1) 129(1): New Derivation of the Dirac Equation, Tetrads as Pauli matrices and Dirac Spinors.

Consider the square of the position vector in Cartesian coordinates:

$$r^2 = (X\underline{i} + Y\underline{j} + Z\underline{k}) \cdot (X\underline{i} + Y\underline{j} + Z\underline{k})$$
$$= X^2 \underline{i}\cdot\underline{i} + Y^2 \underline{j}\cdot\underline{j} + Z^2 \underline{k}\cdot\underline{k} \quad -(1)$$

Eqn (1) may be written as:

$$r^2\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = X^2 \underline{\sigma}_1 \cdot \underline{\sigma}_1 + Y^2 \underline{\sigma}_2 \cdot \underline{\sigma}_2 + Z^2 \underline{\sigma}_3 \cdot \underline{\sigma}_3 \quad -(2)$$

where:

$$\underline{\sigma}_1 = \sigma_1 \underline{i} \quad -(3)$$
$$\underline{\sigma}_2 = \sigma_2 \underline{j} \quad -(4)$$
$$\underline{\sigma}_3 = \sigma_3 \underline{k} \quad -(5)$$

Here, the Pauli matrices are:

$$\sigma_0 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\ \sigma_1 = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\ \sigma_2 = \begin{bmatrix}0 & -i\\ i & 0\end{bmatrix},\ \sigma_3 = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} \quad -(6)$$

In eqs. (3) to (5) the basis elements σ_1 and $\underline{i}$ are related by σ_1 and so on.

Therefore the Pauli matrices are tetrads:

$$q^0_0 = \sigma_0 \quad -(6)$$
$$q^1_X = \sigma_1 \quad -(7)$$
$$q^2_Y = \sigma_2 \quad -(8)$$
$$q^3_Z = \sigma_3 \quad -(9)$$

2) Transpose the tetrads to obtain combinations of rest particle Dirac spinors as follows:

$$q_0^{0\,T} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = u^{(1)}(0) + v^{(2)}(0) \quad -(10)$$

$$q_x^{1\,T} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = u^{(2)}(0) + v^{(1)}(0) \quad -(11)$$

$$i q_y^{2\,T} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} = u^{(2)}(0) - v^{(1)}(0) \quad -(12)$$

$$q_z^{3\,T} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} = u^{(1)}(0) - v^{(2)}(0) \quad -(13)$$

The rest spinors are given by L. H. Ryder, "Quantum Field Theory" (CUP, 2nd. ed. 1996).

Therefore:

$$u^{(1)}(0) = \frac{1}{2}\left(q_0^{0\,T} + q_z^{3\,T}\right) \quad -(14)$$

$$u^{(2)}(0) = \frac{1}{2}\left(q_x^{1\,T} + i q_y^{2\,T}\right) \quad -(15)$$

$$v^{(2)}(0) = \frac{1}{2}\left(q_0^{0\,T} - q_z^{3\,T}\right) \quad -(16)$$

$$v^{(1)}(0) = \frac{1}{2}\left(q_x^{1\,T} - i q_y^{2\,T}\right) \quad -(17)$$

The complete Dirac spinors for the rest fermion are:

3)

$$\phi_u = u(0)\exp\left(-\frac{imc^2 t}{\hbar}\right), \quad -(18)$$

$$\phi_v = v(0)\exp\left(\frac{imc^2 t}{\hbar}\right) \quad -(19)$$

The Dirac spinor is therefore a combination of tetrads obtained from Pauli matrices, which are themselves tetrads.

The Dirac equation is a limit of the ECE wave equation:

$$(\Box + \kappa^2)\phi = 0 \quad -(20).$$

where:

$$\kappa = \frac{mc}{\hbar}. \quad -(21)$$

Therefore eqns. (18) and (19) are:

$$\phi_u = u(0)\exp(-i\omega t) \quad -(22)$$

$$\phi_v = v(0)\exp(i\omega t) \quad -(23)$$

where:

$$\omega = c\kappa. \quad -(24)$$

Eq. (21) indicates wave particle duality.

The energy is always positive:

$$E_n = \hbar\omega \quad -(25)$$

and so there is no negative energy cascade and no Dirac sea.

1) 129(2): Some Details of the Dirac Equation

The ECE wave equation is:

$$(\Box + kT)\, q^a_\mu = 0 \quad -(1)$$

and the Dirac equation is the limit:

$$kT = \kappa^2 = \left(\frac{mc}{\hbar}\right)^2 \quad -(2)$$

The d'Alembertian is:

$$\Box = \gamma^\mu \gamma^\nu \partial_\mu \partial_\nu \quad -(3)$$

where the Dirac matrices are defined by the Minkowski metric:

$$2g_{\mu\nu} = \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu \quad -(4)$$

The wave equation (1) is therefore:

$$(i\gamma^\mu \partial_\mu - \kappa)(i\gamma^\nu \partial_\nu + \kappa)\, q^a_\mu = 0. \quad -(5)$$

There are two possible solutions:

$$(i\hbar\gamma^\mu \partial_\mu - mc)\, q^a_\mu = 0 \quad -(6)$$

or

$$(i\hbar\gamma^\mu \partial_\mu + mc)\, q^a_\mu = 0 \quad -(7)$$

with:

$$p^\mu = i\hbar\, \partial^\mu \quad -(8)$$

The Dirac equation as originally inferred is eq. (6):

$$\boxed{(\gamma^\mu p_\mu - mc)\, q^a_\mu = 0} \quad -(9)$$

and is a particular case of eq. (1)

2) The Dirac matrix is:

$$\gamma^\mu = (\gamma^0, \gamma^i) \quad -(10)$$

and is made up of Pauli matrices as follows:

$$\gamma^0 = \begin{bmatrix} \sigma^0 & 0 \\ 0 & \sigma^0 \end{bmatrix} \quad -(11)$$

$$\gamma^i = \begin{bmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{bmatrix} \quad -(12)$$

This means that Dirac algebra is a consequence of the tetrad postulate of ECE theory.

We have:

$$\gamma^\mu p_\mu = \gamma^0 p_0 - \gamma^i p_i \quad -(13)$$

so eqn. (9) is:

$$(\gamma^0 p_0 - \gamma^i p_i - mc)\, q^a_\mu = 0 \quad -(14)$$

where:

$$p_\mu = (p_0, -\underline{p}) = \left(\frac{En}{c}, -\underline{p}\right) \quad -(15)$$

Now write:

$$q^a_\mu = \begin{bmatrix} \phi_R \\ \phi_L \end{bmatrix} \quad -(16)$$

where ϕ_R and ϕ_L are the Pauli spinors. Then eq. (14) becomes:

3)
$$\begin{bmatrix} -mc & p_0 + \underline{\sigma} \cdot \underline{p} \\ p_0 - \underline{\sigma} \cdot \underline{p} & -mc \end{bmatrix} \begin{bmatrix} \phi_R \\ \phi_L \end{bmatrix} = 0 \quad -(17)$$

as usually written in textbooks. Note that eq. (17) is more correctly written as:

$$\begin{bmatrix} -mc\sigma^0 & p_0\sigma^0 + \underline{\sigma} \cdot \underline{p} \\ p_0\sigma^0 - \underline{\sigma} \cdot \underline{p} & -mc\sigma^0 \end{bmatrix} \begin{bmatrix} \phi_R \\ \phi_L \end{bmatrix} = 0 \quad -(18)$$

<u>The Weyl Equations</u>

These are the Dirac equation for a rest particle where:
$$p_\mu = (p_0, 0) \quad -(19)$$

so eq. (9) becomes:
$$(\gamma^0 p_0 - mc)\psi^a_\mu = 0 \quad -(20)$$

and eq. (17) is:
$$\begin{bmatrix} -mc & p_0 \\ p_0 & -mc \end{bmatrix} \begin{bmatrix} \phi_R \\ \phi_L \end{bmatrix} = 0 \quad -(21)$$

i.e.
$$p_0 \phi_L = mc\,\phi_R \quad -(22)$$
$$p_0 \phi_R = mc\,\phi_L \quad -(23)$$

or
$$\gamma^0 p_0 \psi^a_\mu = mc\,\psi^a_\mu \quad -(24)$$

where $p_0 = \frac{i\hbar}{c}\frac{\partial}{\partial t}$ $-(25)$

4) and $$\gamma^0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \quad -(26)$$

Now we $$\gamma^0\gamma^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad -(27)$$

The Weyl equation is therefore:

$$\boxed{i\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \frac{d\,{}^aV_\mu}{dt} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \left(\frac{mc^2}{\hbar}\right) {}^aV_\mu} \quad -(28)$$

There are four solutions:

1) $${}^aV_1^R = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \exp\left(-i\frac{mc^2}{\hbar}t\right) \quad -(29)$$

2) $${}^aV_2^R = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \exp\left(-i\frac{mc^2}{\hbar}t\right) \quad -(30)$$

3) $${}^aV_1^L = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \exp\left(-i\frac{mc^2}{\hbar}t\right) \quad -(31)$$

4) $${}^aV_2^L = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \exp\left(-i\frac{mc^2}{\hbar}t\right) \quad -(32)$$

5) So from eq. (28):

$$i\frac{\partial q_1^R}{\partial t} = \frac{mc^2}{\hbar} q_1^L \quad -(33)$$

$$i\frac{\partial q_2^R}{\partial t} = \frac{mc^2}{\hbar} q_2^L \quad -(34)$$

$$i\frac{\partial q_1^L}{\partial t} = \frac{mc^2}{\hbar} q_1^R \quad -(35)$$

$$i\frac{\partial q_2^L}{\partial t} = \frac{mc^2}{\hbar} q_2^R \quad -(36)$$

Eqs. (33) and (34) are the same as eq. (23), and eqs. (35) and (36) are the same as eq. (22). It is seen that eqs. (29) to (32) obey eq. (1) in the case:

$$\square \rightarrow \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \quad -(37)$$

corresponding to a rest particle. Therefore:

$$(\square + \kappa^2)\begin{bmatrix} q_1^R & q_2^R \\ q_1^L & q_2^L \end{bmatrix} = 0 \quad -(38)$$

If the column vectors in eqs. (29) to (32) are rearranged as 2×2 matrices, for example:

$$q_1^R = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \exp\left(-\frac{imc^2}{\hbar}t\right) \quad -(39)$$

and so on, then eq. (1) remains true. The eigenfunction of the Dirac equation is a tetrad.

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Notation

$$\boxed{i \frac{\partial \phi_R}{\partial t} = \frac{mc^2}{\hbar} \phi_L} \rightarrow \boxed{i \frac{\partial}{\partial t} \begin{bmatrix} q_1^R \\ q_2^R \end{bmatrix} = \frac{mc^2}{\hbar} \begin{bmatrix} q_1^L \\ q_2^L \end{bmatrix}}$$

So:

$$\boxed{\begin{bmatrix} \phi_R \\ \phi_L \end{bmatrix} = \begin{bmatrix} q_1^R \\ q_2^R \\ q_1^L \\ q_2^L \end{bmatrix} = \phi}$$

Usually ϕ is known as the Dirac spinor.

ECE gives much more information about the internal structure of ϕ.

No Negative Energy Problem

There is no negative energy problem if the Dirac equation is properly interpreted as a limit of the tetrad postulate. The correct interpretation of eq. (9) is:

$$\boxed{\gamma^\mu p_\mu q_\mu^a = \frac{E_0}{c} q_\mu^a} \quad -(40)$$

where

$$E_0 = mc^2 > 0 \quad -(41)$$

and

$$p_\mu = i\hbar \partial_\mu . \quad -(42)$$

129(3): The Weyl Equation and Rest Spinors

The Weyl equation is essentially the quantization of

$$E_0 = mc^2 \quad -(1)$$

It is useful to write out the procedure by which the Weyl equation is obtained from gravity. The first step is to define the tetrad of the Weyl equation:

$$\begin{bmatrix} V^R \\ V^L \end{bmatrix} = \begin{bmatrix} q^R_1 & q^R_2 \\ q^L_1 & q^L_2 \end{bmatrix} \begin{bmatrix} V^1 \\ V^2 \end{bmatrix} \quad -(2)$$

This is an example of $V^a = q^a_\mu V^\mu$ $-(3)$

The physical meaning of the tetrad is to be determined.

The next step is to use the tetrad postulate:

$$D_\mu q^a_\nu = 0, \quad -(4)$$

which can be rewritten as:

$$\Box q^a_\nu = R q^a_\nu \quad -(5)$$

where $R = q^\lambda_a \partial^\mu \left(\Gamma^\nu_{\mu\lambda} q^a_\nu - \omega^a_{\mu b} q^b_\lambda \right)$

$-(6)$

By hypothesis, (which can be called the third ECE hypothesis):

$$R = -kT \quad -(7)$$

where k is the Einstein constant in units of metres

2) per kilogram. Here R has the units of m^{-2}, so T has the units of kgm per cubic metre, which is mass density.

Therefore eq. (5) becomes:

$$\left(\Box + kT\right) q^a_\mu = 0 \quad -(8)$$

which is the ECE wave equation. In the limit where the fermion field becomes that of a free fermion:

$$kT \rightarrow \left(\frac{mc}{\hbar}\right)^2 \quad -(9)$$

where

$$\lambda_c = \frac{\hbar}{mc} \quad -(10)$$

is the Compton wavelength. In this limit eq. (8) becomes:

$$\left(\Box + \left(\frac{mc}{\hbar}\right)^2\right) q^a_\mu = 0 \quad -(11)$$

which is a wave equation whose eigenfunction is the tetrad. Eq. (11) is:

$$\left(\Box + \left(\frac{mc}{\hbar}\right)^2\right)\begin{bmatrix} q^R_1 & q^R_2 \\ q^L_1 & q^L_2 \end{bmatrix} = 0 \quad -(12)$$

i.e.

$$\left(\Box + \left(\frac{mc}{\hbar}\right)^2\right) q^R_1 = 0 \quad -(13)$$

$\vdots$

3) $$\left(\Box + \left(\frac{mc}{\hbar}\right)^2\right) q_2^L = 0 \quad -(14)$$

There are four wave equations in q_1^R, q_2^R, q_1^L, q_2^L. These four tetrad eigenfunctions are to be determined. In the format (13) to (14) these are Klein Gordon equations, and also correspond to the wave format of the Dirac equation. This is seen by writing eqs (13) to (14) in the form of a column four vector, which is the Dirac spinor:

$$\left(\Box + \left(\frac{mc}{\hbar}\right)^2\right)\begin{bmatrix} q_1^R \\ q_2^R \\ q_1^L \\ q_2^L \end{bmatrix} = 0 \quad -(15)$$

The two Pauli spinors are therefore

$$\phi^R = \begin{bmatrix} q_1^R \\ q_2^R \end{bmatrix}, \quad \phi^L = \begin{bmatrix} q_1^L \\ q_2^L \end{bmatrix} \quad (16)$$

The way in which the Dirac equation inter-relates the two Pauli spinors is found by developing the d'Alembertian operator $\Box$ as follows:

4)

$$\Box = \partial^\mu \partial_\mu = \gamma^\mu \gamma^\nu \partial_\mu \partial_\nu \quad -(17)$$

Here:

$$\gamma^\mu = (\gamma^0, \gamma^i) \quad -(18)$$

is the Dirac matrix. This is a 4×4 matrix made up of the Pauli matrices as follows:

$$\gamma^0 = \begin{bmatrix} \sigma^0 & 0 \\ 0 & \sigma^0 \end{bmatrix}, \quad \gamma^i = \begin{bmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{bmatrix} \quad -(19)$$

where

$$i = 1, 2, 3 \quad -(20)$$

The four Pauli matrices are:

$$\sigma^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad -(21)$$

and the Dirac spinor is:

$$\psi = \begin{bmatrix} \phi^R \\ \phi^L \end{bmatrix}. \quad -(22)$$

From eq. (16) it is seen that ψ is made up of four tetrad elements arranged in a column vector.

Eqs. (17) to (21) are the matrix equations, and introduce the Pauli matrices from the d'Alembertian.

The three space-like Pauli matrices obey

5) the cyclical relations:

$$\left[\frac{\sigma^1}{2}, \frac{\sigma^2}{2}\right] = i\frac{\sigma^3}{2} \quad -(23)$$

et cyclicum

and this is an $SU(2)$ symmetry relation. The factor $1/2$ is the geometrical origin of half-integral spin.

With these definitions we obtain the Dirac equation:

$$\boxed{\left(i\gamma^\mu \partial_\mu - \frac{mc}{\hbar}\right)\psi = 0} \quad -(24)$$

It is seen that eq. (24) is an equation of geometry.

In this notation:

$$\gamma^\mu \partial_\mu = \gamma^0 \partial_0 + \gamma^i \partial_i \quad -(25)$$

where:

$$\partial_0 = \frac{1}{c}\frac{\partial}{\partial t}, \quad \partial_1 = \frac{\partial}{\partial X}, \ldots, \partial_3 = \frac{\partial}{\partial Z} \quad -(26)$$

The de-Broglie wave particle dualism is introduced as:

$$p^\mu = i\hbar \partial^\mu \quad -(27)$$

where the four-momentum is

$$p^\mu = \left(\frac{E}{c}, \underline{p}\right) \quad -(28)$$

where E is energy.

6) The classical Einstein energy equation is:

$$p^\mu p_\mu = m^2 c^2 \quad -(29)$$

where

$$p^\mu p_\mu = \frac{E^2}{c^2} - p^2 \quad -(30)$$

Therefore:

$$E^2 = c^2 p^2 + m^2 c^4 \quad -(31)$$

The Weyl equation is obtained when:

$$\underline{p} = \underline{0} \quad -(32)$$

This means that there is no particle momentum. In this case the particle is called "the rest particle", for which:

$$E_0^2 = m^2 c^4 \quad -(33)$$

and:

$$\gamma^\mu \partial_\mu = \gamma^0 \partial_0 \quad -(34)$$

Using eqs. (24) and (27), the Dirac equation becomes:

$$\boxed{(\gamma^\mu p_\mu - mc)\psi = 0} \quad -(35)$$

The Weyl equation is therefore:

$$\boxed{i\gamma^0 \partial_0 \psi = \frac{mc}{\hbar}\psi} \quad -(36)$$

or

$$\boxed{\gamma^0 p_0 \psi = mc\psi} \quad -(37)$$

1) 129(4): Development of the Weyl Equation

The equation may be written as:

$$\gamma^\circ E_\circ \phi = mc^2 \phi \quad - (1)$$

or as:

$$i\gamma^\circ \frac{\partial \phi}{\partial t} = \left(\frac{mc^2}{\hbar}\right)\phi . \quad - (2)$$

Expanding out the matrices:

$$i\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \frac{\partial \phi}{\partial t} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \left(\frac{mc^2}{\hbar}\right)\phi \quad - (3)$$

These equations denote the quantization of

$$E_\circ = mc^2 . \quad - (4)$$

This energy, the rest energy $E_\circ$, is considered positive. Therefore the rest mass m is positive. The spinor is:

$$\phi = \begin{bmatrix} \phi^R \\ \phi^L \end{bmatrix} . \quad - (5)$$

The following are four solutions to eq. (3).

1

$$i\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \frac{\partial}{\partial t} e^{-i\omega t} = \omega \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} e^{-i\omega t} \quad - (6)$$

2

$$i\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \frac{\partial}{\partial t} e^{-i\omega t} = \omega \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} e^{-i\omega t} \quad - (7)$$

(3)
$$i\begin{bmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{bmatrix}\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\frac{\partial}{\partial t}e^{-i\omega t}=\omega\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}0\\0\\1\\0\end{bmatrix}e^{-i\omega t}\quad -(8).$$

(4)
$$i\begin{bmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{bmatrix}\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\frac{\partial}{\partial t}e^{-i\omega t}=\omega\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}0\\0\\0\\1\end{bmatrix}e^{-i\omega t}\quad -(9)$$

where
$$\omega = \frac{mc^2}{\hbar}. \quad -(10)$$

It is seen that the equation inter-relates different column vectors, as follows:

$$(1)\ \begin{bmatrix}0\\0\\1\\0\end{bmatrix}\leftrightarrow\begin{bmatrix}1\\0\\0\\0\end{bmatrix};\quad (2)\ \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\leftrightarrow\begin{bmatrix}0\\1\\0\\0\end{bmatrix};\quad -(11)$$

$$(3)\ \begin{bmatrix}1\\0\\0\\0\end{bmatrix}\leftrightarrow\begin{bmatrix}0\\0\\1\\0\end{bmatrix};\quad (4)\ \begin{bmatrix}0\\1\\0\\0\end{bmatrix}\leftrightarrow\begin{bmatrix}0\\0\\0\\1\end{bmatrix}.$$

Eqns (6) to (9) are:

(1)
$$i\frac{\partial}{\partial t}\begin{bmatrix}\phi_1^L\\0\\0\\0\end{bmatrix}=\omega\begin{bmatrix}\phi_1^R\\0\\0\\0\end{bmatrix}\quad -(12)$$

where
$$\phi_1^L=\phi_1^R=e^{-i\omega t}\quad -(13)$$

3)

(2)
$$i\frac{\partial}{\partial t}\begin{bmatrix}0\\ \phi_2^L\\ 0\\ 0\end{bmatrix} = \omega\begin{bmatrix}0\\ \phi_2^R\\ 0\\ 0\end{bmatrix} \quad -(14)$$

where:
$$\phi_2^L = \phi_2^R = e^{-i\omega t}. \quad -(15)$$

(3)
$$i\frac{\partial}{\partial t}\begin{bmatrix}0\\ 0\\ \phi_1^R\\ 0\end{bmatrix} = \omega\begin{bmatrix}0\\ 0\\ \phi_1^L\\ 0\end{bmatrix} \quad -(16)$$

where
$$\phi_1^R = \phi_1^L = e^{-i\omega t}. \quad -(17)$$

(4)
$$i\frac{\partial}{\partial t}\begin{bmatrix}0\\ 0\\ 0\\ \phi_2^R\end{bmatrix} = \omega\begin{bmatrix}0\\ 0\\ 0\\ \phi_2^L\end{bmatrix} \quad -(18)$$

where
$$\phi_2^R = \phi_2^L = e^{-i\omega t}. \quad -(19)$$

Now add eqns. (12), (14), (16) and (18):
$$i\frac{\partial}{\partial t}\begin{bmatrix}\phi_1^L\\ \phi_2^L\\ \phi_1^R\\ \phi_2^R\end{bmatrix} = \omega\begin{bmatrix}\phi_1^R\\ \phi_2^R\\ \phi_1^L\\ \phi_2^L\end{bmatrix} \quad -(20)$$

or
$$\boxed{i\frac{\partial}{\partial t}\begin{bmatrix}\phi^L\\ \phi^R\end{bmatrix} = \omega\begin{bmatrix}\phi^R\\ \phi^L\end{bmatrix}.} \quad -(21)$$

where
$$\boxed{\phi^R = \phi^L} \quad -(22)$$

4) Eq. (22) means that the rest spinors of a particle are indistinguishable. The reason is a particle at rest has no helicity. The helicity is generated only when the momentum $\underline{p}$ of a particle is non-zero. There is actually nothing in the Weyl equation to indicate right and left helicity. The electric charge does not enter into the analysis at all. Eq. (3) contains only the zeroith Pauli matrix:

$$\sigma^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad -(23)$$

so

$$i \begin{bmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{bmatrix} \frac{\partial \phi}{\partial t} = \begin{bmatrix} \sigma^0 & 0 \\ 0 & \sigma^0 \end{bmatrix} \left(\frac{mc^2}{\hbar} \right) \phi \quad -(24)$$

i.e.

$$i \begin{bmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} \phi^R \\ \phi^L \end{bmatrix} = \begin{bmatrix} \sigma^0 & 0 \\ 0 & \sigma^0 \end{bmatrix} \left(\frac{mc^2}{\hbar} \right) \begin{bmatrix} \phi^R \\ \phi^L \end{bmatrix} \quad -(25)$$

The mass m is positive throughout the analysis, and the rest energy E_0 is positive throughout. There is no negative energy problem and no indication of the existence of an anti-particle because there is no helicity. Electric charge does not enter into the analysis.

129(5): Tetrad Representation of the Weyl Spinors of a Rest Particle

The Weyl spinor of a rest particle is:

$$\phi = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} e^{-i\omega t} \quad -(1)$$

where:

$$\omega = mc^2 / \hbar . \quad -(2)$$

Thus:

$$\left(\Box + \left(\frac{mc}{\hbar} \right)^2 \right) \phi = 0 \quad -(3)$$

where

$$\Box = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} . \quad -(4)$$

Eq. (3) is true if:

$$\phi = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} e^{-i\omega t} \quad -(5)$$

$$= \begin{bmatrix} \phi_1^R & \phi_2^R \\ \phi_1^L & \phi_2^L \end{bmatrix} \quad -(6)$$

Now use:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \sigma^0 + \sigma^1 \quad -(7)$$

where

$$\sigma^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad -(8)$$

are Pauli matrices.

Therefore:

2)

$$\phi = q^a_\mu = (\sigma^0 + \sigma^1) e^{-i\omega t} \quad -(9)$$

The tetrad representation of the spinor of a rest particle is therefore:

$$\boxed{q^a_\mu = (\sigma^0 + \sigma^1) e^{-i\omega t}} \quad -(10)$$

and so:

$$\boxed{q^a_\mu = \begin{bmatrix} \phi^R_1 & \phi^R_2 \\ \phi^L_1 & \phi^L_2 \end{bmatrix} = (\sigma^0 + \sigma^1) e^{-i\omega t}} \quad -(11)$$

Q.E.D.

The Weyl equation is a special case of

$$(\Box + kT) q^a_\mu = 0, \quad -(12)$$

the ECE wave equation.

Mathematical Results

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{2}(\sigma^0 + \sigma^3) \quad -(13)$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \frac{1}{2}(\sigma^1 + i\sigma^2) \quad -(14)$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \frac{1}{2}(\sigma^1 - i\sigma^2) \quad -(15)$$

3) $$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{2}\left(\sigma^0 - \sigma^3\right) \quad -(16)$$

We have:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad -(17)$$

so eqns. (13) to (16) are tetrad / spinor relations. Cartan discovered spinors in 1913 and tetrads in the early twenties.

Therefore:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \sigma^0 + \sigma^3 & \sigma^1 + i\sigma^2 \\ \sigma^1 - i\sigma^2 & \sigma^0 - \sigma^3 \end{bmatrix} \quad -(18)$$

As shown in 129(1), the Pauli matrices are themselves tetrads. The Dirac matrices are also tetrads because: $\quad -(19)$

$$g_{\mu\nu} = q^a_\mu q^b_\nu \eta_{ab} = \frac{1}{2}\left(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu\right) \quad -(20)$$

where

$$g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1) \quad -(21)$$

$$\eta_{ab} = \mathrm{diag}(1, -1, -1, -1) \quad -(22)$$

1.) 129(6): Continuity Equation for Probability in Relativistic Quantum Mechanics.

This type of theory is described in both Ryder and Atkins. For the Klein Gordon, Weyl and Dirac equations a Minkowski spacetime is used. The probability four-current in the correct S.I. units is

$$j^\mu = (c\rho, \underline{j}) \quad -(1)$$

and the continuity equation is:

$$\partial_\mu j^\mu = 0. \quad -(2)$$

In vector notation, eq. (2) is:

$$\frac{1}{c}\frac{\partial\rho}{\partial t} + \underline{\nabla}\cdot\underline{j} = 0. \quad -(3)$$

The operator equivalence is:

$$p^\mu = i\hbar\,\partial^\mu \quad -(4)$$

A real valued current density is obtained from the eigenfunction ϕ by using the sum of the function and its complex conjugate. So, for example, the ~~charge~~ probability density is:

$$\rho = \frac{i\hbar}{2mc^2}\phi^*\frac{\partial\phi}{\partial t} - \frac{i\hbar}{2mc^2}\phi\frac{\partial\phi^*}{\partial t} \quad -(5)$$

$$\boxed{\rho = \frac{i\hbar}{2mc^2}\left(\phi^*\frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^*}{\partial t}\right)} \quad -(6)$$

The S.I. units of $\hbar$ are J s, so ρ is unitless as required. The eigenfunction ϕ is complex valued and unitless. From eq. (6):

$$\frac{\partial}{\partial t}\left(\phi^*\frac{\partial\phi}{\partial t}-\phi\frac{\partial\phi^*}{\partial t}\right)=\frac{\partial\phi^*}{\partial t}\frac{\partial\phi}{\partial t}+\phi^*\frac{\partial^2\phi}{\partial t^2}-\frac{\partial\phi}{\partial t}\frac{\partial\phi^*}{\partial t}-\phi\frac{\partial^2\phi^*}{\partial t^2}$$

$$=\phi^*\frac{\partial^2\phi}{\partial t^2}-\phi\frac{\partial^2\phi^*}{\partial t^2} \quad -(7)$$

Therefore:

$$\boxed{\partial_\mu j^\mu = i\frac{\hbar}{2mc^2}\left(\phi^*\Box\phi-\phi\Box\phi^*\right)} \quad -(8)$$

where

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2 \quad -(9)$$

The Klein Gordon equation is:

$$\left(\Box+\left(\frac{mc}{\hbar}\right)^2\right)\phi = 0 \quad -(10)$$

for a particle without spin. Also:

$$\left(\Box+\left(\frac{mc}{\hbar}\right)^2\right)\phi^* = 0 \quad -(11)$$

The probability density of the equation is:

$$\rho = \frac{i\hbar}{2mc^2}\left(\phi^*\frac{\partial\phi}{\partial t}-\phi\frac{\partial\phi^*}{\partial t}\right) \quad -(12)$$

3)

For a particle at rest, a solution of eqns. (10) and (11) is:

$$\phi = \exp\left(-\frac{imc^2}{\hbar}t\right) \quad -(13)$$

$$\phi^* = \exp\left(\frac{imc^2}{\hbar}t\right) \quad -(14)$$

In this case:

$$\frac{\partial\phi}{\partial t} = -\frac{imc^2}{\hbar}\phi, \quad -(15)$$

$$\frac{\partial\phi^*}{\partial t} = \frac{imc^2}{\hbar}\phi^* \quad -(16)$$

and $\boxed{\rho = 1}$ $-(17)$

There is 100% probability of finding the particle at rest. Otherwise the Klein Gordon equation is a second order differential equation. It needs initial condition on ϕ and $\partial\phi / \partial t$. So in general the probability density ρ may be negative. In quantum field theory this problem is circumvented by second quantization, so the Klein Gordon equation is no longer regarded as a single particle equation. The Weyl equation does not have ~~this~~ problem for the rest particle as shown in the next note.

1. 129(7). Probability Density of the Weyl Equation.

The wavefunction or spinor of the Weyl equation has been shown in previous notes to be derivable from Cartan geometry. Therefore the probability density given by the equation also has a geometrical origin, as has the expectation values. These are all therefore properties of spacetime itself.

Some basic mathematical concepts are given first for ease of reference. The first concept is that of a hermitian matrix. This is a square matrix which is not changed by taking the transpose of its complex conjugate. For example, if:

$$\tilde{A}^* = A. \quad -(1)$$

$$A = \begin{bmatrix} 1 & 1+i \\ 1-i & 3 \end{bmatrix},$$

The transpose of a column vector is a row vector. For example, if:

$$A = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \quad \tilde{A} = [a_1 \quad a_2 \quad a_3] \quad -(2)$$

A row vector multiplied by a column vector is a scalar:

$$[a_1 \quad a_2 \quad a_3]\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = a_1^2 + a_2^2 + a_3^2. \quad -(3)$$

The Dirac spinor is

$$\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix} \quad -(4)$$

and its transposed complex conjugate is:

$$\psi^+ = [\psi_1^* \quad \psi_2^* \quad \psi_3^* \quad \psi_4^*]. \quad -(5)$$

The probability density of the Dirac and Weyl equations is:

2)

$$\rho = \gamma\, \psi^+ \psi \quad -(6)$$

Using the rule (3):

$$\boxed{\rho = \psi_1 \psi_1^* + \psi_2 \psi_2^* + \psi_3 \psi_3^* + \psi_4 \psi_4^*} \quad -(7)$$

This is positive definite and can therefore be interpreted as a probability in quantum mechanics. The probability density of relativistic quantum mechanics is therefore defined by tetrad elements of Cartan geometry.

In Dirac algebra and the theory of the Dirac equation, the adjoint spinor $\overline{\psi}$ is used. This is defined by:

$$\overline{\psi} = \psi^+ \gamma^0 \quad -(8)$$

where γ^0 is the zeroth or time-like Dirac matrix:

$$\gamma^0 = \begin{bmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad -(9)$$

where

$$\sigma^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad -(10)$$

is the zero order Pauli matrix, the 2×2 unit matrix. So

$$\overline{\psi} = \begin{bmatrix} \psi_1^* & \psi_2^* & \psi_3^* & \psi_4^* \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad -(11)$$

3) i.e

$$\boxed{\bar{\psi} = [\psi_3^* \; \psi_4^* \; \psi_1^* \; \psi_2^*]} \quad -(12)$$

It follows that:

$$\rho = \bar{\psi} \gamma^0 \psi = \psi^+ \psi \quad -(13)$$

The purpose of introducing the adjoint spinor $\bar{\psi}$ is that the probability density ρ is the expectation value of γ^0. Therefore the expectation value of quantum mechanics can be traced to the geometry of spacetime. The expectation value was originally introduced by Born in an empirical way. ECE theory gives much more insight to it.

More generally, the probability four current j^μ is the expectation value of the Dirac matrix γ^μ:

$$\boxed{j^\mu = \bar{\psi} \gamma^\mu \psi} \, . \quad -(14)$$

This current is conserved:

$$\partial_\mu j^\mu = 0 . \quad -(15)$$

Therefore:

$$\partial_\mu \left(\bar{\psi} \gamma^\mu \psi \right) = 0 . \quad -(16)$$

using the Leibnitz Theorem:

$$\partial_\mu \left(\bar{\psi} \gamma^\mu \psi \right) = \left(\partial_\mu \bar{\psi} \right) \gamma^\mu \psi + \bar{\psi} \gamma^\mu \partial_\mu \psi \quad -(17)$$

4.) The probability four current is:

$$j^{\mu} = \begin{bmatrix} \psi_3^* & \psi_4^* & \psi_1^* & \psi_2^* \end{bmatrix} \gamma^{\mu} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix} \qquad -(18)$$

where: $$\gamma^{\mu} = (\gamma^0, \gamma^i), \quad \gamma^0 = \begin{bmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{bmatrix}, \quad \gamma^i = \begin{bmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{bmatrix} \qquad -(19)$$

$$\sigma^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \qquad -(20)$$

Therefore:

$$j^0 = \begin{bmatrix} \psi_3^* & \psi_4^* & \psi_1^* & \psi_2^* \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix} \qquad -(21)$$

$$= \begin{bmatrix} \psi_3^* & \psi_4^* & \psi_1^* & \psi_2^* \end{bmatrix} \begin{bmatrix} \psi_3 \\ \psi_4 \\ \psi_1 \\ \psi_2 \end{bmatrix} \qquad -(22)$$

$$\boxed{j^0 = \psi_1 \psi_1^* + \psi_2 \psi_2^* + \psi_3 \psi_3^* + \psi_4 \psi_4^*}$$

$$j^1 = \begin{bmatrix} \psi_3^* & \psi_4^* & \psi_1^* & \psi_2^* \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix} \qquad -(23)$$

$$= \begin{bmatrix} \psi_3^* & \psi_4^* & \psi_1^* & \psi_2^* \end{bmatrix} \begin{bmatrix} -\psi_4 \\ -\psi_3 \\ \psi_2 \\ \psi_1 \end{bmatrix}$$

5) $$\boxed{j_1 = \psi_1\psi_2^* + \psi_2\psi_1^* - \psi_3\psi_4^* - \psi_4\psi_3^*} \quad -(24)$$

$$j_2 = \left[\psi_3^* \ \psi_4^* \ \psi_1^* \ \psi_2^*\right] \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix} \quad -(25)$$

$$= i\left[\psi_3^* \ \psi_4^* \ \psi_1^* \ \psi_2^*\right] \begin{bmatrix} \psi_4 \\ -\psi_3 \\ -\psi_2 \\ \psi_1 \end{bmatrix}$$

$$\boxed{j_2 = i\left(\psi_1\psi_3^* - \psi_2\psi_4^* - \psi_3\psi_1^* + \psi_4\psi_2^*\right)} \quad -(26)$$

$$j_3 = \left[\psi_3^* \ \psi_4^* \ \psi_1^* \ \psi_2^*\right] \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix} \quad -(27)$$

$$= \left[\psi_3^* \ \psi_4^* \ \psi_1^* \ \psi_2^*\right] \begin{bmatrix} -\psi_3 \\ \psi_4 \\ \psi_1 \\ -\psi_2 \end{bmatrix}$$

$$\boxed{j_3 = \psi_1\psi_1^* - \psi_2\psi_2^* - \psi_3\psi_3^* + \psi_4\psi_4^*} \quad -(28)$$

We must now test directly whether:

$$\left(\partial_\mu \bar{\psi}\right)\gamma^\mu \psi + \bar{\psi}\gamma^\mu \partial_\mu \psi = 0 \quad -(29)$$

129(8): Conservation of Probability Density in the Weyl Equation.

In the last note it was shown that:

$$\partial_\mu j^\mu = (\partial_\mu \bar{\phi}) \gamma^\mu \phi + \bar{\phi} \gamma^\mu \partial_\mu \phi \quad -(1)$$

The continuity equation is:

$$\partial_\mu j^\mu = 0. \quad -(2)$$

For the Weyl equation of a rest particle only the γ^0 matrix appears. Therefore the continuity eq. (2) is:

$$\partial_0 j^0 = 0. \quad -(3)$$

This is analogous to conservation of charge. In this case it may be checked directly as follows that:

$$(\partial_0 \bar{\phi}) \gamma^0 \phi + \bar{\phi} \gamma^0 \partial_0 \phi = 0. \quad -(4)$$

We have

$$\gamma^0 \phi = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{bmatrix} = \begin{bmatrix} \phi_3 \\ \phi_4 \\ \phi_1 \\ \phi_2 \end{bmatrix} \quad -(5)$$

and

$$\partial_0 \bar{\phi} = [\partial_0 \phi_3^* \;\; \partial_0 \phi_4^* \;\; \partial_0 \phi_1^* \;\; \partial_0 \phi_2^*] \quad -(6)$$

So:

$$(\partial_0 \bar{\phi}) \gamma^0 \phi = \phi_1 \partial_0 \phi_1^* + \phi_2 \partial_0 \phi_2^* + \phi_3 \partial_0 \phi_3^* + \phi_4 \partial_0 \phi_4^* \quad -(7)$$

Similarly:

$$\gamma^0 \partial_0 \phi = \begin{bmatrix} \partial_0 \phi_3 \\ \partial_0 \phi_4 \\ \partial_0 \phi_1 \\ \partial_0 \phi_2 \end{bmatrix} \quad -(8)$$

2)

$$\bar{\psi}\gamma^0\partial_0\psi = \psi_1^*\partial_0\psi_1 + \psi_2^*\partial_0\psi_2 + \psi_3^*\partial_0\psi_3 + \psi_4^*\partial_0\psi_4 \quad -(9)$$

So:

$$\boxed{\begin{aligned}\partial_0 j^0 = {} & \left(\psi_1\partial_0\psi_1^* + \psi_1^*\partial_0\psi_1\right) + \left(\psi_2\partial_0\psi_2^* + \psi_2^*\partial_0\psi_2\right) \\ & + \left(\psi_3\partial_0\psi_3^* + \psi_3^*\partial_0\psi_3\right) + \left(\psi_4\partial_0\psi_4^* + \psi_4^*\partial_0\psi_4\right)\end{aligned}} \quad -(10)$$

For the Weyl equation:

$$\psi_1 = \psi_2 = \psi_3 = \psi_4 = \exp\left(-\frac{imc^2}{\hbar}t\right) \quad -(11)$$

$$\psi_1^* = \psi_2^* = \psi_3^* = \psi_4^* = \exp\left(\frac{imc^2}{\hbar}t\right) \quad -(12)$$

Therefore

$$\partial_0\psi_1^* = \frac{i}{c}\psi_1^* \text{ etc.} \quad -(13)$$

$$\partial_0\psi_1 = -\frac{i}{c}\psi_1 \text{ etc.} \quad -(14)$$

So

$$\psi_1\partial_0\psi_1^* + \psi_1^*\partial_0\psi_1 = 0 \quad -(15)$$

etc.

Q.E.D.

Remarks

The probability density of the Weyl equation is rigorously conserved. The spinors are rigorously defined in previous notes to paper 129. In this analysis negative energy is rejected. The classical rest energy is always:

$$E_0 = mc^2 \quad -(16)$$

for ϕ_1, ϕ_2, ϕ_3 and ϕ_4. The wave equation is obeyed:

$$\left(\Box + \left(\frac{mc}{\hbar}\right)^2 \right) \phi = 0 \quad -(17)$$

$$\left(\Box + \left(\frac{mc}{\hbar}\right)^2 \right) \phi^* = 0 \quad -(18)$$

For a rest particle:

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \quad -(19)$$

The Dirac equations for ϕ and $\overline{\phi}$ are:

$$\left(i\gamma^\mu \partial_\mu - \frac{mc}{\hbar} \right) \phi = 0 \quad -(20)$$

and

$$\overline{\phi} \left(i\gamma^\mu \partial_\mu + \frac{mc}{\hbar} \right) = 0 \quad -(21)$$

The Weyl equations are:

$$\left(i\gamma^0 \partial_0 - \frac{mc}{\hbar} \right) \phi = 0 \quad -(22)$$

$$\overline{\phi} \left(i\gamma^0 \partial_0 + \frac{mc}{\hbar} \right) = 0 \quad -(23)$$

4) If we multiply eq. (22) and eq. (23) and use

$$\Box = \gamma^\mu \gamma^\nu \partial_\mu \partial_\nu \quad -(24)$$

$$\frac{1}{c} \frac{\partial^2}{\partial t^2} = \gamma^0 \gamma^0 \partial_0 \partial_0 \quad -(25)$$

Then:

$$\boxed{\bar{\psi} \left(\Box + \frac{m^2 c^2}{\hbar^2} \right) \psi = 0} \quad -(26)$$

This may be written:

$$\bar{\psi} \Box \psi = -\bar{\psi} \frac{m^2 c^2}{\hbar^2} \psi \quad -(27)$$

$$\boxed{\bar{\psi} \Box \psi = -\frac{m^2 c^2}{\hbar^2} \bar{\psi} \psi} \quad -(28)$$

This means:

$$\bar{\psi} \frac{\partial^2 \psi}{\partial t^2} = -\frac{m^2 c^4}{\hbar^2} \bar{\psi} \psi \quad -(29)$$

Now use:

$$p^\mu = i \hbar \partial^\mu \quad -(30)$$

so

$$\frac{E_0^2}{c^2} \; p^0 p_0 = -\hbar^2 \partial^0 \partial_0 = -\frac{\hbar^2}{c^2} \frac{\partial^2}{\partial t^2}$$

and

$$E_0^2 \bar{\psi} \psi = m^2 c^4 \bar{\psi} \psi \quad -(31)$$

$$\boxed{E_0 = mc^2}$$

129(9): Conservation of Probability Four-Current in the Dirac Equation.

This is demonstrated by using:

$$\left(i\gamma^\mu \partial_\mu - \frac{mc}{\hbar}\right)\psi = 0 \quad -(1)$$

$$\bar{\psi}\left(i\gamma^\mu \partial_\mu + \frac{mc}{\hbar}\right) = 0. \quad -(2)$$

Therefore:

$$\gamma^\mu \partial_\mu \psi = -\frac{imc}{\hbar}\psi \quad -(3)$$

$$\gamma^\mu \partial_\mu \bar{\psi} = \frac{imc}{\hbar}\bar{\psi} \quad -(4)$$

It follows that:

$$\partial_\mu\left(\bar{\psi}\gamma^\mu\psi\right) = \left(\partial_\mu\bar{\psi}\right)\gamma^\mu\psi + \bar{\psi}\gamma^\mu\left(\partial_\mu\psi\right)$$

$$= \left(\gamma^\mu\partial_\mu\bar{\psi}\right)\psi + \bar{\psi}\left(\gamma^\mu\partial_\mu\psi\right)$$

$$= \frac{imc}{\hbar}\bar{\psi}\psi - \frac{imc}{\hbar}\bar{\psi}\psi \quad -(5)$$

$$= 0$$

So $\boxed{\partial_\mu j^\mu = 0}$ Q.E.D. $-(6)$

Here $\boxed{j^\mu = \bar{\psi}\gamma^\mu\psi}$ $-(7)$

2)

Eqn. (7) shows clearly that the Dirac equation is based on geometry, because:

$$\Box = \gamma^{\mu}\gamma^{\nu}\partial_{\mu}\partial_{\nu} \quad -(8)$$

The probability four current of relativistic quantum mechanics is geometrical in origin. We now know that the origin of the equation is the tetrad postulate:

$$D_{\mu} q^{a}_{\nu} = 0, \quad -(9)$$

which can be rewritten as:

$$\Box q^{a}_{\nu} = R q^{a}_{\nu}. \quad -(10)$$

The origin of the Dirac spinor ψ is the Cartan tetrad q^{a}_{ν}. Philosophically, this means that there is no indeterminacy, because there is nothing about geometry that is absolutely unknowable.

Therefore the Copenhagen interpretation of quantum mechanics is rejected.

Any wave equation of physics is also a Dirac equation. This is because eq. (10) is

$$\boxed{\gamma^{\mu}\partial_{\mu}\gamma^{\sigma}\partial_{\sigma} q^{a}_{\nu} = R q^{a}_{\nu}} \quad -(11)$$

Eq. (11) gives:

3)

$$\boxed{\left(i\gamma^\mu \partial_\mu - R^{1/2}\right) q^a_{\ \sim} = 0} \quad -(12)$$

which is the Dirac equation in any spacetime.

Similarly: $$\bar{q}^a_{\ \sim}\left(i\gamma^\mu \partial_\mu + R^{1/2}\right) = 0 \quad -(13)$$

where $\bar{q}^a_{\ \sim}$ is the adjoint tetrad.

Finally eq (6) written out in full is:

$$\begin{aligned}
&\partial_0\left(\phi_1\phi_1^* + \phi_2\phi_2^* + \phi_3\phi_3^* + \phi_4\phi_4^*\right)\\
&+ \partial_1\left(\phi_1\phi_2^* + \phi_2\phi_1^* - \phi_3\phi_4^* - \phi_4\phi_3^*\right)\\
&+ i\,\partial_2\left(\phi_1\phi_3^* - \phi_2\phi_4^* - \phi_3\phi_1^* + \phi_4\phi_2^*\right)\\
&+ \partial_3\left(\phi_1\phi_1^* - \phi_2\phi_2^* - \phi_3\phi_3^* + \phi_4\phi_4^*\right)\\
&= 0
\end{aligned} \quad -(14)$$

More generally, the probability current is the expectation value:

$$\boxed{j^\mu = \bar{q}^a_{\ \sim}\,\gamma^\mu\, q^{\sim}_{\ a}} \quad -(15)$$

) 129(10) : Symmetry of the Weyl Equation.

The Weyl equation may be written as :

$$i \frac{\partial \phi^L}{\partial t} = \frac{mc^2}{\hbar} \phi^R \quad -(1)$$

$$i \frac{\partial \phi^R}{\partial t} = \frac{mc^2}{\hbar} \phi^L \quad -(2)$$

where ϕ^R and ϕ^L are the Pauli spinors. The Dirac spinor is :

$$\phi = \begin{bmatrix} \phi^R \\ \phi^L \end{bmatrix} . \quad -(3)$$

The fundamental symmetry operators are : $\hat{C}$, $\hat{P}$, $\hat{T}$, $C\hat{P}$, $C\hat{T}$, $\hat{P}T$ and $C\hat{P}T$. Here $\hat{C}$ is the charge conjugation operator :

$$\hat{C}(e) = -e \quad -(4)$$

which reverses the sign of electric charge e. P is the parity operator, which has the effect :

$$\hat{P}(\underline{r}) = -\underline{r} \quad -(5)$$

where $\underline{r}$ is the position vector, and $\hat{T}$ is the motion reversal operator :

$$\hat{T}(\underline{p}) = -\underline{p} \quad -(6)$$

which reverses the momentum. The Weyl equation (1) and (2) do not involve electric charge, so automatically conserve $\hat{C}$. They do not involve momentum, because they are for a particle at rest, but under $\hat{T}$, we have

$$\hat{T}(t) = -t \quad -(7)$$

where t is the time. The spinors of the Weyl equation

2) are:
$$\phi_1^R = \phi_2^R = \phi_1^L = \phi_2^L$$
$$= \exp\left(-\frac{imc^2}{\hbar}t\right) \quad -(8)$$

and are interpreted in ECE theory as spinors of "the particle with spin." More accurately this should be "the particle with helicity". When the particle is at rest however the helicity is zero, because it has no momentum. From eqs. (1), (2) and (8):
$$i\frac{\partial}{\partial t}\left(\exp\left(-\frac{imc^2}{\hbar}t\right)\right) = \frac{mc^2}{\hbar}\exp\left(-\frac{imc^2}{\hbar}t\right). \quad -(9)$$

The application of $\hat{p}$ leaves the equation unchanged, because it does not contain $\underline{r}$. This is true, however, only for one sense of frame. In the Cartesian system, the frame sense (chirality) is defined by:
$$\underline{i} \times \underline{j} = \underline{k} \quad -(10)$$
et cyclicum

but we may also have all the equations of physics written in the frame of opposite chirality or handedness:
$$\underline{i} \times \underline{j} = -\underline{k} \quad -(11)$$
et cyclicum

For the Pauli matrices:
$$\left[\frac{\sigma^1}{2}, \frac{\sigma^2}{2}\right] = i\frac{\sigma^3}{2} \quad -(12)$$
et cyclicum

3) or: $\left[\frac{\sigma^1}{2}, \frac{\sigma^2}{2} \right] = -i \frac{\sigma^3}{2}$ — (13)

et cyclicum.

Eq. (11) is generated from eq. (10) by:

$$\hat{P}(\underline{k}) = -\underline{k} \quad — (14)$$

and eq. (13) is generated from eq. (12) by:

$$\hat{P}(\sigma^3) = -\sigma^3, \quad — (15)$$

i.e.

$$\boxed{\hat{P} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}} \quad — (16)$$

In ECE theory it will be investigated whether eq. (16) generates the antiparticle and not the Dirac sea. If so, eq. (16) is preferred by Ockham's Razor.

Reversing the sign of t in eq. (9):

$$-i \frac{\partial}{\partial t} \left(\exp \left(\frac{imc^2}{\hbar} t \right) \right) = \frac{mc^2}{\hbar} \exp \left(\frac{imc^2}{\hbar} t \right) \quad — (17)$$

So the Weyl equation conserves $\hat{T}$.

Therefore the Weyl equation conserves $\hat{C}, \hat{P}, \hat{T}$, $\hat{CP}$, $\hat{CT}$, $\hat{PT}$ and $\hat{CPT}$. It also conserves these operators if eq. (16) is applied. This is automatic because it involves only $\underline{\sigma}$.