135(1): Antisymmetry in Weak and Strong Field Theory.

Consider the first Cartan-Maurer structure equation in tensor notation:

$$T^a_{\mu\nu} = \partial_\mu q^a_\nu - \partial_\nu q^a_\mu + \omega^a_{\mu b} q^b_\nu - \omega^a_{\nu b} q^b_\mu \quad -(1)$$

where:

$$a = (1), (2), (3) \quad -(2)$$

Eq. (1) in vector notation is:

$$\underline{T}_{\mu\nu} = \partial_\mu \underline{q}_\nu - \partial_\nu \underline{q}_\mu + \underline{\omega}_{\mu b} q^b_\nu - \underline{\omega}_{\nu b} q^b_\mu \quad -(3)$$

where:

$$\underline{T}_{\mu\nu} = T^{(1)}_{\mu\nu} \underline{e}^{(1)} + T^{(2)}_{\mu\nu} \underline{e}^{(2)} + T^{(3)}_{\mu\nu} \underline{e}^{(3)} \quad -(4)$$

$$\underline{q}_\nu = q^{(1)}_\nu \underline{e}^{(1)} + q^{(2)}_\nu \underline{e}^{(2)} + q^{(3)}_\nu \underline{e}^{(3)} \quad -(5)$$

$$\underline{\omega}_{\mu b} = \omega^{(1)}_{\mu b} \underline{e}^{(1)} + \omega^{(2)}_{\mu b} \underline{e}^{(2)} + \omega^{(3)}_{\mu b} \underline{e}^{(3)} \quad -(6)$$

The electromagnetic field is therefore:

$$\underline{F}_{\mu\nu} = \partial_\mu \underline{A}_\nu + \underline{\omega}_{\mu b} A^b_\nu - \left(\partial_\nu \underline{A}_\mu + \underline{\omega}_{\nu b} A^b_\mu\right) \quad -(7)$$

with antisymmetry constraint:

$$\partial_\mu \underline{A}_\nu + \underline{\omega}_{\mu b} A^b_\nu + \partial_\nu \underline{A}_\mu + \underline{\omega}_{\nu b} A^b_\mu = \underline{0} \quad -(8)$$

In this notation:

$$A^{(0)} \underline{\omega}_{\mu\nu} = \underline{\omega}_{\mu b} A^b_\nu . \quad -(9)$$

The magnetic field is:

$$\underline{B}_{\mu\nu} = \partial_\mu \underline{A}_\nu - \partial_\nu \underline{A}_\mu + \underline{\omega}_{\mu b} A^b_\nu - \underline{\omega}_{\nu b} A^b_\mu \quad -(10)$$

$$\mu, \nu = 1, 2, 3$$

i.e.

$$\underline{B}_{\mu\nu} = \partial_\mu \underline{A}_\nu - \partial_\nu \underline{A}_\mu + A^{(0)} \left(\underline{\omega}_{\mu\nu} - \underline{\omega}_{\nu\mu} \right) \quad -(11)$$

and the electric field is:

$$\underline{E}_{0i} = c \left(\partial_0 \underline{A}_i - \partial_i \underline{A}_0 + A^{(0)} \left(\underline{\omega}_{0i} - \underline{\omega}_{i0} \right) \right) \quad -(12)$$

In this format the theory is <u>the generalization of gauge theory</u>, in which:

3)

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - ig\left(A^b_\mu A^c_\nu - A^b_\nu A^c_\mu\right) \quad -(13)$$

In eq. (13) the internal indices a, b and c are abstract, while in eq. (7) they are indices of the circular complex basis. In ECE theory, gauge theory is replaced by general relativity. Therefore

$$A^{(0)}\omega^a_{\mu\nu} = -ig A^b_\mu A^c_\nu, \quad -(14)$$

$$A^{(0)}\omega^a_{\nu\mu} = -ig A^b_\nu A^c_\mu \quad -(15)$$

$$B^a_{\mu\nu} = A^{(0)}\left(\omega^a_{\mu\nu} - \omega^a_{\nu\mu}\right) \quad -(16)$$

$$= -ig\left(A^b_\mu A^c_\nu - A^b_\nu A^c_\mu\right)$$

i.e.

$$\underline{B}_{\mu\nu} = -ig\, \underline{A}_\mu \times \underline{A}_\nu \quad -(17)$$

or

$$\underline{B}^{(3)*} = -ig\, \underline{A}^{(1)} \times \underline{A}^{(2)}. \quad -(18)$$

<u>This is the</u> $\underline{B}^{(3)}$ <u>field.</u> From antisymmetry:

$$\partial_\mu A^a_\nu - ig A^b_\mu A^c_\nu = -\left(\partial_\nu A^a_\mu - ig A^b_\nu A^c_\mu\right) \quad -(19)$$

This method can now be extended to generalize all gauge field theory.

1) 135(2): Replacement of Gauge Theory by ECE

As discussed by Carroll on his pp 147 ff the idea of gauge theory is based on an internal 3-D vector space with structure group $SO(3)$, for example. In this case the gauge field is $\phi^A(x^\mu)$, $A = 1, 2, 3$. It is an internal, abstract, three-vector unrelated to spacetime. The gauge transform is:

$$\phi^A \rightarrow O^{A'}{}_A \phi^A \quad -(1)$$

Gauge theories are severely limited by eqn. (1). In these theories the connection is the connection on the fibre bundle, and is denoted $A^A_{\mu B}$. Under gauge transform:

$$A^{A'}_{\mu B'} = O^{A'}_A O^B_{B'} A^A_{\mu B} - O^C_{B'} \partial_\mu O^{A'}_C \quad -(2)$$

The gauge covariant derivative is:

$$D_\mu \phi^A = \partial_\mu \phi^A + A^A_{\mu B} \phi^B \quad -(3)$$

Fibre bundles are completely abstract, not geometrical. The torsion tensor is not defined for any gauge theory connections. The tetrad cannot be used in gauge theory. In addition, the new antisymmetry law of ECE prohibits gauge freedom.

2)

In ECE theory, the tetrad is defined in a geometrical context, and so the basis is defined. In ECE, basis vectors point along coordinate axes, and the coordinate basis may be used:

$$\hat{e}_{(\mu)} = \partial_\mu . \quad -(4)$$

The tetrad in ECE is defined by:

$$\hat{e}_{(a)} = q^{\mu}_{\ a} \partial_\mu = q^{\mu}_{\ a} \hat{e}_{(\mu)} \quad -(5)$$

or:

$$\hat{e}_{(\mu)} = q^{a}_{\ \mu} \hat{e}_{(a)} \quad -(6)$$

where $\hat{e}_{(a)}$ is another orthogonal basis.

<u>ECE</u> therefore has <u>major advantages over the now obsolete</u> <u>gauge theory</u>.

To illustrate this consider the plane wave:

$$\underset{\sim}{r}(\phi) = (\underline{i} - i\underline{j})e^{i\phi} \quad -(7)$$

Its <u>Frenet tangent vector</u> is:

$$\underset{\sim}{T} = d\underline{r}/d\phi = i\underline{r} \quad -(8)$$

$$= (i\underline{i} + \underline{j})e^{i\phi} \quad -(9)$$

Its components are:

$$\left.\begin{array}{l} T_X = -T_1 = ie^{i\phi}, \\ T_Y = -T_2 = e^{i\phi} \end{array}\right\} \quad -(10)$$

Its Frenet normal vector is :

$$\underline{N} = \frac{d\underline{T}}{d\phi} = \frac{d^2\underline{r}}{d\phi^2} = -\underline{r} \quad -(11)$$

Its Frenet binormal vector is:

$$\underline{B} = \underline{T} \times \underline{N} = \underline{0} \quad -(12)$$

We have:

$$\left.\begin{aligned} \underline{B} &= \underline{T} \times \underline{N} \\ \underline{N} &= \underline{B} \times \underline{T} \\ \underline{T} &= \underline{N} \times \underline{B} \end{aligned}\right\} \quad -(13)$$

If we consider:

$$Re(\underline{r}(\phi)) = (\cos\phi, \sin\phi, 0) \quad -(14)$$

then

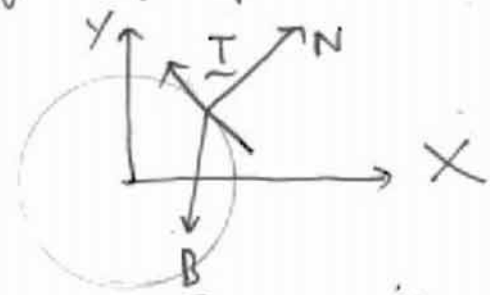
$$\underline{T} = (-\sin\phi, \cos\phi, 0) \quad -(15)$$

$$\underline{N} = (-\cos\phi, -\sin\phi, 0) \quad -(16)$$

$$\underline{B} = (0, 0, 1) \quad -(17)$$

Helix

Circle



4)

The $(\underset{\sim}{T}, \underline{N}, \underline{B})$ frame goes around in a circle defined by eq. (14), the $\underline{B}$ vector is in the Z direction.

The tangent vector $\underline{N}$ is defined in this case by $d/d\phi$ acting on $\underset{\sim}{r}$. Therefore T_x and T_y can be thought of as a frame of reference going around in a circle. <u>This is an example of the basis (4)</u>.

In a <u>propagating</u> plane wave:

$$\underset{\sim}{r} = (\underline{i} - i\underline{j}) \exp(i(\omega t - KZ)) \quad -(18)$$

and so:

$$\underset{\sim}{r} = \underset{\sim}{r}(Z). \quad -(19)$$

This is a curve in differential geometry with parameter Z. The tangent is:

$$\boxed{\underset{\sim}{T} = \frac{d\underset{\sim}{r}}{dZ} = -iK\underset{\sim}{r}} \quad -(20)$$

$$\text{So:} \; -T_x = T_1 = \frac{\partial r_1}{\partial Z} = \partial_3 r_1 = iKe^{i\phi} \quad -(21)$$

$$-T_y = T_2 = \frac{\partial r_2}{\partial Z} = \partial_3 r_2 = Ke^{i\phi} \quad -(22)$$

<u>Here ∂_3 is an example of ∂_μ in eq. (4)</u>

Cartan generalized this Frenet differential geometry. In order to define a dimensionless tetrad it is convenient to use the dimensionless components (10) and compare them with the static Cartesian frame:

$$e(a) = (1, 1). \quad -(23)$$

Thus:

$$T(\mu) = q^a_{\ \mu}\, e(a) \quad -(24)$$

$$\begin{bmatrix} T_X \\ T_Y \end{bmatrix} = \begin{bmatrix} q^X_{\ X} & q^X_{\ Y} \\ q^Y_{\ X} & q^Y_{\ Y} \end{bmatrix} \begin{bmatrix} e_X \\ e_Y \end{bmatrix} \quad -(25)$$

i.e.

$$\begin{bmatrix} i \\ 1 \end{bmatrix} e^{i\phi} = \begin{bmatrix} q^X_{\ X} & q^X_{\ Y} \\ q^Y_{\ X} & q^Y_{\ Y} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad -(26)$$

$$\left.\begin{aligned} q^X_{\ X} &= i e^{i\phi}, \quad q^Y_{\ Y} = e^{i\phi} \\ q^X_{\ Y} &= q^Y_{\ X} = 0 \end{aligned}\right\} \quad -(27)$$

<u>This</u> tetrad <u>cannot</u> be <u>defined in gauge theory</u>.

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1)

135(3): Commutator Expansion of Basic Equations in Field Theory.

In Riemann geometry:

$$[D_\mu, D_\nu] V^\rho = [\partial_\mu, \partial_\nu] V^\rho + [\partial_\mu, \Gamma^\rho_{\nu\sigma}] V^\sigma - [\partial_\nu, \Gamma^\rho_{\mu\sigma}] V^\sigma - (\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}) D_\lambda V^\rho + [\Gamma^\rho_{\mu\lambda}, \Gamma^\lambda_{\nu\sigma}] V^\sigma \quad -(1)$$

where

$$\Gamma^\lambda_{[\mu,\nu]} := \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}. \quad -(2)$$

By definition, all these terms are individually antisymmetric in μ and ν, because they are all commutators.

The standard model is refuted immediately from

$$\Gamma^\lambda_{\mu\nu} = - \Gamma^\lambda_{\nu\mu}. \quad -(3)$$

It is therefore trivial to show that the standard theory of gravitation is incorrect, because it uses:

$$\Gamma^\lambda_{\mu\nu} = ? \; \Gamma^\lambda_{\nu\mu} \quad -(4)$$

In electromagnetism, on the $U(1)$ level:

$$[D_\mu, D_\nu] \phi = -ig [\partial_\mu, \partial_\nu] \phi - ig [A_\mu, \partial_\nu] \phi - ig [\partial_\mu, A_\nu] \phi - g^2 [A_\mu, A_\nu] \phi \quad -(5)$$

and every term is antisymmetric because every term is a commutator.

2)

Therefore U(1) electrodynamics is trivially refuted as follows. In U(1) e/m:

$$[D_\mu, D_\nu]\phi \overset{?}{=} -ig(\partial_\mu A_\nu - \partial_\nu A_\mu)\phi \quad -(6)$$

but

$$\partial_\mu A_\nu = -\partial_\nu A_\mu \quad -(7)$$

Therefore:

$$\frac{\partial \underline{A}}{\partial t} = \underline{\nabla}\phi. \quad -(8)$$

However:

$$\underline{\nabla} \times \frac{\partial \underline{A}}{\partial t} = \underline{\nabla} \times \underline{\nabla}\phi = \underline{0}. \quad -(9)$$

In U(1):

$$\underline{E} = -\underline{\nabla}\phi - \frac{\partial \underline{A}}{\partial t}, \quad -(10)$$

$$\underline{B} = \underline{\nabla} \times \underline{A} \quad -(11)$$

Eq. (9) means that:

$$\underline{\nabla} \times \underline{E} = \underline{0} \quad -(12)$$

and

$$\frac{\partial \underline{B}}{\partial t} = \underline{0} \quad -(13)$$

Eqs (12) and (13) mean that $\underline{E}$ and $\underline{B}$ are always static. In U(1) in this case:

$$\underline{A} = \underline{0} \quad -(14)$$

so

$$\underline{E} = \underline{B} = \underline{0} \quad -(15)$$

Also, U(1) gauge theory is trivially refuted as follows.

3) On the u(1) level:

$$\partial_\mu F^{\mu\nu} = j^\nu / \epsilon_0 \quad -(16)$$

where

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \quad -(17)$$

Therefore:

$$\partial_\mu \left(\partial^\mu A^\nu - \partial^\nu A^\mu \right) = j^\nu / \epsilon_0 - (18)$$

i.e.

$$\Box A^\nu - \partial^\nu \left(\partial_\mu A^\mu \right) = j^\nu / \epsilon_0 - (19)$$

The u(1) model uses the Lorenz gauge:

$$\partial_\mu A^\mu = ? \; 0 \quad -(20)$$

in order to derive the d'Alembert equation:

$$\Box A^\nu = j^\nu / \epsilon_0 \quad -(21)$$

This is completely arbitrary and meaningless.

The correct method is to use:

$$\partial^\mu A^\nu = - \partial^\nu A^\mu \quad -(22)$$

whereupon:

$$\Box A^\nu = \frac{1}{2} j^\nu / \epsilon_0 - (23)$$

Q.E.D. The use of eq. (20) is incorrect, it is based on the assumption that A^μ is arbitrary. This is because if:

$$A^\mu \rightarrow A^\mu + \partial^\mu \chi \quad -(24)$$

4) then $F^{\mu\nu}$ is not changed because:

$$\partial^\mu \partial^\nu \chi = \partial^\nu \partial^\mu \chi = 0 \quad -(25)$$

In 4(i) errors are compounded by the incorrect assertion that ϕ is arbitrary. It is trivially clear that this assertion is incorrect, because if we accept (20), then:

$$\Box \phi = 0 \quad -(26)$$

and ϕ cannot be arbitrary, reductio ad absurdum.

From eq. (5):

$$\partial^\mu \partial^\nu \chi = -\partial^\nu \partial^\mu \chi \quad -(27)$$

The only possible solution of eqns. (25) and (27) is:

$$\boxed{\partial^\nu \partial^\mu \chi = \pm \partial^\mu \partial^\nu \chi = 0} \quad -(28)$$

Therefore there is no gauge freedom and the whole of twentieth century gauge theory is refuted.

Note 135(4): Commutator Structures of Curvature and Torsion, Proof of the $R = D \wedge \omega$ Theorem

The fundamental commutator of Riemann geometry is:

$$[D_\mu, D_\nu] V^\rho = D_\mu (D_\nu V^\rho) - D_\nu (D_\mu V^\rho) \quad -(1)$$

$$= \partial_\mu (D_\nu V^\rho) - \Gamma^\lambda_{\mu\nu} D_\lambda V^\rho + \Gamma^\rho_{\mu\sigma} D_\nu V^\sigma - (\mu \leftrightarrow \nu) \quad -(2)$$

All terms change sign when μ and ν interchange. All terms are therefore commutators by definition.

Use:

$$D_\nu V^\rho = \partial_\nu V^\rho + \Gamma^\rho_{\nu\sigma} V^\sigma \quad -(3)$$

$$D_\lambda V^\rho = \partial_\lambda V^\rho + \Gamma^\rho_{\lambda\sigma} V^\sigma \quad -(4)$$

$$D_\nu V^\sigma = \partial_\nu V^\sigma + \Gamma^\sigma_{\nu\lambda} V^\lambda \quad -(5)$$

and:

$$\Gamma^\rho_{\mu\sigma} \Gamma^\sigma_{\nu\lambda} V^\lambda = \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} V^\sigma \quad -(6)$$

Thus:

$$\begin{aligned}
[D_\mu, D_\nu] V^\rho = {} & \partial_\mu \partial_\nu V^\rho + (\partial_\mu \Gamma^\rho_{\nu\sigma}) V^\sigma + \Gamma^\rho_{\nu\sigma} \partial_\mu V^\sigma \\
& - \Gamma^\lambda_{\mu\nu} \partial_\lambda V^\rho - \Gamma^\lambda_{\mu\nu} \Gamma^\rho_{\lambda\sigma} V^\sigma \\
& + \Gamma^\rho_{\mu\sigma} \partial_\nu V^\sigma + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} V^\sigma \\
& - \partial_\nu \partial_\mu V^\rho - (\partial_\nu \Gamma^\rho_{\mu\sigma}) V^\sigma - \Gamma^\rho_{\mu\sigma} \partial_\nu V^\sigma \\
& + \Gamma^\lambda_{\nu\mu} \partial_\lambda V^\rho + \Gamma^\lambda_{\nu\mu} \Gamma^\rho_{\lambda\sigma} V^\sigma \\
& - \Gamma^\rho_{\nu\sigma} \partial_\mu V^\sigma - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma} V^\sigma
\end{aligned} \quad -(7)$$

Now use:

2) $$[\partial_\mu, \partial_\nu] V^\rho = 0 \quad -(8)$$

Therefore:

$$[D_\mu, D_\nu] V^\rho = \left(\partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}\right) V^\sigma$$

$$- \left(\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}\right)\left(\partial_\lambda V^\rho + \Gamma^\rho_{\lambda\sigma} V^\sigma\right) \quad -(9)$$

$$= R^\rho{}_{\sigma\mu\nu} V^\sigma - T^\lambda_{\mu\nu} D_\lambda V^\sigma \quad -(10)$$

The commutator structure of the fundamental equation of Riemann geometry is:

$$\boxed{[D_\mu, D_\nu] V^\rho = \left([\partial_\mu, \Gamma^\rho_{\nu\sigma}] - [\partial_\nu, \Gamma^\rho_{\mu\sigma}] + [\Gamma^\rho_{\mu\lambda}, \Gamma^\lambda_{\nu\sigma}]\right) V^\sigma - \Gamma^\lambda_{[\mu,\nu]} D_\lambda V^\rho \quad -(11)}$$

For example:

$$[\partial_\mu, \Gamma^\rho_{\nu\sigma}] V^\sigma = \partial_\mu \left(\Gamma^\rho_{\nu\sigma} V^\sigma\right) - \Gamma^\rho_{\nu\sigma} \partial_\mu V^\sigma$$

$$= \Gamma^\rho_{\nu\sigma} \partial_\mu V^\sigma + \left(\partial_\mu \Gamma^\rho_{\nu\sigma}\right) V^\sigma - \Gamma^\rho_{\nu\sigma} \partial_\mu V^\sigma$$

$$= \left(\partial_\mu \Gamma^\rho_{\nu\sigma}\right) V^\sigma \quad -(12)$$

Q. E. D.

By definition:

3)
$$[\partial_\mu, \Gamma^\rho_{\nu\sigma}] V^\sigma = - [\partial_\nu, \Gamma^\rho_{\mu\sigma}] V^\sigma \quad -(13)$$
$$[\Gamma^\rho_{\mu\lambda}, \Gamma^\lambda_{\nu\sigma}] V^\sigma = - [\Gamma^\rho_{\nu\lambda}, \Gamma^\lambda_{\mu\sigma}] V^\sigma \quad -(14)$$
$$\Gamma^\lambda_{[\mu,\nu]} = - \Gamma^\lambda_{[\nu,\mu]} \quad -(15)$$

The connection transforms as:
$$\left(\Gamma^\lambda_{\mu\nu}\right)' = \left(\frac{\partial x^\mu}{\partial x^{\mu'}}\right)\left(\frac{\partial x^\nu}{\partial x^{\nu'}}\right)\left(\frac{\partial x^{\lambda'}}{\partial x^\lambda}\right)\Gamma^\lambda_{\mu\nu} - \left(\frac{\partial x^\mu}{\partial x^{\mu'}}\right)\left(\frac{\partial x^\nu}{\partial x^{\nu'}}\right)\left(\frac{\partial^2 x^{\lambda'}}{\partial x^\mu \partial x^\nu}\right) \quad -(16)$$
$$= - \left(\Gamma^\lambda_{\mu\nu}\right)'.$$

The torsion transforms as:
$$\left(T^\lambda_{\mu\nu}\right)' = \left(\frac{\partial x^\mu}{\partial x^{\mu'}}\right)\left(\frac{\partial x^\nu}{\partial x^{\nu'}}\right)\left(\frac{\partial x^{\lambda'}}{\partial x^\lambda}\right) T^\lambda_{\mu\nu} \quad -(17)$$
$$- \left(\frac{\partial x^\mu}{\partial x^{\mu'}}\right)\left(\frac{\partial x^\nu}{\partial x^{\nu'}}\right)[\partial_\mu, \partial_\nu] x^{\lambda'}$$
$$= \left(\frac{\partial x^\mu}{\partial x^{\mu'}}\right)\left(\frac{\partial x^\nu}{\partial x^{\nu'}}\right)\left(\frac{\partial x^{\lambda'}}{\partial x^\lambda}\right) T^\lambda_{\mu\nu}$$

because:
$$[\partial_\mu, \partial_\nu] x^{\lambda'} = 0 \quad -(18)$$
$$= - [\partial_\nu, \partial_\mu] x^{\lambda'}$$

As is well known, the connection is not a

4) tensor, because it does not transform as a tensor. This is because of second, inhomogeneous, term of eq. (16). The torsion is a tensor and transforms as a tensor in eq. (17). The commutator antisymmetry of Minkowski spacetime is defined by eq. (18). The commutator in this special case is zero.

<u>Proof of the Second Cartan Structure Equation.</u>

The Cartan geometry also has a commutator structure. To prove this consider:

$$[D_\mu, D_\nu] V^a = D_\mu (D_\nu V^a) - D_\nu (D_\mu V^a)$$

$$= \partial_\mu (D_\nu V^a) - \Gamma^\lambda_{\mu\nu} D_\lambda V^a + \omega^a_{\ \mu b} D_\nu V^b - (\mu \leftrightarrow \nu) \quad -(19)$$

Use:

$$D_\nu V^a = \partial_\nu V^a + \omega^a_{\ \nu b} V^b \quad -(20)$$

$$D_\lambda V^a = \partial_\lambda V^a + \omega^a_{\ \lambda b} V^b \quad -(21)$$

$$D_\nu V^b = \partial_\nu V^b + \omega^b_{\ \nu c} V^c \quad -(22)$$

$$\omega^a_{\ \mu b} \omega^b_{\ \nu c} V^c = \omega^a_{\ \mu c} \omega^c_{\ \nu b} V^b \quad -(23)$$

Thus:

$$\begin{aligned}[D_\mu, D_\nu] V^a = {} & \partial_\mu \partial_\nu V^a + (\partial_\mu \omega^a_{\ \nu b}) V^b + \omega^a_{\ \nu b} \partial_\mu V^b \\ & - \Gamma^\lambda_{\mu\nu} \partial_\lambda V^a - \Gamma^\lambda_{\mu\nu} \omega^a_{\ \lambda b} V^b \\ & + \omega^a_{\ \mu b} \partial_\nu V^b + \omega^a_{\ \mu c} \omega^c_{\ \nu b} V^b \quad -(24) \\ & - (\mu \leftrightarrow \nu)\end{aligned}$$

5)

$$= \left(\partial_\mu \omega^a_{\nu b} - \partial_\nu \omega^a_{\mu b} + \omega^a_{\mu c}\omega^c_{\nu b} - \omega^a_{\nu c}\omega^c_{\mu b}\right) V^b - \left(\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}\right)\left(\partial_\lambda V^a + \omega^a_{\lambda b} V^b\right) \quad -(25)$$

$$= R^a{}_{b\mu\nu} V^b - T^\lambda_{\mu\nu} D_\lambda V^a \quad -(26)$$

This proves the second Cartan Maurer equation:

$$R^a{}_{b\mu\nu} = \partial_\mu \omega^a_{\nu b} - \partial_\nu \omega^a_{\mu b} + \omega^a_{\mu c}\omega^c_{\nu b} - \omega^a_{\nu c}\omega^c_{\mu b} \quad -(27)$$

<u>Q.E.D.</u>

In commutator format:

$$\boxed{R^a{}_{b\mu\nu} V^b = \left([\partial_\mu, \omega^a_{\nu b}] - [\partial_\nu, \omega^a_{\mu b}] + [\omega^a_{\mu c}, \omega^c_{\nu b}]\right) V^b} \quad -(28)$$

where each commutator is antisymmetric by definition. By definition of the tetrad:

$$V^a = q^a_\mu V^\mu \quad -(29)$$

$$= q^a_\nu V^\nu \quad -(30)$$

Using the tetrad postulate:

$$D_\mu V^a = q^a_\nu D_\mu V^\nu \quad -(31)$$

6) The structure equation (37) in the notation of differential geometry is:

$$R^a{}_b = d \wedge \omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \quad -(32)$$

In addition we now have the powerful new antisymmetry constraints:

$$\boxed{\begin{aligned} \partial_\mu \omega^a{}_{\nu b} &= - \partial_\nu \omega^a{}_{\mu b} && -(33) \\ \omega^a{}_{\mu c} \omega^c{}_{\nu b} &= - \omega^a{}_{\nu c} \omega^c{}_{\mu b} && -(34) \\ \Gamma^\lambda_{\mu\nu} &= - \Gamma^\lambda_{\nu\mu} && -(35) \end{aligned}}$$

The tetrad postulate is:

$$D_\mu q^a_\lambda = \partial_\mu q^a_\lambda + \omega^a_{\mu b} q^b_\lambda - \Gamma^\nu_{\mu\lambda} q^a_\nu = 0 \quad -(36)$$

By definition:

$$\omega^a_{\mu\lambda} = \omega^a_{\mu b} q^b_\lambda \quad -(37)$$

$$\Gamma^a_{\mu\lambda} = \Gamma^\nu_{\mu\lambda} q^a_\nu \quad -(38)$$

so:

$$\partial_\mu q^a_\nu = \Gamma^a_{\mu\nu} - \omega^a_{\mu\nu} \quad -(39)$$

$$\boxed{\Gamma^a_{\mu\nu} = - \Gamma^a_{\nu\mu} = \partial_\mu q^a_\nu + \omega^a_{\mu\nu}} \quad -(40)$$

7) Therefore the antisymmetric constraint equation of Cartan geometry is:

$$\boxed{\partial_\mu q^a_{\ \nu} + \partial_\nu q^a_{\ \mu} + \omega^a_{\ \mu\nu} + \omega^a_{\ \nu\mu} = 0} \quad -(41)$$

The first Cartan Maurer structure equation is:

$$T^a_{\ \mu\nu} = \partial_\mu q^a_{\ \nu} - \partial_\nu q^a_{\ \mu} + \omega^a_{\ \mu\nu} - \omega^a_{\ \nu\mu}. \quad -(42)$$

which is equivalent to:

$$T^\lambda_{\ \mu\nu} = \Gamma^\lambda_{\ \mu\nu} - \Gamma^\lambda_{\ \nu\mu}. \quad -(43)$$

using eq. (36) and:

$$T^a_{\ \mu\nu} = q^a_{\ \lambda} T^\lambda_{\ \mu\nu} \quad -(44)$$

$$T^\lambda_{\ \mu\nu} = q^\lambda_{\ a} T^a_{\ \mu\nu} \quad -(45)$$

$$q^a_{\ \lambda} q^\lambda_{\ a} = \delta^\lambda_{\ \lambda}. \quad -(46)$$

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1) 135(5): Self-Consistency of the Commutator Method.

In note 135(4) it was shown that:

$$[D_\mu, D_\nu] V^a = R^a{}_{b\mu\nu} V^b - T^\lambda_{\mu\nu} D_\lambda V^a \quad -(1)$$

and that

$$[D_\mu, D_\nu] V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma - T^\lambda_{\mu\nu} D_\lambda V^\rho \quad -(2)$$

It will be shown in these notes that eqns. (1) and (2) imply:

$$R^a{}_{b\mu\nu} = q^a_\rho q^\sigma_b R^\rho{}_{\sigma\mu\nu}. \quad -(3)$$

Proof By definition:

$$V^a = q^a_\rho V^\rho \quad -(4)$$

therefore in eq. (1):

$$[D_\mu, D_\nu](q^a_\rho V^\rho) = R^a{}_{b\mu\nu}(q^b_\sigma V^\sigma) - T^\lambda_{\mu\nu} D_\lambda (q^a_\rho V^\rho) \quad -(5)$$

$$= R^a{}_{\sigma\mu\nu} V^\sigma - q^a_\rho T^\lambda_{\mu\nu} D_\lambda V^\rho. \quad -(6)$$

This is because, by definition:

$$R^a{}_{\sigma\mu\nu} = q^b_\sigma R^a{}_{b\mu\nu} \quad -(7)$$

and

$$D_\lambda (q^a_\rho V^\rho) = q^a_\rho D_\lambda V^\rho + (D_\lambda q^a_\rho) V^\rho \quad -(8)$$

by the Leibniz theorem. However, the tetrad

2) postulate is: $D_\lambda q^a_\rho = 0$ — (a)

thus giving the right hand side of eq. (6).

The left hand side of eq. (6) is:

$$[D_\mu, D_\nu](q^a_\rho V^\rho) = D_\mu(D_\nu(q^a_\rho V^\rho)) - D_\nu(D_\mu(q^a_\rho V^\rho)) \quad (10)$$

$$= q^a_\rho [D_\mu, D_\nu] V^\rho \quad (11)$$

using again the Leibniz theorem and tetrad postulate. Therefore

$$q^a_\rho [D_\mu, D_\nu] V^\rho = R^a{}_{\sigma\mu\nu} V^\sigma - q^a_\rho T^\lambda{}_{\mu\nu} D_\lambda V^\rho \quad (11)$$

Multiply all terms by q^ρ_a to find:

$$[D_\mu, D_\nu] V^\rho = q^\rho_a R^a{}_{\sigma\mu\nu} V^\sigma - T^\lambda{}_{\mu\nu} D_\lambda V^\rho$$

$$= R^\rho{}_{\sigma\mu\nu} V^\sigma - T^\lambda{}_{\mu\nu} D_\lambda V^\rho \quad (12)$$

This proves eq. (3), Q.E.D.

Therefore Cartan geometry is self-

3) consistent. Differential forms and tensors are generated by the commutator $[D_\mu, D_\nu]$. In his original theory, Cartan used a for a tangent Minkowski spacetime at point P to a base manifold. In ECE theory this concept has been extended so that a and μ represent two different representations of the same spacetime, with one representation moving with respect to the other.

The concept of field of force is therefore geometry within a proportionality. The most fundamental concept of the geometry is the commutator. Therefore the most fundamental concept in field theory is the commutator; the torsion and curvature are equally fundamental by definition:

$$[D_\mu, D_\nu] V^a := -[D_\nu, D_\mu] V^a \quad -(13)$$

$$[D_\mu, D_\nu] V^\rho := -[D_\nu, D_\mu] V^\rho \quad -(14)$$

introducing powerful new antisymmetry laws.

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135 (6): Mathematical Fundamentals

In previous notes to this paper it has been shown that $[D_\mu, D_\nu]$ in any forms as well as tensors are generated by spacetime of any dimension. This introduces new antisymmetry laws throughout field theory. It is important to define the mathematical fundamentals in view of these advances, and this is the purpose of this note.

At each point in spacetime associate a set of vectors located at that point. This is the tangent space at P, denoted T_P. The aim is to relate the tangent space to objects that can be constructed from the base manifold. The set of tangent spaces of a manifold M is the tangent bundle $T(M)$. Any vector is a linear combination of basis vectors, and no vector in the basis is a linear combination of the basis vectors.

In a four dimensional space, a basis of four vectors $\hat{e}_{(\mu)}$ is set up at each tangent space and is adapted to the coordinates x^μ. The basis vector $\hat{e}_{(1)}$ for example points along the x axis. However, the basis set need not be adapted to the coordinate system. In n dimensional spaces, a vector is an abstract geometrical entity.

A curve or path in spacetime is specified by the coordinates as a function of the parameter λ,

d) e.g. $x^\mu(\lambda)$. The tangent vector is then:

$$V^\mu = \frac{dx^\mu}{d\lambda} \quad — (1)$$

The tangent space at point P in a manifold M must be constructed using things that are intrinsic to M. The V^μ vector is coordinate dependent. In the theory of spaces the directional derivative operator (ddo) is used to eliminate coordinate dependence from differential geometry. The ddo maps:

$$f \rightarrow \frac{df}{d\lambda} \quad — (2)$$

at P. The tangent space T_p is identified with the space of ddo's along curves through P. The ddo's form a vector space with the same dimensionality as M. In eq. (2) for example the derivative operator is $d/d\lambda$.

The ddo acts linearly on functions and obeys the Leibniz Theorem, or product rule. The set of ddo's is a vector space. Any dd is a sum of real numbers times partial derivatives. Thus for example:

$$\left(a\frac{d}{d\lambda} + b\frac{d}{d\eta}\right)(fg) = af\frac{dg}{d\lambda} + ag\frac{df}{d\lambda} + bf\frac{dg}{d\eta} + bg\frac{df}{d\eta}$$

$$= \left(a\frac{df}{d\lambda} + b\frac{df}{d\eta}\right)g + \left(a\frac{dg}{d\lambda} + b\frac{dg}{d\eta}\right)f$$

$$— (3)$$

3) The product rule is satisfied, and the set of d.d's is a vector space. This vector space is the tangent space. For coordinates x^μ the set of n directional derivatives at P is the set of partial derivatives ∂_μ at P. The set $\{\partial_\mu\}$ at P is a basis set for the tangent space T_p. The number of basis vectors is n, so T_n is n-dimensional. The components of a tangent vector are real numbers multiplied by partial derivatives.

To see this, expand the vector or operator $d/d\lambda$ in terms of the partials ∂_μ. To do this, use the chain rule:

$$\frac{\partial}{\partial x^a} = \sum_b \frac{\partial y^b}{\partial x^a} \frac{\partial}{\partial y^b} \quad -(4)$$

so:

$$\frac{df}{d\lambda} = \frac{dx^\mu}{d\lambda} \partial_\mu f \quad -(5)$$

i.e

$$\boxed{\frac{d}{d\lambda} = \left(\frac{dx^\mu}{d\lambda}\right) \partial_\mu} \quad -(6)$$

The partial operators ∂_μ act as the basis set for the directional derivative $d/d\lambda$.

The vector represented by the operator

4) $d/d\lambda$ is the tangent vector to the curve with parameter λ. Thus eq. (6) is a restatement of

$$V^{\mu} = \frac{dx^{\mu}}{d\lambda} \quad -(7)$$

but eq. (6) generalizes eq. (7) to an arbitrary manifold, in which the basis vectors are:

$$\boxed{\hat{e}_{(\mu)} = \partial_{\mu}} \quad -(8)$$

This basis is the coordinate basis for T_p, the tangent spacetime at P to a base manifold. This idea is an abstraction or generalization of setting up the basis vectors to point along the coordinate axes. In three dimensional Euclidean space the basis vectors of the Cartesian system, for example, are $\underline{i}$, $\underline{j}$ and $\underline{k}$ in X, Y and Z.

Tangent vectors can be represented by any orthonormal basis. An example is the complex circular basis, which is made up of combinations of $\underline{i}$, $\underline{j}$ and $\underline{k}$:

$$\underline{e}^{(1)} = \frac{1}{\sqrt{2}}\left(\underline{i} - i\underline{j}\right) \quad -(9)$$

$$\underline{e}^{(2)} = \frac{1}{\sqrt{2}}\left(\underline{i} + i\underline{j}\right) \quad -(10)$$

$$\underline{e}^{(3)} = \underline{k} \quad -(11)$$

5) This basis is convenient for polarization of radiation in physics. In general, tangent vectors can be represented by any orthonormal curvilinear coordinates, e.g. spherical polar, cylindrical polar, etc. If for example the indices of the complex circular rep are a, its basis vectors are $\hat{e}_{(a)}$. If the indices of the Cartesian rep are μ, its basis vectors are $\hat{e}_{(\mu)}$. Both a and μ refer to the tangent space of the tetrad $V^a_{\ \mu}$ is then defined by

$$\boxed{\hat{e}_{(\mu)} = V^a_{\ \mu}\,\hat{e}_{(a)}} \quad -(12)$$

The basis set $\hat{e}_{(a)}$ need not be a coordinate basis at all. It can be for example the basis made up of Pauli matrices. This is orthonormal and is used in the SU(2) theory of spinors, i.e. the SU(2) representation space. The latter was introduced by Cartan in 1913. Cartan's purpose in introducing the tetrad in the early twenties was to represent spinors in the arbitrary manifold.

6)

One of the uses of tetrads is to define the metric tensor in the arbitrary manifold as:

$$\boxed{g_{\mu\nu} = q^a_\mu \, q^b_\nu \, \eta_{ab}} \quad -(13)$$

where η_{ab} is the Minkowski metric:

$$\eta_{ab} = \eta^{ab} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad -(14)$$

So if we know q^a_μ and q^b_ν we can deduce $g_{\mu\nu}$.

In ECE theory the metrics $g_{\mu\nu}$ are completely different from those of the Einstein field equation, now well known to be incorrect.

The tetrad in ECE theory may also be defined in terms of coordinates as:

$$x^a = q^a_\mu \, x^\mu \quad -(15)$$

The complete vector field is:

$$X = x^a \, \hat{e}_{(a)} = x^\mu \, \hat{e}_{(\mu)} . \quad -(16)$$

The concept of field of force in physics

7) is introduced by making one coordinate system move with respect to a static coordinate system. The tetrad of the circularly polarized electromagnetic field of force for example is defined by:

$$e^{i\phi}\, x^a = q^a_\mu\, x^\mu \quad -(17)$$

where:

$$\phi = \omega t - \kappa Z \quad -(18)$$

is the electromagnetic phase. Thus:

$$\left.\begin{aligned} q^{(1)} &= e^{(1)}\, e^{i\phi} \\ q^{(2)} &= e^{(2)}\, e^{-i\phi} \\ q^{(3)} &= e^{(3)} \end{aligned}\right\} \quad -(19)$$

The fundamental hypothesis is:

$$A^a_\mu = A^{(0)}\, q^a_\mu \quad -(20)$$

where A^a_μ is the electromagnetic potential.

Similarly:

$$\Phi^a_\mu = \Phi^{(0)}\, q^a_\mu \quad -(21)$$

where $\Phi^{(0)}$ is the gravitational potential.

1) 135 (8): Minkowski Metric of the Complex Circular Basis

The unit vectors of this basis are defined by:

$$\underline{e}^{(1)} = \frac{1}{\sqrt{2}}\left(\underline{i} - i\underline{j}\right),\ \underline{e}^{(2)} = \frac{1}{\sqrt{2}}\left(\underline{i} + i\underline{j}\right),\ \underline{e}^{(3)} = \underline{k} \quad -(1)$$

so

$$\underline{i} = \frac{\sqrt{2}}{2}\left(\underline{e}^{(1)} + \underline{e}^{(2)}\right),\ \underline{j} = \frac{\sqrt{2}}{2}\, i\left(\underline{e}^{(1)} - \underline{e}^{(2)}\right) \quad -(2)$$

$$\underline{k} = \underline{e}^{(3)}$$

and:

$$\underline{r} = X\underline{i} + Y\underline{j} + Z\underline{k}$$

$$= \frac{\sqrt{2}}{2}\left(X + iY\right)\underline{e}^{(1)} + \frac{\sqrt{2}}{2}\left(X - iY\right)\underline{e}^{(2)} + Z\,\underline{e}^{(3)} \quad -(3)$$

Let:

$$u_1 = \frac{\sqrt{2}}{2}\left(X + iY\right) \quad -(4)$$

$$u_2 = \frac{\sqrt{2}}{2}\left(X - iY\right) \quad -(5)$$

$$u_3 = Z \quad -(6)$$

so

$$\underline{r} = u_1\,\underline{e}^{(1)} + u_2\,\underline{e}^{(2)} + u_3\,\underline{e}^{(3)} \quad -(7)$$

The metric coefficients are:

$$h_1 = \left|\frac{\partial \underline{r}}{\partial u_1}\right| = \left|\underline{e}^{(1)}\right| \quad -(8)$$

$$h_2 = \left|\frac{\partial \underline{r}}{\partial u_2}\right| = \left|\underline{e}^{(2)}\right| \quad -(9)$$

$$h_3 = \left|\frac{\partial \underline{r}}{\partial u_3}\right| = \left|\underline{e}^{(3)}\right| \quad -(10)$$

In the theory of complex numbers, if

2)

$$z = x + iy$$

then

$$|z| = zz^* = (x+iy)(x-iy) \qquad -(11)$$

$$= x^2 + y^2 .$$

Similarly:

$$|\underline{e}^{(1)}| = \frac{1}{\sqrt{2}}(\underline{i} + i\underline{j}) \cdot \frac{1}{\sqrt{2}}(\underline{i} - i\underline{j}) = 1 \qquad -(12)$$

So:

$$h_1 = h_2 = h_3 = 1 . \qquad -(13)$$

The metric elements are

$$g_{11} = h_1^2 = 1 \quad \text{etc.} \qquad -(14)$$

The Minkowski metric in the complex circular basis is therefore:

$$\eta_{ab} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad -(15)$$

The general metric is:

$$g_{\mu\nu} = q^a_\mu q^b_\nu \eta_{ab} \qquad -(16)$$

where q^a_μ and q^b_ν are tetrads.

For a plane wave:

$$q^{(1)}_x = \frac{1}{\sqrt{2}} e^{i\phi} \qquad -(17)$$

$$q^{(1)}_y = \frac{i}{\sqrt{2}} e^{i\phi} \qquad -(18)$$

$$q^{(2)}_x = \frac{1}{\sqrt{2}} e^{-i\phi} \qquad -(19)$$

$$q^{(2)}_y = -\frac{i}{\sqrt{2}} e^{-i\phi} \qquad -(20)$$

3) From eq. (15):

$$\eta_{(0)(0)} = 1, \; \eta_{(i)(i)} = -1; \; i = 1, 2, 3 \quad -(21)$$

The Metric of a Plane Wave

We have:

$$g_{xx} = q^{(1)}_x q^{(1)}_x \eta_{(1)(1)} + q^{(1)}_x q^{(2)}_x \eta_{(1)(2)} + \dots + q^{(2)}_x q^{(2)}_x \eta_{(2)(2)} \quad -(22)$$

$$= -\frac{1}{2}\left(e^{2i\phi} + e^{-2i\phi}\right) \quad -(23)$$

$$\boxed{g_{xx} = -\frac{1}{2}\cos(2\phi)} \quad -(24).$$

$$g_{xy} = q^{(1)}_x q^{(1)}_y \eta_{(1)(1)} + q^{(2)}_x q^{(2)}_y \eta_{(2)(2)}$$

$$= -\frac{i}{2} e^{2i\phi} + \frac{i}{2} e^{-2i\phi}$$

$$= -\frac{i}{2}\left(e^{2i\phi} - e^{-2i\phi}\right)$$

$$\boxed{g_{xy} = \frac{1}{2}\sin(2\phi)} \quad -(25)$$

$$g_{yx} = g_{xy} = \frac{1}{2}\sin(2\phi) \quad -(26)$$

$$g_{yy} = q^{(1)}_y q^{(1)}_y \eta_{(1)(1)} + q^{(2)}_y q^{(2)}_y \eta_{(2)(2)}$$

4)

$$g_{YY} = \frac{1}{2}\cos(2\phi) \quad -(27)$$

Therefore:

$$\boxed{g_{\mu\nu} = \frac{1}{2}\begin{bmatrix} -\cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & \cos(2\phi) \end{bmatrix}} \quad -(28)$$

It is seen that this is not the metric of a flat spacetime as in Maxwell Heaviside theory. The MH metric of reference is:

$$g_{\mu\nu}(MH) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad -(29)$$

In eq. (28):

$$\phi = \omega t - \kappa Z \quad -(30)$$

where ω is the angular frequency at t and where κ is the wavenumber at Z.

135(8): Definition of Tetrad by Superimposing Frames

1) Static Tetrad

Consider the spacelike position vector $\underline{r}$ in Cartesian coordinates:

$$\underline{r} = X\,\underline{i} + Y\,\underline{j} + Z\,\underline{k} \quad -(1)$$

and complex circular coordinates:

$$\underline{r} = r^{(1)}\underline{e}^{(1)} + r^{(2)}\underline{e}^{(2)} + r^{(3)}\underline{e}^{(3)} \quad -(2)$$

Eqs. (1) and (2) are examples of x^μ and x^a, where:

$$x^\mu = (ct, \underline{r}_1) \quad -(3)$$

$$x^a = (ct, \underline{r}_2) \quad -(4)$$

Therefore:

$$x^\mu = (ct, X, Y, Z) \quad -(5)$$

$$x^a = (ct, r^{(1)}, r^{(2)}, r^{(3)}) \quad -(6)$$

The tetrad is defined by:

$$x^a = q^a_{\ \mu}\, x^\mu. \quad -(7)$$

Here:

$$r^{(1)} = \frac{1}{\sqrt{2}}(X + iY) \quad -(8)$$

$$r^{(2)} = \frac{1}{\sqrt{2}}(X - iY) \quad -(9)$$

$$r^{(3)} = Z \quad -(10)$$

Restricting attention to transverse components:

$$x^a = \frac{1}{\sqrt{2}}\begin{bmatrix} X + iY \\ X - iY \end{bmatrix}, \quad x^\mu = \begin{bmatrix} X \\ Y \end{bmatrix} \quad -(11)$$

2) Therefore:

$$\frac{1}{\sqrt{2}}\begin{bmatrix} X+iY \\ X-iY \end{bmatrix} = \begin{bmatrix} q_1^{(1)} & q_2^{(1)} \\ q_1^{(2)} & q_2^{(2)} \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} \quad -(12)$$

i.e.

$$\frac{1}{\sqrt{2}}(X+iY) = q_1^{(1)} X + q_2^{(1)} Y \quad -(13)$$

$$\frac{1}{\sqrt{2}}(X-iY) = q_1^{(2)} X + q_2^{(2)} Y \quad -(14)$$

A solution of eqs. (13) and (14) is:

$$\left.\begin{array}{l} q_1^{(1)} = \frac{1}{\sqrt{2}}, \quad q_2^{(1)} = \frac{i}{\sqrt{2}} \\ q_1^{(2)} = \frac{1}{\sqrt{2}}, \quad q_2^{(2)} = -\frac{i}{\sqrt{2}} \end{array}\right\} -(15)$$

This may be expressed in vector format as:

$$\underline{q}^{(1)} = \frac{1}{\sqrt{2}}(\underline{i} + i\underline{j}) \quad -(16)$$

$$\underline{q}^{(2)} = \frac{1}{\sqrt{2}}(\underline{i} - i\underline{j}) \quad -(17)$$

2) Dynamic Tetrad

The $\underline{r}$ vector in eq. (1) is spun around Z and moved forward:

$$\underline{r} = (X\underline{i} + Y\underline{j})e^{i(\omega t - kZ)} \quad -(18)$$

so:

$$\underline{q}^{(1)} = \frac{1}{\sqrt{2}}(\underline{i} + i\underline{j})e^{i(\omega t - kZ)} \quad -(19)$$

$$\underline{q}^{(2)} = \frac{1}{\sqrt{2}}(\underline{i} - i\underline{j})e^{i(\omega t - kZ)} \quad -(20)$$

3) Eqns. (19) and (20) represent right and left handed circular polarization:

$$\underline{q}_R^{(1)} = \frac{1}{\sqrt{2}}\left(\underline{i} + i\underline{j}\right) e^{i\phi} \quad -(21)$$

$$\underline{q}_L^{(2)} = \frac{1}{\sqrt{2}}\left(\underline{i} - i\underline{j}\right) e^{i\phi} \quad -(22)$$

The complex conjugates of eqns. (21) and (22) are:

$$\underline{q}_R^{(2)} = \underline{q}_R^{(1)*} = \frac{1}{\sqrt{2}}\left(\underline{i} - i\underline{j}\right) e^{-i\phi} \quad -(23)$$

$$\underline{q}_L^{(1)} = \underline{q}_L^{(2)*} = \frac{1}{\sqrt{2}}\left(\underline{i} + i\underline{j}\right) e^{-i\phi} \quad -(24)$$

The electromagnetic potential is:

$$A_\mu^a = A^{(0)} q_\mu^a \quad -(25)$$

The conjugate products are:

$$\underline{q}_R^{(1)} \times \underline{q}_R^{(2)} = \frac{1}{2}\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & i & 0 \\ 1 & -i & 0 \end{vmatrix} \quad -(26)$$

$$= -i$$

$$= \underline{q}_L^{(1)} \times \underline{q}_L^{(2)}$$

Note that in eq. (18) the frame itself is spun and moved forward along Z:

$$\boxed{\underline{i} \rightarrow e^{i\phi}\underline{i}, \; \underline{j} \rightarrow e^{i\phi}\underline{j}} \quad -(27)$$

4) Therefore the tetrad can be derived using unit vectors instead of position vectors. This means

$$X = Y = 1 \quad -(21)$$

other coordinate systems must be used e.g. the spherical polar (r, θ, ϕ):

$$\left.\begin{aligned} X &= r \sin\theta \cos\phi \\ Y &= r \sin\theta \sin\phi \\ Z &= r \cos\theta \end{aligned}\right\} -(22)$$

In this case:

$$\begin{bmatrix} r \sin\theta \cos\phi \\ r \sin\theta \sin\phi \end{bmatrix} = \begin{bmatrix} q^1_1 & q^1_2 \\ q^2_1 & q^2_2 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} -(23)$$

i.e.

$$\left.\begin{aligned} r \sin\theta \cos\phi &= q^1_1 X + q^1_2 Y = X \\ r \sin\theta \sin\phi &= q^2_1 X + q^2_2 Y = Y \end{aligned}\right\} -(24)$$

i.e.

$$\left.\begin{aligned} q^1_1 &= 1, \quad q^1_2 = 0 \\ q^2_1 &= 0, \quad q^2_2 = 1 \end{aligned}\right\} -(25)$$

In this case the tetrad means simply that the cartesian and spherical polar systems are defined by eq. (22). The complex circular representation represents circular polarization.

1) 135(9): Link Between Scale Factor and Tetrad.

The Cartan tetrad is a generalization of the idea of scale factor in differential geometry. The scale factor is the rate at which arc length s_i increases along the ith coordinate curve with respect to u_i in the theory of curvilinear coordinates:

$$h_i = \frac{ds_i}{du_i} \quad -(1)$$

In curvilinear coordinates a curve $\underline{r}$ is parameterized by:

$$\underline{r} = \underline{r}(u_1, u_2, u_3) \quad -(2)$$

so:

$$ds = |d\underline{r}| = \left| \frac{\partial \underline{r}}{\partial u_1} du_1 + \frac{\partial \underline{r}}{\partial u_2} du_2 + \frac{\partial \underline{r}}{\partial u_3} du_3 \right| \quad -(3)$$

so:

$$h_i = \left| \frac{\partial \underline{r}}{\partial u_i} \right| \quad -(4)$$

The unit tangent vector to the curve u_i at point P is:

$$\underline{e}_i = \frac{1}{h_i} \frac{\partial \underline{r}}{\partial u_i} \quad -(5)$$

i.e.

$$\boxed{\frac{\partial \underline{r}}{\partial u_i} = h_i \, \underline{e}_i} \quad -(6)$$

In Cartan geometry this is generalized to:

$$\boxed{\partial_\mu = q^a_{\ \mu} \, e_a} \quad -(7)$$

In curvilinear coordinates the metric is

2) is defined as:

$$\boxed{g_{ij} = \frac{\partial \underline{r}}{\partial u_i} \cdot \frac{\partial \underline{r}}{\partial u_j}} \quad -(8)$$

which is generalized in Cartan geometry to:

$$\boxed{g_{\mu\nu} = q^a_\mu \, q^b_\nu \, \eta_{ab}} \quad -(9)$$

Cartan geometry is true whether the coordinate system is orthogonal or not. Curvilinear coordinate analysis is ~~the~~ restricted to orthogonal coordinates, and is coordinate adapted. The Cartan system can be used for basis elements that are not coordinate adapted. An example of a coordinate adapted position vector is:

$$\underline{r} = X\,\underline{i} + Y\,\underline{j} + Z\,\underline{k} \quad -(10)$$

The components are X, Y and Z and the basis elements are the Cartesian unit vectors $\underline{i}$, $\underline{j}$ and $\underline{k}$. The vector field is $\underline{r}$. The Pauli matrices may also be basis elements of $\underline{r}$:

$$\sigma^0 r = X\sigma^1 + Y\sigma^2 + Z\sigma^3 \quad -(11)$$

where:

$$\sigma^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$-(12)$$

3) The vector field is independent of the way in which it is represented in eqs. (10) and (11). From eq. (10):

$$r^2 = X^2 + Y^2 + Z^2 \quad -(13)$$

and from eq. (11):

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} r^2 = (X^2 + Y^2 + Z^2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad -(14)$$

The Cartan tetrad can be defined as:

$$\sigma^a = q^a_\mu e^\mu \quad -(15)$$

where this notation is, in general:

$$\begin{bmatrix} \sigma^0 \\ \sigma^1 \\ \sigma^2 \\ \sigma^3 \end{bmatrix} = \begin{bmatrix} q^0_0 & q^0_1 & q^0_2 & q^0_3 \\ q^1_0 & q^1_1 & q^1_2 & q^1_3 \\ q^2_0 & q^2_1 & q^2_2 & q^2_3 \\ q^3_0 & q^3_1 & q^3_2 & q^3_3 \end{bmatrix}\begin{bmatrix} e^0 \\ e^1 \\ e^2 \\ e^3 \end{bmatrix} \quad -(16)$$

Here:

$$e^0 = e^1 = e^2 = e^3 = 1 \quad -(17)$$

If the tetrad is assumed to be diagonal:

$$q^0_0 = \sigma^0, \; q^1_1 = \sigma^1, \; q^2_2 = \sigma^2, \; q^3_3 = \sigma^3 \quad -(18)$$

The metric is then:

$$\left.\begin{aligned} g_{00} &= (q^0_0)^2 \\ g_{ii} &= -(q^i_i)^2 \end{aligned}\right\} -(19)$$

i.e.

$$g_{00} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; g_{ii} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad -(20)$$

4) This is the Minkowski metric with each element multiplied by σ^0, a result which reflects the fact that:

$$\underline{i} \times \underline{j} = \underline{k} \quad -(21)$$

etc.

and

$$\left[\frac{\sigma^1}{2}, \frac{\sigma^2}{2} \right] = i \frac{\sigma^3}{2} \quad -(22)$$

etc.

The Gell-Mann Matrices

The well known Pauli matrices have been shown to be tetrads. As in pages 129 and 130 the Dirac theory of the electron can be developed as a theory of Cartan geometry. The Gell-Mann matrices are defined similarly to eq. (22). The latter is:

$$\left[\frac{\sigma^a}{2}, \frac{\sigma^b}{2} \right] = i \epsilon^{abc} \frac{\sigma^c}{2} \quad -(23)$$

where:

$$\epsilon^{123} = 1, \; \epsilon^{312} = 1, \; \epsilon^{231} = 1 \quad -(24)$$

and so on. This is the SU(2) group's structure constant. In this notation there is no summation over the repeated c index.

The Gell-Mann matrices are used

5)

in nuclear strong force theory and are defined by:

$$\left[\frac{\lambda^a}{2}, \frac{\lambda^b}{2} \right] = i f^{abc} \frac{\lambda^c}{2} \quad -(25)$$

where f^{abc} is the structure constant of the SU(3) group:

$$f^{123} = 1, \; f^{147} = -f^{156} = f^{246} = f^{257} = f^{345} = -f^{367} = \frac{1}{2}, \quad -(26)$$

$$f^{458} = f^{678} = \frac{\sqrt{3}}{2}$$

For example:

$$\left[\frac{\lambda^1}{2}, \frac{\lambda^2}{2} \right] = i \frac{\lambda^3}{2} \quad -(27)$$

where:

$$\lambda^1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; \lambda^2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; \lambda^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad -(28)$$

It is seen that these are extended Pauli matrices, cf.:

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \; \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \; \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad -(29)$$

with

$$\left[\frac{\sigma^1}{2}, \frac{\sigma^2}{2} \right] = i \frac{\sigma^3}{2} \quad -(30)$$

6) It follows that the Gell-Mann matrices are also tetrads. Assuming diagonal tetrads:

$$\boxed{\lambda^1 = q^1_1, \quad \lambda^2 = q^2_2, \quad \lambda^3 = q^3_3} \quad -(31)$$

The other Gell-Mann matrices are:

$$\lambda^4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \lambda^5 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}, \quad \lambda^6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$\lambda^7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \quad -(33)$$

so:

$$\boxed{\begin{array}{l} \lambda^4 = q^4_4, \quad \lambda^5 = q^5_5, \quad \lambda^6 = q^6_6 \\ \lambda^7 = q^7_7, \quad \lambda^8 = q^8_8 \end{array}} \quad -(34)$$

The potential of the strong field is:

$$A^a_\mu = A^{(0)} q^a_\mu \quad -(35)$$

and the strong field is:

$$F^a_{\mu\nu} = (d \wedge A^a)_{\mu\nu} + (\omega^a{}_b \wedge A^b)_{\mu\nu} \quad -(36)$$

in ECE theory. In gauge theory, torsion is not defined.

Note 135(10): Field Equation in SU(2), the field of the Dirac Spinor.

In paper 129 it was shown that the ECE equation of the rest fermion is:

$$(i\sigma^1 \partial_0 - \kappa)\phi = 0 \quad -(1)$$

and is defined entirely by tetrads. The wave function ϕ is:

$$\phi = (q_0^0 + q_1^1) e^{-i\phi} \quad -(2)$$

where

$$q_0^0 = \sigma^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad -(3)$$

$$q_1^1 = \sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \quad -(4)$$

Here:

$$\phi = \frac{mc^2}{\hbar} t \quad -(5)$$

and

$$\phi = \begin{bmatrix} q_1^R & q_2^R \\ q_1^L & q_2^L \end{bmatrix} \quad -(6)$$

Therefore there are two types of tetrad:

1) The tetrads defined by the Pauli matrices in eqs. (3) and (4).
2) The tetrad defined in a two dimensional representation space:

$$\begin{bmatrix} V^R \\ V^L \end{bmatrix} = \begin{bmatrix} q_1^R & q_2^R \\ q_1^L & q_2^L \end{bmatrix} \begin{bmatrix} V^1 \\ V^2 \end{bmatrix} \quad -(7)$$

Thus:

$$\boxed{q_\mu^a = \begin{bmatrix} q_1^R & q_2^R \\ q_1^L & q_2^L \end{bmatrix} = (q_0^0 + q_1^1) e^{-i\phi}} \quad -(8)$$

2)

Here q_μ^a is the wave-function of the rest fermion and also the potential of the rest fermion.

In ECE electromagnetic theory, the potential is:

$$A_\mu^a = A^{(0)} q_\mu^a . \quad - (9)$$

If the rest fermion is an electron at rest, its potential is, from eqs. (8) and (9):

$$A_\mu^a = A^{(0)} \left(q_0^0 + q_1^1 \right) e^{-i\phi} \quad - (10)$$

and described by:

$$\left(i \sigma^1 \partial_0 - \kappa \right) A_\mu^a = 0 . \quad - (11)$$

This means:

$$i \frac{\partial A_1^R}{\partial t} = \left(\frac{mc^2}{\hbar} \right) A_1^L \quad - (12)$$

$$i \frac{\partial A_2^R}{\partial t} = \left(\frac{mc^2}{\hbar} \right) A_2^L \quad - (13)$$

$$i \frac{\partial A_1^L}{\partial t} = \left(\frac{mc^2}{\hbar} \right) A_1^R \quad - (14)$$

$$i \frac{\partial A_2^L}{\partial t} = \left(\frac{mc^2}{\hbar} \right) A_2^R \quad - (15)$$

In ECE theory the potential A_μ^a will generate the field of the rest electron:

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + \omega_{\mu b}^a A_\nu^b - \omega_{\nu b}^a A_\mu^b \quad - (16)$$

with antisymmetry constraint:

3)

$$\partial_\mu A^a_\nu + \partial_\nu A^a_\mu + \omega^a_{\mu b} A^b_\nu + \omega^a_{\nu b} A^b_\mu = 0 \quad -(17)$$

This exercise illustrates that the force field of the rest electron can be worked out using geometry and a combination of quantum mechanics and classical field theory.

For example:

$$A^R_1 = e^{-i\phi} \quad -(18)$$

and:

$$F^R_{01} = \partial_0 A^R_1 - \partial_1 A^R_0 + \omega^R_{0b} A^b_1 - \omega^R_{1b} A^b_0. \quad -(19)$$

with:

$$\partial_0 A^R_1 + \partial_1 A^R_0 + \omega^R_{0b} A^b_1 + \omega^R_{1b} A^b_0 = 0 \quad -(20)$$

If it is assumed that:

$$\partial_0 A^R_1 + \omega^R_{0b} A^b_1 = -\left(\partial_1 A^R_0 + \omega^R_{1b} A^b_0\right) \quad -(21)$$

then:

$$F^R_{01} = 2\left(\frac{1}{c}\frac{\partial A^R_1}{\partial t} + \omega^R_{0b} A^b_1\right) \quad -(22)$$

$$= 2\left(-\frac{imc}{\hbar} A^{(0)} e^{-i\phi} + \omega^R_{0b} A^b_1\right) \quad -(23)$$

Averaging over time:

$$\langle F^R_{01} \rangle = 2\omega^R_{0b} A^b_1 \quad -(24)$$

4)

The average electric field is:

$$\boxed{\langle E^R_1 \rangle = 2c\omega^R{}_{0b} A^b_1 = -2c\omega^R{}_{1b} A^b_0} \quad -(25)$$

The expectation value of the field is a more meaningful quantity to calculate in the context of quantum mechanics. Writing:

$$E^R_1 = 2\left(\frac{\partial A^R_1}{\partial t} + c\omega^R{}_{0b} A^b_1 \right) \quad -(26)$$

$$= 2A^{(0)} \left(\frac{\partial q^R_1}{\partial t} + c\omega^R{}_{0b} q^b_1 \right) \quad -(27)$$

then the expectation value is:

$$\langle\langle E^R_1 \rangle\rangle = 2A^{(0)} \left(q^{R*}_1 \frac{\partial q^R_1}{\partial t} + c q^{b*}_1 \omega^R{}_{0b} q^b_1 \right) \quad -(28)$$

The real and physical part of it is:

$$\boxed{\begin{aligned} \text{Real } \langle\langle E^R_1 \rangle\rangle &= 2A^{(0)} c\omega^R{}_{0b} \\ &= -2A^{(0)} c\omega^R{}_{1b} \end{aligned}} \quad -(29)$$

THIS IS THE ELECTRIC FIELD OF THE COULOMB LAW.

135(11): Field of the ECE Spinor for a Free Fermion

In paper 130, eq. (39), it was shown that the equation of the free fermion (e.g. electron) is an equation in tetrad elements:

$$\left(\sigma^0 E - c \sigma^3 \underline{\sigma} \cdot \underline{p} \right) q^a_\mu = mc^2 \sigma^1 q^a_\mu \quad -(1)$$

where:

$$q^a_\mu = \begin{bmatrix} q^R_1 & q^R_2 \\ q^L_1 & q^L_2 \end{bmatrix} \quad -(2)$$

Eq. (1) is:

$$\boxed{\left.\begin{aligned} \left(E - c \underline{\sigma} \cdot \underline{p} \right) q^R_1 &= mc^2 q^L_1 \\ \left(E - c \underline{\sigma} \cdot \underline{p} \right) q^R_2 &= mc^2 q^L_2 \\ \left(E + c \underline{\sigma} \cdot \underline{p} \right) q^L_1 &= mc^2 q^R_1 \\ \left(E + c \underline{\sigma} \cdot \underline{p} \right) q^L_2 &= mc^2 q^R_2 \end{aligned} \right\} \quad -(3)}$$

This is an SU(2) representation space because of the presence of the Pauli matrices $\underline{\sigma}$.

The potential of the free fermion is therefore:

$$A^a_\mu = A^{(0)} q^a_\mu \quad -(4)$$

so

$$\boxed{\left(\sigma^0 E - c \sigma^3 \underline{\sigma} \cdot \underline{p} \right) A^a_\mu = mc^2 \sigma^1 A^a_\mu} \quad -(5)$$

where:

$$A^a_\mu = \begin{bmatrix} A^R_1 & A^R_2 \\ A^L_1 & A^L_2 \end{bmatrix} \quad -(6)$$

Eq. (5) is a factorization of:

2)
$$\left(\Box + \left(\frac{mc}{\hbar} \right)^2 \right) A^a_\mu = 0 \quad -(7)$$

where m is the mass of the fermion (electron). Therefore the moving electron generates the quantized electromagnetic potential A^a_μ in an SU(2) representation space.

Note carefully that the electromagnetic potential used in Maxwell Heaviside theory is written in an O(3) representation space for its spacelike component. This is part of a Minkowski spacetime. Since A^a_μ is quantized Cartan tetrad, it may be used to generate the quantized Cartan torsion in SU(2) representation space, and therefore the quantized electromagnetic field of a moving electron in SU(2):

$$F^a = d \wedge A^a + \omega^a{}_b \wedge A^b \quad -(8)$$

$$= d \wedge A^a + A^{(0)} \omega^a \quad -(9)$$

In tensor notation:

$$\left.\begin{aligned} F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \omega^a_{\mu b} A^b_\nu - \omega^a_{\nu b} A^b_\mu \\ &= \partial_\mu A^a_\nu + \partial_\nu A^a_\mu + A^{(0)} \left(\omega^a_{\mu\nu} - \omega^a_{\nu\mu} \right) \end{aligned}\right\} -(10)$$

In vector notation:

$$\underline{E}^a = -\underline{\nabla} \phi^a - \frac{\partial \underline{A}^a}{\partial t} - c\omega^a_{0b} \underline{A}^b + c A^b_0 \underline{\omega}^a{}_b \quad -(11)$$

$$\underline{B}^a = \underline{\nabla} \times \underline{A}^a - \underline{\omega}^a{}_b \times \underline{A}^b \quad -(12)$$

Eq. (11) is:

$$\underline{E}^a = -\underline{\nabla}\phi^a - \frac{\partial \underline{A}^a}{\partial t} + 2cA^{(0)}\underline{\omega}^a \quad -(13)$$

The vector representation is always used in three dimensions. A vector cross product cannot be defined in four dimensions, and is not familiar in SU(2) representation space. Therefore in SU(2), the field of the moving electron is most clearly defined in tensor notation from eq. (10):

$$\boxed{F^a_{\mu\nu} - A^{(0)}\left(\omega^a_{\mu\nu} - \omega^a_{\nu\mu}\right) = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu} \quad -(14)$$

To proceed, analytical approximations of eq. (7) are needed. This is the wave form of the Dirac equation multiplied by $A^{(0)}$. There are many such approximations available, both analytical and computational. For example there are approximations in the free Dirac electron, and for the Dirac electron in the H atom. As is well known, eq. [illegible] gives the observed H atom spectrum quite accurately, except for the Lamb shift. The

4) latter requires the radiative correction due to the vacuum $eA^{(0)}$ as in paper 85. This is an extra momentum via the minimum prescription:

$$p^{(0)} = eA^{(0)} \quad - (15)$$

Therefore to find the electromagnetic field of a moving, and therefore radiating, electron these analytical approximations of eq. (7) are used in eq. (14). This may be done computationally or analytically.

This is a novel methodology in quantized radiation theory.

As is well known, the classical limit of eq. (7) is the Einstein energy equation:

$$p^\mu p_\mu = m^2 c^2 \quad - (16)$$

where:

$$p^\mu = i\hbar \partial^\mu \quad - (17)$$

$$p_\mu = i\hbar \partial_\mu . \quad - (18)$$

Here:

$$p^\mu = \left(\frac{E}{c}, \underline{p} \right) \quad - (19)$$

$$p_\mu = \left(\frac{E}{c}, -\underline{p} \right) \quad - (20)$$

$$\partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\underline{\nabla} \right) \quad - (21)$$

$$\partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \underline{\nabla} \right) \quad - (22)$$

5) and $$\Box = \partial^\mu \partial_\mu \quad -(23)$$

Therefore:
$$p^\mu p_\mu = \frac{E^2}{c^2} - p^2 = m^2 c^2 \quad -(24)$$

and
$$E^2 = c^2 p^2 + m^2 c^4 \quad -(25)$$

Also:
$$p^\mu p_\mu = -\hbar^2 \Box \quad -(26)$$

In eq. (26), $\Box$ is a second order differential operator:
$$\Box = -\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \quad -(27)$$

known as the d'Alembertian. The $\Box$ acts on the potential tetrad wave function, which is also a potential wave function in ECE electrodynamics. Therefore from eqs. (16) and (26):
$$-\hbar^2 \Box A^a_\mu = m^2 c^2 A^a_\mu \quad -(28)$$

which is eq. (7), Q. E. D.

In the usual method of deriving for example ESR, NMR, MRI, and the inverse Faraday effect (IFE), the minimal prescription is used:
$$p^\mu \rightarrow p^\mu + eA^\mu \quad -(29)$$
$$p_\mu \rightarrow p_\mu + eA_\mu \quad -(30)$$

giving the relativistic Hamilton Jacobi equation:

$$(p^\mu + eA^\mu)(p_\mu + eA_\mu) = m^2 c^2 \quad -(31)$$

This was used for example in "The Enigmatic Photon", volume 1. This is a classical equation which however gives neither the properties of the half integral spin of the electron (ESR, NMR, MRI, etc.) nor the IFE, nor the $\underline{B}^{(3)}$ field.

All this information is contained in eq. (5). The latter recognizes that the potential A^a_μ must be quantized and must contain an internal index a indicating states of polarization. In eq. (31) the potential is classical and the states of polarization of A_μ are not recognized, so the $\underline{B}^{(3)}$ field:

$$\underline{B}^{(3)*} = -ig\, \underline{A}^{(1)} \times \underline{A}^{(2)} \quad -(32)$$

is not defined in the classical Hamilton-Jacobi equation. The states of polarization here are (1), (2), (3).

The next note will deal with the derivation of the half integral spin properties such as ESR, NMR and MRI etc. from eq. (5), in which the equation of the free electron is written in terms of 2×2 matrices

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135(12): Basic Structure of ECE Fermion Equation and Quantum Electrodynamics

From eq. (3) of the last note the basic structure of the ECE fermion equation is:

$$(\sigma^0 p_0 - \underline{\sigma}\cdot\underline{p})\, q_1^R = mc\,\sigma^0 q_1^L \quad -(1)$$

$$(\sigma^0 p_0 - \underline{\sigma}\cdot\underline{p})\, q_2^R = mc\,\sigma^0 q_2^L \quad -(2)$$

$$(\sigma^0 p_0 + \underline{\sigma}\cdot\underline{p})\, q_1^L = mc\,\sigma^0 q_1^R \quad -(3)$$

$$(\sigma^0 p_0 + \underline{\sigma}\cdot\underline{p})\, q_2^L = mc\,\sigma^0 q_2^R \quad -(4)$$

Eqns (1) and (2) are:

$$\boxed{\begin{aligned}\sigma^\mu p_\mu\, q_1^R &= mc\,\sigma^0 q_1^L \\ \sigma^\mu p_\mu\, q_2^R &= mc\,\sigma^0 q_2^L\end{aligned}} \quad \begin{aligned}-(5)\\-(6)\end{aligned}$$

Eqns (3) and (4) follow from eqs. (1) and (2) as follows. Multiply both sides of eq. (1) for example by $(\sigma^0 p_0 + \underline{\sigma}\cdot\underline{p})$:

$$(\sigma^0 p_0 - \underline{\sigma}\cdot\underline{p})(\sigma^0 p_0 + \underline{\sigma}\cdot\underline{p})\, q_1^R \quad -(7)$$

$$= mc\,\sigma^0 (\sigma^0 p_0 + \underline{\sigma}\cdot\underline{p})\, q_1^L$$

Use:

$$\sigma^0 p_0 \sigma^0 p_0 = \sigma^0 p_0^2 \quad -(8)$$

$$(\underline{\sigma}\cdot\underline{p})(\underline{\sigma}\cdot\underline{p}) = \sigma^0\, \underline{p}\cdot\underline{p} + i\,\underline{\sigma}\cdot\underline{p}\times\underline{p}$$

$$-(9)$$

2)
If $\underline{p}$ is real:

$$(\underline{\sigma} \cdot \underline{p})(\underline{\sigma} \cdot \underline{p}) = \sigma^0 \, \underline{p} \cdot \underline{p}, \quad -(10)$$

so:

$$(\sigma^0)^2 \left(p_0^2 - \underline{p} \cdot \underline{p} \right) \phi_1^R = mc\sigma^0 \left(\sigma^0 p_0 + \underline{\sigma} \cdot \underline{p} \right) \phi_1^L \quad -(11)$$

However:

$$p_0^2 - \underline{p} \cdot \underline{p} = p^\mu p_\mu = m^2 c^2 \quad -(12)$$

so:

$$\left(\sigma^0 p_0 + \underline{\sigma} \cdot \underline{p} \right) \phi_1^L = \sigma^0 mc \, \phi_1^R \quad -(13)$$

Q.E.D.

Therefore the ECE equation of the fermion simplifies to:

$$\boxed{\sigma^\mu p_\mu \phi^R = mc\sigma^0 \phi^L} \quad -(14)$$

where

$$\phi^R = \left[\phi_1^R \ \ \phi_2^R \right] \quad -(15)$$

$$\phi^L = \left[\phi_1^L \ \ \phi_2^L \right] \quad -(16)$$

Eq. (14) means:

$$\boxed{i\sigma^\mu \partial_\mu \phi^R = mc\sigma^0 \phi^L} \quad -(17)$$

using

$$p_\mu = i\hbar \partial_\mu \quad -(18)$$

The fundamental ECE hypothesis is:

$$A^R = A^{(0)} \phi^R \quad -(19)$$
$$A^L = A^{(0)} \phi^L \quad -(20)$$

So:

$$\boxed{i \sigma^\mu \partial_\mu A^R = mc \sigma^0 A^L} \quad -(21)$$

Written out in full:

$$\boxed{\begin{aligned} i\sigma^\mu \partial_\mu A_1^R &= mc\sigma^0 A_1^L \\ i\sigma^\mu \partial_\mu A_2^R &= mc\sigma^0 A_2^L \end{aligned}} \quad \begin{aligned} &-(22) \\ &-(23) \end{aligned}$$

These are the equations of the electromagnetic potential due to the electron.

They are equivalent to:

$$\left(\Box + \left(\frac{mc}{\hbar} \right)^2 \right) A_\mu^a = 0 \quad -(24)$$

where

$$A_\mu^a = \begin{bmatrix} A_1^R & A_2^R \\ A_1^L & A_2^L \end{bmatrix} \quad -(25)$$

This is an SU(2) representation of quantum electrodynamics. Eqs. (22) and (23) are descriptions of the electromagnetic radiation due to an electron.

The same description can be arrived at by using the minimal prescription:

4)

$$p_\mu \to p_\mu + eA_\mu \quad -(26)$$

which means:

$$\boxed{i\partial_\mu \to i\partial_\mu + \frac{e}{\hbar}A_\mu} \quad -(27)$$

In eqs. (5) and (6):

$$\boxed{\begin{aligned} \sigma^\mu(p_\mu + eA_\mu)\phi_1^R &= mc\,\sigma^0\phi_1^L \quad -(28) \\ \sigma^\mu(p_\mu + eA_\mu)\phi_2^R &= mc\,\sigma^0\phi_2^L \quad -(29) \end{aligned}}$$

This is a <u>semi-classical</u> approach because p_μ is quantized but A_μ is classical. Eqs. (22) and (23) on the other hand <u>are fully quantized</u>. As is well known, the semi-classical equations (28) and (29) give ESR, NMR, MRI and the Zeeman effect. These effects come from the left hand side of eq. (7), through ~~the term:~~ energy term:

$$H\phi_1^R = i\frac{e}{2m}\underline{\sigma}\cdot(\underline{p}\times\underline{A} + \underline{A}\times\underline{p})\phi_1^R \quad -(30)$$

$$= \frac{e\hbar}{2m}\underline{\sigma}\cdot\underline{B}\,\phi_1^R$$

(see M. W. Evans and L. B. Crowell, p. 27).

5) This is because the classical kinetic energy in the non-relativistic limit is:

$$H = \frac{1}{2m}\,\underline{\sigma}\cdot\left(\underline{p} + e\underline{A}\right)\underline{\sigma}\cdot\left(\underline{p} + e\underline{A}\right) \quad -(31)$$

However there are unsatisfactory aspects to this semi-classical approach because it does not give the inverse Faraday effect. The correct approach to the interaction of the e/m field and the electron is:

$$\left(\sigma^\circ p_\circ - \underline{\sigma}\cdot\underline{p}\right) q_1^R = mc\,\sigma^\circ q_1^L \quad -(32)$$

$$\left(\sigma^\circ p_\circ - \underline{\sigma}\cdot\underline{p}\right) q_2^R = mc\,\sigma^\circ q_2^L \quad -(33)$$

$$\left(\sigma^\circ p_\circ + \underline{\sigma}\cdot\underline{p}\right) A_1^L = mc\,\sigma^\circ A_1^R \quad -(34)$$

$$\left(\sigma^\circ p_\circ + \underline{\sigma}\cdot\underline{p}\right) A_2^L = mc\,\sigma^\circ A_2^R \quad -(35)$$

and these are solved simultaneously. If we define:

$$p_\circ A_1^L = eA_\circ q_1^L \quad -(36)$$

$$\underline{p}\, A_1^L = e\underline{A}\, q_1^L \quad -(37)$$

$$mc A_1^R = eA_\circ q_1^R \quad -(38)$$

then eqs. (34) and (35) are:

$$\boxed{\begin{aligned}\left(\sigma^\circ A_\circ + \underline{\sigma}\cdot\underline{A}\right) q_1^L &= A_\circ \sigma^\circ q_1^R \quad -(39)\\ \left(\sigma^\circ A_\circ + \underline{\sigma}\cdot\underline{A}\right) q_2^L &= A_\circ \sigma^\circ q_2^R \quad -(40)\end{aligned}}$$

giving the Zeeman effect, ESR, NMR and MRI