1) Note 118(7): Some Key Mathematics

Antisymmetric Tensor in 4-D

$$\bar{\epsilon}_{\mu\nu\rho\sigma} = |g|^{1/2}\,\epsilon_{\mu\nu\rho\sigma} \qquad -(1)$$

where $|g|^{1/2}$ is the square root of the determinant of the metric and:

$$\epsilon_{\mu\nu\rho\sigma} = \begin{cases} +1 & \text{if } \mu\nu\rho\sigma \text{ is an even permutation of } 0123 \\ -1 & \text{if } \mu\nu\rho\sigma \text{ is an odd permutation of } 0123 \\ 0 & \text{otherwise} \end{cases}$$

A permutation is an ordering of numbers 0, 1, 2, 3 which is defined as starting with 0123 and exchanging two of the digits. An even permutation is an even number of such exchanges, and an odd permutation is an odd number of exchanges.

Examples

$$\epsilon_{1023} = -1, \quad \epsilon_{0132} = -1, \quad \epsilon_{0312} = 1, \quad \epsilon_{0321} = -1.$$

Wedge Product

Given a p-form A and a q-form B, the wedge product is a $(p+q)$ form obtained from the antisymmetrized tensor product:

$$(A \wedge B)_{\mu_1 \cdots \mu_{p+q}} = \frac{(p+q)!}{p!\,q!} A_{[\mu_1 \cdots \mu_p} B_{\mu_{p+1} \cdots \mu_{p+q}]} \qquad -(2)$$

For example:

$$\begin{aligned} (A \wedge B)_{\mu\nu} &= 2A_{[\mu}B_{\nu]} \\ &= A_\mu B_\nu - A_\nu B_\mu \end{aligned} \qquad -(3)$$

2)

The wedge product has the property:

$$A \wedge B = (-1)^{pq} B \wedge A \quad -(4)$$

So for example:

$$R \wedge q = q \wedge R \quad -(5)$$

<u>Exterior derivative</u>

This is defined as:

$$(d \wedge A)_{\mu_1 \ldots \mu_{p+1}} = (p+1)\, \partial_{[\mu_1} A_{\mu_2 \ldots \mu_{p+1}]} \quad -(6)$$

and differentiates a $p$-form to give a $(p+1)$-form.

<u>Hodge Dual</u>

This is a map in an $n$-dimensional manifold from a $p$ form to an $(n-p)$ form:

$$\tilde{A}_{\mu_1 \ldots \mu_{n-p}} = \frac{1}{p!} \epsilon^{\nu_1 \ldots \nu_p}{}_{\mu_1 \ldots \mu_{n-p}} A_{\nu_1 \ldots \nu_p} \quad -(7)$$

<u>Transformation Laws</u>

$$\partial_{\mu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \partial_{\mu} \quad -(8)$$

$$V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} V^{\mu} \quad -(9)$$

$$V = V^{\mu} \partial_{\mu} = V^{\mu'} \partial_{\mu'} \quad -(10)$$

3)

$$T^{\mu_1'\cdots\mu_k'}{}_{\nu_1'\cdots\nu_l'} = \frac{\partial x^{\mu_1'}}{\partial x^{\mu_1}} \cdots \frac{\partial x^{\mu_k'}}{\partial x^{\mu_k}} \cdot \frac{\partial x^{\nu_1}}{\partial x^{\nu_1'}} \cdots \frac{\partial x^{\nu_l}}{\partial x^{\nu_l'}} T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l} \qquad -(11)$$

<u>Covariant Derivative (D)</u>

This is an operator that reduces to the ordinary derivative in Minkowski spacetime but transforms as a tensor in the arbitrary manifold. Thus $D$ maps in such a way that:

1) $$D(T+S) = DT + DS \qquad -(12)$$

2) $$D(T \otimes S) = (DT) \otimes S + T \otimes (DS) \qquad -(13).$$

For each direction $\mu$, the covariant derivative $D_\mu$ will be given by $\partial_\mu$ plus a <u>connection</u> $(\Gamma_\mu)^\rho{}_\sigma$, which is an $n \times n$ matrix for each $\mu$, where $n$ is the dimension of the manifold. Thus:

$$D_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda \qquad -(14)$$

which transforms covariantly:

$$D_{\mu'} V^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} D_\mu V^\nu \qquad -(15)$$

For the general tensor:

$$D_\sigma T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} = \partial_\sigma T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} \qquad -(16)$$

$$+ \Gamma^{\mu_1}_{\sigma\lambda} T^{\lambda\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} + \Gamma^{\mu_2}_{\sigma\lambda} T^{\mu_1\lambda\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} + \cdots$$

$$- \Gamma^\lambda_{\sigma\nu_1} T^{\mu_1\mu_2\cdots\mu_k}{}_{\lambda\nu_2\cdots\nu_l} - \Gamma^\lambda_{\sigma\nu_2} T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\lambda\cdots\nu_l} - \cdots$$

4) Torsion Tensor

$$T^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu} \quad -(17)$$

for any connection.

Metric Compatibility

$$D_\rho g_{\mu\nu} = D_\rho g^{\mu\nu} = 0 \quad -(18)$$

Action of Commutator

$$[D_\mu, D_\nu] V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma - T^{\lambda}_{\mu\nu} D_\lambda V^\rho \quad -(19)$$

where the curvature tensor is:

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} \quad -(20)$$

Eq (19) is true for any connection and irrespective of metric compatibility. Also:

$$[D_\mu, D_\nu] X^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_l} = - T^{\lambda}_{\rho\sigma} D_\lambda X^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_l} + R^{\mu_1}{}_{\lambda\rho\sigma} X^{\lambda\mu_2 \dots \mu_k}_{\nu_1 \dots \nu_l} + R^{\mu_2}{}_{\lambda\rho\sigma} X^{\mu_1\lambda \dots \mu_k}_{\nu_1 \dots \nu_l} + \dots - R^{\lambda}{}_{\nu_1\rho\sigma} X^{\mu_1 \dots \mu_k}_{\lambda\nu_2 \dots \nu_l} - R^{\lambda}{}_{\nu_2\rho\sigma} X^{\mu_1 \dots \mu_k}_{\nu_1\lambda \dots \nu_l} - \dots \quad -(21)$$

Therefore torsion is as basic as curvature.

## The Tetrad

$$V^a = q^a_\mu V^\mu \quad -(22)$$

$$g_{\mu\nu} = q^a_\mu q^b_\nu \eta_{ab} \quad -(23)$$

## Use of Tetrad

$$T^a_{\mu\nu} = q^a_\lambda T^\lambda_{\mu\nu} \quad -(24)$$

$$R^a{}_{b\mu\nu} = q^a_\lambda q^\sigma_b R^\lambda{}_{\sigma\mu\nu} \quad -(25)$$

etc.

## Tetrad postulate

$$D_\mu q^a_\nu = \partial_\mu q^a_\nu + \omega^a_{\mu b} q^b_\nu - \Gamma^\lambda_{\mu\nu} q^a_\lambda \quad -(26)$$

$$= 0$$

## Exterior Covariant Derivative

$$(D \wedge X)^a_{\mu\nu} = \partial_\mu X^a_\nu - \partial_\nu X^a_\mu + \omega^a_{\mu b} X^b_\nu - \omega^a_{\nu b} X^b_\mu \quad -(27)$$

## Cartan Structure Equations $\quad -(28)$

$$T^a = D \wedge q^a = d \wedge q^a + \omega^a{}_b \wedge q^b$$

$$R^a{}_b = D \wedge \omega^a{}_b = d \wedge \omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \quad -(29)$$

## Bianchi Identity

$$D \wedge T^a := R^a{}_b \wedge q^b \quad -(30)$$

These are all given by Carroll. Two new identities have been developed in ECE theory.

Dual Identity

$$D \wedge \tilde{T}^a := \tilde{R}^a{}_b \wedge q^b \quad -(31)$$

Derivative Identity

$$D \wedge (D \wedge T^a) := D \wedge (R^a{}_b \wedge q^b) \quad -(32)$$

Metric and Inverse Metric

These are respectively $g_{\mu\nu}$ and $g^{\mu\nu}$, with:

$$g_{\mu\nu} g^{\mu\nu} = 4 \quad -(33)$$

ECE Lemma

$$D^\mu (D_\mu q^a_\nu) := 0 \quad -(34)$$

which is equivalent to:

$$\Box q^a_\mu = R q^a_\mu \quad -(35)$$

which is the geometrical basis of all wave equations of quantum mechanics.