1) Notes 115(2): Origin of the Cartan Geometry.

The origin of the Cartan geometry is the action of the commutator of covariant derivatives $[D_\mu, D_\nu]$ on the vector V^ρ. In a spacetime without a connection, the covariant derivative reduces to the four derivative ∂_μ. Such a spacetime is referred to as a flat spacetime, in which:

$$[\partial_\mu, \partial_\nu] = -[\partial_\nu, \partial_\mu] = 0. \quad -(1)$$

In a spacetime with a non-zero connection:

$$D_\mu V^\rho = \partial_\mu V^\rho + \Gamma^\rho_{\mu\lambda} V^\lambda \quad -(2)$$

where in general the connection is asymmetric:

$$\Gamma^\rho_{\mu\lambda} \neq \Gamma^\rho_{\lambda\mu}. \quad -(3)$$

The connection is the mathematical device that moves the frame of reference itself. It follows directly from eq. (2) that:

$$[D_\mu, D_\nu] V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma - T^\lambda_{\mu\nu} D_\lambda V^\rho \quad -(4)$$

where <u>the curvature tensor</u> is:

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma} \quad -(5)$$

and <u>the torsion tensor</u> is

$$T^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} \quad -(6)$$

2)

There cannot be any a priori reason to assume that one or the other of (5) and (6) must be zero.

They originate directly from the definition itself of the covariant derivative. No other assumption is used in the derivation of (5) and (6). The Cartan structure equations re-express eqns (5) and (6) as:

$$R^a{}_b = D \wedge \omega^a{}_b \quad -(7)$$

and

$$T^a = D \wedge q^a \quad -(8)$$

where q^a is the Cartan tetrad and $\omega^a{}_b$ is the spin connection. It is well known in mathematics that eqns. (7) and (8) are equivalent to eqns. (5) and (6) given the tetrad postulate:

$$D_\mu q^a_\nu = 0. \quad -(9)$$

The latter expresses the fundamental fact that a vector field is independent of its coordinates.

So the origin of Cartan geometry is the commutator of covariant derivatives $[D_\mu, D_\nu]$.

Cartan introduced an orthonormal frame at a point P in the base manifold, the spacetime of the orthonormal frame is a Minkowski spacetime. The relation between objects such as the torsion form and torsion tensor is given by:

$$T^a_{\mu\nu} = q^a_\kappa T^\kappa_{\mu\nu}. \quad -(10)$$

Another example is:

$$R^a{}_{b\mu\nu} = q^a_\kappa q^\sigma_b R^\kappa{}_{\sigma\mu\nu}. \quad -(11)$$

It is also well known in mathematics that the Cartan structure equations imply the Bianchi identity:

$$D \wedge T^a := R^a{}_b \wedge q^b \quad -(12)$$

whose equivalent in tensor notation is given in paper 112, eqs. (15) and (16). The Bianchi identity is a true identity which is derived directly from the definition of covariant derivative (2).

The so called "first Bianchi identity" of the standard model is:

$$R^a{}_b \wedge q^b = 0, \quad -(13)$$

$$T^a = 0, \quad -(14)$$

which in tensor notation is:

$$R^\lambda{}_{\rho\mu\nu} + R^\lambda{}_{\mu\nu\rho} + R^\lambda{}_{\nu\rho\mu} = 0, \quad -(15)$$

$$T^\kappa_{\mu\nu} = 0. \quad -(16)$$

The Hodge dual identity:

$$D \wedge \tilde{T}^a := \tilde{R}^a{}_b \wedge q^b \quad -(17)$$

is an example of eq. (12).

Any field theory of relativity must obey both eqs. (12) and (17).

The Failure of the Einstein Field Equation

The Einstein field equation is:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = kT_{\mu\nu}, \quad -(18)$$

$$\Gamma^\kappa_{\mu\nu} = \Gamma^\kappa_{\nu\mu}. \quad -(19)$$

It has been found by computer algebra in papers 93 ff of www.aias.us that eqs. (18) and (19) do not obey eq. (17). They give the result:

$$\tilde{R}^a{}_b \wedge q^b \neq 0, \quad -(20)$$

$$\tilde{T}^a = 0. \quad -(21)$$

The only exception occurs when:

$$G_{\mu\nu} = 0, \quad -(22)$$

giving: $\tilde{T}^a = \tilde{R}^a{}_b \wedge q^b = 0. \quad -(23)$

However, eq. (22) is just pure geometry, because there is no matter present: $T_{\mu\nu} = 0. \quad -(24)$