1) Note 118(8): Exterior Derivatives and Wedge Products of the Bianchi Identity.

The Bianchi identity is:

$$D \wedge T^a := R^a{}_b \wedge q^b \quad -(1)$$

where in general:

$$(d \wedge A)_{\mu_1 \cdots \mu_{p+1}} = (p+1)\,\partial_{[\mu_1} A_{\mu_2 \cdots \mu_{p+1}]} \quad -(2)$$

and:

$$(A \wedge B)_{\mu_1 \cdots \mu_{p+q}} = \frac{(p+q)!}{p!\,q!} A_{[\mu_1 \cdots \mu_p} B_{\mu_{p+1} \cdots \mu_{p+q}]} \quad -(3)$$

The [ ] brackets denote antisymmetrization, defined by:

$$T_{[\mu_1 \mu_2 \cdots \mu_n]\rho}{}^{\sigma} = \frac{1}{n!}\left(T_{\mu_1 \mu_2 \cdots \mu_n \rho}{}^{\sigma} + \text{alternating sum over permutations of indices } \mu_1 \cdots \mu_n\right) \quad -(4).$$

e.g.

$$T_{[\mu\nu\rho]\sigma} = \frac{1}{6}\left(T_{\mu\nu\rho\sigma} - T_{\mu\rho\nu\sigma} + T_{\rho\mu\nu\sigma} - T_{\nu\mu\rho\sigma} + T_{\nu\rho\mu\sigma} - T_{\rho\nu\mu\sigma}\right) \quad -(5)$$

In the Bianchi identity $R^a{}_b$ is a 2-form and q^b is a one form. So in eq (3):

$$p = 2,\ q = 1,\ \mu_1 = \mu;\ \mu_2 = \nu;\ \mu_3 = \rho \quad -(6)$$

so:

$$(R^a{}_b \wedge q^b)_{\mu\nu\rho} = \frac{3!}{2!\,6}\left(R^a{}_{\mu\nu\rho} - R^a{}_{\mu\rho\nu} + R^a{}_{\rho\mu\nu} - R^a{}_{\nu\mu\rho} + R^a{}_{\nu\rho\mu} - R^a{}_{\rho\nu\mu}\right)$$

$$= R^a{}_{\mu\nu\rho} + R^a{}_{\rho\mu\nu} + R^a{}_{\nu\rho\mu} \quad -(7)$$

using:

$$R^a{}_{\mu\nu\rho} = -R^a{}_{\mu\rho\nu} \quad -(8)$$

etc.

2) Similarly T^a is a 2-form, so in eq. (2):

$$p = 1,\ \mu_1 = \mu,\ \mu_2 = \nu,\ \mu_3 = \rho \quad -(9)$$

and

$$(d \wedge T^a)_{\mu\nu\rho} = \frac{3}{3!}\, \partial_{[\mu} T^a{}_{\nu\rho]} \quad -(10)$$

$$= \partial_\mu T^a{}_{\nu\rho} + \partial_\rho T^a{}_{\mu\nu} + \partial_\nu T^a{}_{\rho\mu} \quad -(11)$$

the result is the same for the covariant exterior derivative, so eqn. (1) is:

$$\boxed{D_\mu T^a{}_{\nu\rho} + D_\rho T^a{}_{\mu\nu} + D_\nu T^a{}_{\rho\mu} := R^a{}_{\mu\nu\rho} + R^a{}_{\rho\mu\nu} + R^a{}_{\nu\rho\mu}}$$

$-(12)$.

as often used in ECE theory, in which eq. (12) has been proven to be the exact identity consisting of the curvature tensor identically equal to the cyclic sum of definitions. Using the definitions:

$$T^a{}_{\nu\rho} = q^a{}_\kappa T^\kappa{}_{\nu\rho} \quad -(13)$$

$$R^a{}_{\mu\nu\rho} = q^a{}_\kappa R^\kappa{}_{\mu\nu\rho} \quad -(14)$$

and the tetrad postulate:

$$D_\mu q^a{}_\kappa = 0 \quad -(15)$$

we obtain eq. (12) in the base manifold:

$$D_\mu T^\kappa{}_{\nu\rho} + D_\rho T^\kappa{}_{\mu\nu} + D_\nu T^\kappa{}_{\rho\mu}$$

$$:= R^\kappa{}_{\mu\nu\rho} + R^\kappa{}_{\rho\mu\nu} + R^\kappa{}_{\nu\rho\mu}$$

$-(16)$

3) which is the same as:

$$\boxed{D_\mu \tilde{T}^{\kappa\mu\nu} := \tilde{R}^{\kappa}{}_{\mu}{}^{\mu\nu}} \quad -(17)$$

This is the most concise and useful form of the Bianchi identity. Similarly, the dual identity is:

$$\boxed{D_\mu T^{\kappa\mu\nu} := R^{\kappa}{}_{\mu}{}^{\mu\nu}} \quad -(18)$$

In this format it becomes clear that the torsion cannot be neglected and that there is Hodge duality invariance. This means that taking Hodge duals does not change the form of the two identities.

Failure of the Einstein Field Equation

The root cause is the use of a symmetric connection, which leads to:

$$\boxed{\begin{array}{l} \tilde{T}^{\kappa\mu\nu} = 0\,; \quad \tilde{R}^{\kappa}{}_{\mu}{}^{\mu\nu} = 0\,; \\ T^{\kappa\mu\nu} = 0, \quad \underline{\text{but}} \quad R^{\kappa}{}_{\mu}{}^{\mu\nu} \neq 0 \end{array}} \quad -(19)$$

Eq. (18) is equivalent to:

$$D_\mu \tilde{T}^{\kappa}{}_{\nu\rho} + D_\rho \tilde{T}^{\kappa}{}_{\mu\nu} + D_\nu \tilde{T}^{\kappa}{}_{\rho\mu} = \tilde{R}^{\kappa}{}_{\mu\nu\rho} + \tilde{R}^{\kappa}{}_{\rho\mu\nu} + \tilde{R}^{\kappa}{}_{\nu\rho\mu} \quad -(20)$$

i.e. $$D \wedge \tilde{T}^{a} := \tilde{R}^{a}{}_{b} \wedge q^{b} \quad -(21)$$

This is the geometry of the field equations of ECE.