

in order to arrive at the Birkhoff / Jebsen theorem:

$$\boxed{ds^2 = -m(r)c^2dt^2 + n(r)dr^2 + r^2d\Omega^2} \quad -(15)$$

This is the standard model procedure, and from eq(14) it can be seen that it is a purely geometrical procedure. Clearly, mass M does not appear in eq. (14). The solution of eq. (14) was first discovered by Schwarzschild in 1916. The latter gave it as:

$$ds^2 = -\left(1+\frac{\mu}{R}\right)c^2dt^2 + \left(1+\frac{\mu}{R}\right)^{-1}dr^2 + r^2d\Omega^2. \quad -(16)$$

In general R is not the same as the radial coordinate r. Eq. (16) is pure geometry, a solution of the purely geometrical equation (14). In 1916, Schwarzschild did not use mass M in eq. (16).

Later workers made the identification:

$$\frac{\mu}{R} := -\frac{2MG}{c^2r} \quad -(17)$$

in order to reproduce orbital phenomena such as light deflection due to gravity and perihelion

5)

precession. The identification (17) is phenomenological, it follows the data. It is now known that the Einstein field equation:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = k T_{\mu\nu} \quad - (18)$$

is geometrically incorrect because of its neglect of torsion. Therefore the procedure (14) to (17) is meaningless.

In paper 111 it was replaced by eq (1), in which μ is recognized as the constant of integration. Thus:

$$n r = \int dr = r + \mu \quad - (19)$$

$$\frac{r}{m} = \int dr = r + \mu, \quad - (20)$$

i.e.

$$n = 1 + \frac{\mu}{r}, \quad - (21)$$

$$m = \left(1 + \frac{\mu}{r}\right)^{-1}. \quad - (22)$$

This leads to the line element:

$$ds^2 = -\left(1 + \frac{\mu}{r}\right) c^2 dt^2 + \left(1 + \frac{\mu}{r}\right)^{-1} dr^2 + r^2 d\Omega^2. \quad - (23)$$

In paper 108 it was found that all orbits are described by:

$$\mu = -\frac{2MG}{c^2} + \frac{a}{r} \quad -(24)$$

where a is a very small perturbation, observable only in binary pulsars.

Orbital equations are all derivable from eqs (19) and (20), and all orbital phenomena, including the orbits of binary pulsars. This is therefore the simplest and most powerful statement of the Birkhoff / Jebsen Theorem:

$$\boxed{n(r)\,r = \frac{r}{m(r)} = \int dr = r + \mu} \quad -(25)$$

Eq. (25) can be viewed as the fundamental equation of all relativistic orbits. The pure geometry of eq (25) is turned into physics using data. This gives eq. (24). The vast majority of orbits are described by

$$a \rightarrow 0. \quad -(26)$$

Finally, eq. (25) can be generalized to:

$$n(R)\,R = \frac{R}{m(R)} = \int dR = R + \mu \quad -(27)$$

where R is the radius used by Schwarzschild.