117(6): Magnetic Dipole Approximation in the non-Relativistic Limit

In this limit the inverse Lorentz transform gives:

$$\underline{B} = -\frac{1}{c^2}\, \underline{v} \times \underline{E} \quad — (1)$$

The electric dipole field is:

$$\underline{E} = \frac{3\underline{n}(\underline{p}\cdot\underline{n}) - \underline{p}}{4\pi\epsilon_0 |\underline{x} - \underline{x}_0|^3} \quad — (2)$$

So:

$$\underline{B} = \frac{\underline{v} \times (\underline{p} - 3\underline{n}(\underline{p}\cdot\underline{n}))}{4\pi\epsilon_0 |\underline{x} - \underline{x}_0|^3 c^2} \quad — (3)$$

$$= \frac{\mu_0}{4\pi r^3}\left(\underline{v}\times\underline{p} - 3\underline{v}\times\underline{n}\,(\underline{p}\cdot\underline{n})\right)$$

The magnetic dipole moment is defined as:

$$\underline{m} = \frac{1}{2}\int \underline{x}' \times \underline{J}(x')\, d^3x' \quad — (4)$$

in units of $C m^2 s^{-1}$, and the electric dipole moment is

$$\underline{p} = \int x' \rho(x')\, d^3x' \quad — (5)$$

in units of Cm. Therefore define:

$$\underline{m} = \underline{p} \times \underline{v} = \int \underline{x}' \times \underline{J}(x')\, d^3x' \quad — (6)$$

and

$$\underline{B} = \frac{\mu_0}{4\pi r^3}\left(-3\underline{v}\times\underline{n}(\underline{p}\cdot\underline{n}) - \underline{m}\right) \quad — (7)$$

2) In order to simplify this expression it may be investigated under what circumstances we can write:

$$-3\underline{v}\times\underline{n}\left(\underline{p}\cdot\underline{n}\right) = 3\underline{n}\left(\underline{m}\cdot\underline{n}\right) \quad -(8)$$

One example is when $\underline{n}$ is aligned in the $\underline{p}$ axis, so:

$$\underline{p} = p_0\underline{n} \quad -(9)$$

then

$$\underline{p}\cdot\underline{n} = 3p_0 \quad -(10)$$

so

$$\underline{v}\times\left(\underline{n}\left(\underline{p}\cdot\underline{n}\right)\right) = \underline{n}\left(\underline{v}\times\underline{p}\cdot\underline{n}\right) \quad -(11)$$

if

$$3\underline{v}\times\underline{n} = \underline{n}\left(\underline{v}\times\underline{n}\cdot\underline{n}\right) \quad -(12)$$

i.e.

$$3\underline{A} = \underline{n}\,\underline{A}\cdot\underline{n} \quad -(13)$$

i.e.

$$3\left(A_x\underline{i} + A_y\underline{j} + A_z\underline{k}\right) = \left(A_x + A_y + A_z\right)\left(\underline{i} + \underline{j} + \underline{k}\right) \quad -(14)$$

i.e. if

$$A_x = A_y = A_z \quad -(15)$$

or

$$v_x = v_y = v_z \quad -(16)$$

Under conditions (9) and (10):

$$\boxed{\underline{B} = \frac{\mu_0}{4\pi r^3}\left(3\underline{n}\left(\underline{m}\cdot\underline{n}\right) - \underline{m}\right)} \quad -(17)$$

More realistically:

$$\langle v_x\rangle = \langle v_y\rangle = \langle v_z\rangle \quad -(18)$$

on average.