1)

146(1): Using the Gravitational Red Shift and Sagnac Effect to Measure the Gravito Magnetic Field.

From paper 117 the gravitomagnetic field is defined as:

$$\underline{\Omega} = -\frac{1}{c^2}\underline{v}\times\underline{g} \quad -(1)$$

$$= \frac{MG}{c^2R^3}L \quad -(2)$$

at the surface of the Earth. The angular momentum of the Earth is, on average, the angular momentum of the sphere of radius R:

$$L = \frac{2}{5}MR^2\omega_E \quad -(3)$$

where ω_E is the earth's angular velocity at the equator:

$$\omega_E = 7.29\times10^{-5}\ \text{rad}\ s^{-1} \quad -(4)$$

Here

$$M = 5.98\times10^{24}\ \text{kg}$$
$$R = 6.37\times10^{6}\ \text{m}$$

so

$$L = 7.076\times10^{33}\ \text{kg}\ m^2\ s^{-1}.$$

On the earth's surface at the equator:

$$\Omega = \frac{\omega_E}{5}\left(\frac{2MG}{c^2R}\right) \quad -(5)$$

so

$$\boxed{\frac{2MG}{c^2R} = 5\frac{\Omega}{\omega_E}} \quad -(6)$$

2) The gravitational metric is therefore:

$$ds^2 = \left(1 - 5\frac{\Omega}{\omega_E}\right)c^2 dt^2 - \left(1 - 5\frac{\Omega}{\omega_E}\right)^{-1} dr^2 - r^2 d\phi^2 - dZ^2 \quad -(7)$$

and the Sagnac effect is:

$$t = \frac{2\pi}{x\omega_0 \pm \omega} \quad -(8)$$

where

$$x = \left(1 - 5\frac{\Omega}{\omega_E}\right)^{1/2} \quad -(9)$$

Here ω_0 is the angular frequency of the light used in the Sagnac interferometer or ring laser gyro, and ω is the angular frequency of rotation of the platform.

The gravitational red shift is:

$$\omega_0 \rightarrow \left(1 - 5\frac{\Omega}{\omega_E}\right)^{1/2} \omega_0 \quad -(10)$$

$$\sim \omega_0 - \frac{5}{2}\frac{\Omega\,\omega_0}{\omega_E}$$

$$\boxed{\Delta\omega_0 = \frac{5}{2}\frac{\Omega\,\omega_0}{\omega_E}} \quad -(11)$$

3) Therefore from eq. (5):

$$\frac{2MG}{c^2 R} = 1.39 \times 10^{-9}$$

$$\boxed{\Omega = 2.03 \times 10^{-14} \text{ radians per second}}$$

One year is 3.156×10^{7} seconds, so

$$\Omega = 6.41 \times 10^{-7} \text{ radians per year}$$

One radian is 2.06265×10^{5} arcseconds, so:

$$\boxed{\Omega = 0.13 \text{ arcseconds per year}}$$

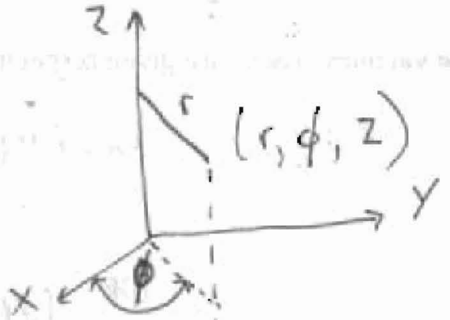
at the Earth's surface at the equator.

This is the gravitomagnetic angular frequency at the Earth's surface at the equator, making the assumption that the Earth is a sphere of mean angular momentum given by eq. (3). There is no need for any satellite experiment to measure Ω. In fact Gravity Probe B failed to measure it, and LAGEOS is disputed. In fact Ω is given simply by eq. (5), all parameters being known. From eq. (1), $\underline{\Omega}$ depends in general on $-\underline{v} \times \underline{g} / c^2$.

46(2): Simple Derivation of the Sagnac Effect.

First set up the cylindrical coordinates (VAPS p. 1023):

$X = r\cos\phi$
$Y = r\sin\phi$
$Z = Z$



The line element in these coordinates is:

$$ds^2 = c^2 dt^2 - dr^2 - r^2 d\phi^2 - dZ^2 \quad - (1)$$

Now rotate in the X-Y plane about Z as follows.

$$d\phi' = d\phi + \omega dt \quad - (2)$$

The rotating line element is:

$$ds'^2 = c^2 dt^2 - dr^2 - r^2 (d\phi + \omega dt)^2 - dZ^2 \quad - (3)$$

and rotates at the angular frequency ω in radians per second.

Now consider the null geodesic:

$$ds'^2 = 0 \quad - (4)$$

appropriate to propagation of light at c around the Sagnac platform. The latter is defined by the X-Y plane:

$$dr = dZ = 0 \quad - (5)$$

where r is the radius of the platform.

2) So: $$\boxed{dt = \pm \left(\frac{r}{c}\right)\left(d\phi + \omega\, dt\right)} \quad - (6)$$

i.e $$\frac{dt}{d\phi} = \frac{1}{\frac{c}{r} \pm \omega} \quad - (7)$$

$$\boxed{\frac{dt}{d\phi} = \frac{1}{\omega_0 \pm \omega}} \quad - (8)$$

where $$\omega_0 = \frac{r}{c}. \quad - (9)$$

therefore: $$\boxed{\frac{d\phi}{dt} = \omega_0 \pm \omega} \quad - (10)$$

This is the Thomas Precession for a null geodesic in a plane, i.e the Thomas precession for a photon travelling at the speed of light. The rotation of the Sagnac platform at the angular frequency ω is the rotation of the Minkowski frame. If the platform is static then:

$$\frac{d\phi}{dt} = \omega_0 \quad - (11)$$

which is precisely the expression for angular frequency.

From eq. (10), the Sagnac effect is an addition or

3)

substraction of angular frequencies. This is exactly the same as the description of the Sagnac effect in ECE theory.

The effect of gravitation on the Sagnac interferometer is found by using the Orbital Theorem of paper 111 to produce the metric:

$$ds^2 = x^2 c^2 dt^2 - \frac{dr^2}{x^2} - r^2 d\phi^2 - dZ^2 \quad - (12)$$

where

$$x = \left(1 - \frac{2GM}{c^2 R}\right)^{1/2} \quad - (13)$$

Here M is the mass of a gravitating object acting on the photon of mass m in the Sagnac interferometer, G is Newton's constant, R is the distance between m and M.

Now rotate the metric (12):

$$ds'^2 = x^2 c^2 dt^2 - \frac{dr^2}{x^2} - r^2 (d\phi + \omega dt)^2 - dZ^2 \quad - (14)$$

For the null geodesic in the X-Y plane, eq. (14) becomes:

$$x\, dt = \pm \left(\frac{r}{c}\right)(d\phi + \omega dt) \quad - (15)$$

i.e.

$$\frac{dt}{d\phi} = \frac{1}{x \frac{c}{r} \pm \omega} \quad - (16)$$

4) or

$$\boxed{\frac{d\phi}{dt} = x\omega_0 \pm \omega} \quad -(17)$$

If the platform is static:

$$\boxed{\frac{d\phi}{dt} = x\omega_0} \quad -(18)$$

and the quantity $d\phi / dt$ is slightly smaller due to the effect of M.

The photon mass m does not appear in the final expression (18) but is implicit in the calculation, because M acts on m through gravitation. Eq. (17) is an expression of the de Sitter precession, which is the Thomas precession in the presence of gravitation.

As in paper 145, new instruments can be constructed on the basis of eq. (17).

146(3): Origin of the Factor Two in Spin - Orbit Interaction

As in P. W. Atkins, "Molecular Quantum Mechanics" (OUP, 2nd. ed. p. 217), the spin orbit interaction Hamiltonian is:

$$H = -\frac{e}{2m^2c^2}\frac{1}{r}\frac{d\phi}{dr} \quad - (1)$$

$$= \xi(r)\,\underline{S}\cdot\underline{L}. \quad - (2)$$

If the reference point is the nucleus, the coordinate system on the electron rotates. In order to calculate the factor 2 in eq. (1), L. Thomas in 1927 used special relativity and the Lorentz transform. The classical calculation in the non-relativistic limit gives twice the result of eq. (1):

$$H(\text{non-relativistic}) = -\frac{e}{m^2c^2}\frac{1}{r}\frac{d\phi}{dr} \quad - (3)$$

where ϕ is a potential. In the non-relativistic result of eq. (3) the angular momentum is:

$$\underline{L} = \underline{r}\times\underline{p} = m\,\underline{r}\times\underline{v} \quad - (4)$$

where m is the mass of an electron moving at close to the speed of light in an orbital around the nucleus. The electron's orbital angular momentum generates a magnetic dipole moment $\underline{m}$.

From the rotation of the Minkowski metric as used in note 146(2), we have:

$$d\phi' = d\phi + \omega\,dt \quad - (5)$$

in the rotating frame. However, for rotation in the X-Y plane:

$$\omega = \frac{d\phi}{dt} \quad - (6)$$

2) so $\boxed{d\phi' = 2d\phi}$ — (7)

The infinitesimal $d\phi'$ in the rotating frame is twice the infinitesimal in the lab frame or observer frame. The rotating frame is the rest frame of the electron, i.e. the electron is at rest in the frame that rotates with the electron.

Thus

$$d\phi = \frac{1}{2} d\phi' \quad -(8)$$

and a calculation based on $d\phi'$ in the frame of the observer must include the factor $1/2$. From eq. (8):

$$\omega = \frac{d\phi}{dt} = \frac{1}{2}\omega' = \frac{1}{2}\frac{d\phi'}{dt} \quad -(9)$$

so the magnitude of angular momentum in the observer frame is

$$L = mr^2\omega = \frac{1}{2}L' = \frac{1}{2}mr^2\omega' \quad -(10)$$

and

$$\boxed{\underline{L} = \frac{1}{2}\underline{L}'} \quad -(11)$$

This is the simplest derivation of the Thomas factor $\frac{1}{2}$.

Derivation from Dirac Equation

Start with the minimal prescription:

$$p^\mu - eA^\mu \quad -(12)$$

where

$$A^\mu = (A_0, \underline{A}) = (\phi/c, \underline{A}) \quad -(13)$$

3.) The Dirac equation is then:

$$(E - ec\phi)\phi^L - c\underline{\sigma}\cdot(\underline{p} - e\underline{A})\phi^R = mc^2\phi^L \quad - (14)$$

$$-(E - ec\phi)\phi^R + c\underline{\sigma}\cdot(\underline{p} - e\underline{A})\phi^L = mc^2\phi^R$$

$$- (15)$$

From eq. (15):

$$\phi^R = \frac{c\underline{\sigma}\cdot(\underline{p} - e\underline{A})\phi^L}{E + mc^2 - ec\phi} \quad - (16)$$

Write the non-relativistic energy as

$$W = E - mc^2 \quad - (17)$$

and write

$$\underline{\pi} = \underline{p} - e\underline{A} \quad - (18)$$

Using eq. (16) in eq. (14):

$$\left(\frac{c^2(\underline{\sigma}\cdot\underline{\pi})(\underline{\sigma}\cdot\underline{\pi})}{E + mc^2 - ec\phi}\right)\phi^L = (W - ec\phi)\phi^L$$

$$- (19)$$

Now write:

$$\frac{c^2}{E - ec\phi + mc^2} = \frac{1}{2m}\left(\frac{2mc^2}{mc^2 + W - ec\phi + mc^2}\right)$$

$$- (20)$$

$$= \frac{1}{2m}\left(1 + \frac{W - ec\phi}{2mc^2}\right)^{-1} \quad - (21)$$

4)

$$\sim \frac{1}{2m}\left(1 - \frac{W - e c \phi}{2mc^2} + \cdots\right) \quad -(22)$$

Therefore:

$$\frac{1}{2m}(\underline{\sigma}\cdot\underline{\pi})(\underline{\sigma}\cdot\underline{\pi})\left(1 - \frac{W - e c\phi}{2mc^2}\right)\phi^L = (W - e c\phi)\phi^L \quad -(23)$$

<u>The</u> Electron g Factor

This may be calculated for the non-relativistic electron in the weak field limit, where

$$E + mc^2 - e c\phi \sim 2mc^2 \quad -(24)$$

In this case, eq. (23) reduces to:

$$W\phi^L = \hat{H}\phi^L \quad -(25)$$

where

$$\hat{H} = \frac{1}{2m}(\underline{\sigma}\cdot\underline{\pi})(\underline{\sigma}\cdot\underline{\pi}) + e c\phi \quad -(26)$$

Therefore

$$\hat{H} = \frac{1}{2m}(\underline{\sigma}\cdot(\underline{p} - e\underline{A}))(\underline{\sigma}\cdot(\underline{p} - e\underline{A})) + e c\phi \quad -(27)$$

The hamiltonian operator $\hat{H}$ operates on the wave-function ϕ^L to give the eigenvalues W.

The operator $\underline{p}$ is:

5)
$$\underline{p} = \frac{\hbar}{i}\underline{\nabla} \quad -(28)$$

Using the algebra of Pauli matrices:

$$(\underset{\sim}{\sigma}\cdot\underline{\pi})(\underset{\sim}{\sigma}\cdot\underline{\pi}) = \pi^2 + i\underset{\sim}{\sigma}\cdot(\underline{\pi}\times\underline{\pi})$$
$$= (\underline{p} - e\underline{A})^2 + i\underset{\sim}{\sigma}\cdot(\underline{p}\times\underline{A} + \underline{A}\times\underline{p}) \quad -(29)$$

Therefore:

$$\hat{H}\phi^L = \frac{1}{2m}(\underline{p} - e\underline{A})^2\phi^L + e\phi\phi^L \quad -(30)$$
$$+ \frac{ie\underset{\sim}{\sigma}}{2m}\cdot(\underline{p}\times\underline{A})\phi^L + \frac{ie\underset{\sim}{\sigma}}{2}\cdot(\underline{A}\times\underline{p})\phi^L$$

The first two terms give the classical hamiltonian.
The last two terms are:

$$\frac{i\underset{\sim}{\sigma}\hbar e}{2m}\cdot\left(\frac{\underline{\nabla}}{i}\times(\underline{A}\phi^L) + \underline{A}\times\frac{\underline{\nabla}}{i}\phi^L\right) \quad -(31)$$
$$= \frac{e\hbar}{2m}\underset{\sim}{\sigma}\cdot\left((\underline{\nabla}\times\underline{A})\phi^L + (\underline{\nabla}\phi^L)\times\underline{A} + \underline{A}\times(\underline{\nabla}\phi^L)\right)$$
$$= \frac{e\hbar}{2m}\underset{\sim}{\sigma}\cdot\underline{B} \quad -(32)$$

Therefore:

6)
$$\hat{H} = \frac{1}{2m}\left(\underset{\sim}{p} - e\underline{A}\right)^2 + ec\phi + \frac{e\hbar}{2m}\,\underline{\sigma}\cdot\underline{B} \quad -(33)$$

where
$$\underset{\sim}{B} = \underline{\nabla}\times\underline{A} \quad -(34)$$

in the standard model.

In ECE the relation (34) must be asserted with the spin connection. The factor $\frac{1}{2}$ before $\underset{\sim}{\sigma}\cdot\underset{\sim}{B}$ in eq. (33) means that the electron's magnetic dipole moment is:

$$\underline{m} = \frac{2e\hbar}{m}\,\underline{\sigma} \quad -(35)$$

giving
$$\boxed{g = 2}, \quad -(36)$$

where g is the electron's g factor. This is corrected by the radiative corrections, see paper 85.

The Thomas Factor of Two

This can be derived from consideration of just the scalar potential ϕ, so eq. (23) simplifies to:

$$\frac{1}{2m}\left(\underset{\sim}{\sigma}\cdot\underset{\sim}{p}\right)\left(\underset{\sim}{\sigma}\cdot\underset{\sim}{p}\right)\left(1 - \frac{W - ec\phi}{2mc^2}\right)\phi^L = \left(W - ec\phi\right)\phi^L \quad -(37)$$

7)

i.e.

$$\left(\frac{p^2}{2m} - \frac{p^2 W}{4m^2c^2} + \frac{ec\,\underline{\sigma}\cdot\underline{p}\,\phi\,\underline{\sigma}\cdot\underline{p}}{4m^2c^2}\right)\phi^L = (W - ec\phi)\phi^L \quad -(38)$$

If it is assumed that

$$p^2 \ll 4m^2c^2, \quad -(39)$$

i.e.

$$v \ll c \quad -(40)$$

then eq. (38) is:

$$\hat{H}\phi^L = W\phi^L \quad -(41)$$

where

$$\hat{H} = \frac{p^2}{2m} + ec\phi + \frac{ec\,\underline{\sigma}\cdot\underline{p}\,\phi\,\underline{\sigma}\cdot\underline{p}}{4m^2c^2} \quad -(41)$$

The first two terms are classical, the third term is the Thomas factor spin orbit interaction with the g is 8 of 1/2.

We have:

$$\underline{\sigma}\cdot\underline{p}\,\phi\,\underline{\sigma}\cdot\underline{p} = \frac{\hbar}{i}\underline{\sigma}\cdot\underline{\nabla}(\phi\,\underline{\sigma}\cdot\underline{p})$$

$$= \frac{\hbar}{i}\underline{\sigma}\cdot\left((\underline{\nabla}\phi)\underline{\sigma}\cdot\underline{p}\right) + \phi(\underline{\sigma}\cdot\underline{p})(\underline{\sigma}\cdot\underline{p})$$

$$= -\frac{\hbar}{i}(\underline{\sigma}\cdot\underline{E})(\underline{\sigma}\cdot\underline{p}) + \phi p^2$$

where

$$\underline{E} = -\underline{\nabla}\phi \quad -(42)$$

is the electric field strength.

8) Again in ECE, the definition (43) must be modified to include the spin connection.

Using the algebra of Pauli matrices:

$$(\underline{\sigma} \cdot \underline{E})(\underline{\sigma} \cdot \underline{p}) = \underline{E} \cdot \underline{p} + i\, \underline{E} \times \underline{p} \quad -(43)$$

So:

$$\underline{\sigma} \cdot \underline{p}\, \phi\, \underline{\sigma} \cdot \underline{p} = i\hbar\, \underline{E} \cdot \underline{p} - \hbar\, \underline{\sigma} \cdot \underline{E} \times \underline{p} + \phi p^2 \quad -(43)$$

The spin-orbit term is:

$$\hat{H}_{so} = \frac{e\hbar}{4m^2c^2}\, \underline{\sigma} \cdot \underline{E} \times \underline{p} \quad -(44)$$

the electric field of the nucleus is:

$$\underline{E} = \frac{e\, \underline{r}}{4\pi\epsilon_0 r^3} \quad -(45)$$

$$= -\nabla \phi$$

where

$$\phi = \frac{e}{4\pi\epsilon_0 r} \quad -(46)$$

so:

$$\hat{H}_{so} = \left(\frac{e}{4\pi\epsilon_0 r^3} \right) \frac{e\hbar}{4m^2c^2}\, \underline{\sigma} \cdot \underline{L} \quad -(47)$$

where

$$\underline{L} = \underline{r} \times \underline{p} \quad -(48)$$

9) is the angular momentum of the electron in the observer frame.

Eq. (47) means that the electron's spin angular momentum interacts with the electron's orbital angular momentum in the observer frame.

The hamiltonian (47) can be written as:

$$\hat{H}_{S.O.} = \psi(r)\,\frac{g e \hbar}{2m^2c^2}\,\underline{\sigma}\cdot\underline{L} \qquad (49)$$

where $g = 2$. — (50)

The result (49) is half that from a classical calculation (Atkins pp. 216 ff.) because of the result (11).

1) 146(4): Derivation of the Sagnac Effect from the Cartan Torsion

The dimensionless phase factor is defined as:

$$\gamma^a := \kappa \int T^a_{\mu\nu}\, d\sigma^{\mu\nu} \quad -(1)$$

where κ is a wavenumber and $d\sigma^{\mu\nu}$ the area in four dimensions. The Cartan torsion is:

$$T^a_{\mu\nu} = \partial_\mu q^a_\nu - \partial_\nu q^a_\mu + \omega^a_{\mu\nu} - \omega^a_{\nu\mu} \quad -(2)$$

Consider the diagonal unit tetrad:

$$q^a_\mu = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad -(3)$$

then

$$T^a_{\mu\nu} = 2\,\omega^a_{\mu\nu}. \quad -(4)$$

Now ~~to~~ identify the torsion in eq (4) as the wavenumber:

$$T^a_{\mu\nu} = \kappa^a_{\mu\nu} = 2\,\omega^a_{\mu\nu} \quad -(5)$$

and consider the special case:

$$T^{(3)}_{12} = \kappa^{(3)}_{12}. \quad -(6)$$

This is the tensor form of the vector component:

$$T^{(3)}_{3} = \kappa^{(3)}_{3} \quad -(7)$$

where

$$\underline{\kappa}^{(3)} = \kappa^{(3)}_{3}\, \underline{k} \quad -(8)$$

2) The complete four-wavenumber is:

$$\kappa_\mu^{(3)} = \left(\kappa_0^{(3)}, -\underline{\kappa}^{(3)}\right) \quad -(9)$$

Therefore eq. (1) may be written as:

$$\gamma^a = \kappa \int \kappa^a_{\mu\nu}\, d\sigma^{\mu\nu} = \kappa \oint \kappa^a_\mu\, dx^\mu \quad -(10)$$

which is a non-Abelian Stokes Theorem

In vector format:

$$\gamma^{(3)} = \kappa \int \underline{\kappa}^{(3)} \cdot d\underline{A_r} \quad -(11)$$

Denote this for ease of notation as:

$$\boxed{\gamma = \kappa^2 A_r} \quad -(12)$$

Now define: $\kappa = \frac{\omega}{c}$ $-(13)$

so

$$\gamma = \frac{\omega^2 A_r}{c^2} \quad -(14)$$

To derive the Sagnac effect as a special case of eq (14) identify the area as

$$A_r = \pi r^2 \quad -(15)$$

where r is the radius of the Sagnac platform.

3) Identify:

$$\omega = \frac{c}{r} \quad -(16)$$

Therefore

$$\boxed{\kappa = \frac{1}{r}} \quad -(17)$$

Write eq. (14) as:

$$\gamma = ct = \omega\left(\frac{\omega A r}{c^2}\right) \quad -(18)$$

then the time taken for the light beam to go around the Sagnac platform is

$$\boxed{t = \frac{\omega A r}{c^2}} \quad -(19)$$

If the platform is rotated at Ω, the difference in time for clockwise and anti-clockwise rotation is calculated from:

$$\Delta\gamma = \frac{Ar}{c^2}\left((\omega+\Omega)^2 - (\omega-\Omega)^2\right) \quad -(20)$$

$$= \frac{4 A r \Omega \omega}{c^2}$$

so

$$\boxed{\Delta t = \frac{4 A r \Omega}{c^2}} \quad -(21)$$

which is the Sagnac effect.

4) In eq. (13), κ and ω are related by the Planck theory of the photon:

$$E = \hbar\omega, \quad p = \hbar\kappa. \quad -(22)$$

Eq. (22) is a special case of the Einstein energy equation:

$$E^2 = c^2 p^2 + m^2 c^4 \quad -(23)$$

when

$$m = 0, \quad -(24)$$

so

$$E = cp. \quad -(25)$$

The Sagnac effect is therefore

$$\boxed{\kappa = \frac{\omega}{c} = \frac{1}{r}}. \quad -(26)$$

This is a special case of eq. (10), which is a general result applicable to all situations. As shown in paper 8, eq. (10) is itself a special case of the ECE phase.

Eq. (26) is the result for the platform at rest:

$$\Omega = 0 \quad -(27)$$

It is also known that the Sagnac ~~effect~~ effect is a special case of the Thomas precession. In the case (27) the Thomas precession

5) metric is the Minkowski metric:

$$ds^2 = c^2 dt^2 - dr^2 - r^2 d\phi^2 - dZ^2 \quad -(28)$$

and the Sagnac effect is the special case of eq. (28) when:

$$ds = dr = dZ = 0 \quad -(29)$$

i.e.

$$r^2 d\phi^2 = c^2 dt^2 \quad -(30)$$

$$\frac{d\phi}{dt} = \pm \frac{c}{r} = \pm \omega \quad -(31)$$

In the light-like condition defined by eq. (29):

$$d\phi^2 = (\kappa c)^2 dt^2 = \omega^2 dt^2 \quad -(32)$$

Eq. (28) is the metric of the Maxwell Heaviside equations. The latter always use a static metric, and so cannot describe the extra effect of Ω on the Sagnac effect.

Since eq. (30) is rigorous, and not confined to the Sagnac effect, there is an effect of rotation on electromagnetic radiation in general. This is a phenomenon of general relativity. The metric for electromagnetic radiation is:

6)

$$ds^2 = c^2 dt^2 - dr^2 - \frac{1}{\kappa^2} d\phi^2 - dZ^2 \quad -(33)$$

$$= 0$$

For matter waves (e.g. electrons):

$$ds^2 \neq 0 \quad -(34)$$

and from eq. (23)

$$\omega^2 = c^2 \kappa^2 + \left(\frac{mc}{\hbar}\right)^2 \quad -(35)$$

so for matter waves eq. (35) replaces eq. (13):

If Minkowski spacetime is rotated at Ω, the electromagnetic metric (33) is changed by

$$d\phi' = d\phi + \Omega\, dt \quad -(36)$$

so

$$\frac{d\phi}{dt} = \omega \pm \Omega \quad -(37)$$

Eq. (37) means that in any situation, the electromagnetic angular velocity is affected by spacetime rotation. An example is a Fourier transform spectrometer on a rotating platform.

1) 146(5): Effect of Rotation on Spectra

Consider the Minkowski metric in cylindrical polar coordinates:

$$ds^2 = c^2 dt^2 - dr^2 - r^2 d\phi^2 - dZ^2 \quad (1)$$

and rotate it at the angular frequency Ω:

$$d\phi' = d\phi + \Omega\, dt \quad (2)$$

then

$$ds^{2\prime} = c^2 dt^2 - dr^2 - r^2 (d\phi + \Omega\, dt)^2 - dZ^2 \quad (3)$$

$$= c^2 dt^2 - dr^2 - r^2 d\phi^2 - 2r^2 \Omega\, dt\, d\phi - r^2 \Omega^2 dt^2 - dZ^2$$

$$= (c^2 - r^2\Omega^2)\, dt^2 - dr^2 - r^2 d\phi^2 - 2r^2\Omega\, dt\, d\phi - dZ^2 .$$

In this equation:

$$v = \Omega R \quad (4)$$

where v is the tangential linear velocity for a rotation of angular frequency Ω. Here R is the radius of rotation. If the radius r in eq. (4) is defined as R, then

$$ds^{2\prime} = (c^2 - v^2)\, dt^2 - dr^2 - r^2 d\phi^2 - 2R^2\Omega\, d\phi\, dt - dZ^2 \quad (5)$$

$$= c^2 dt'^2$$

Now write eq. (5) as:

$$c^2 dt'^2 = \left(1 - \frac{v^2}{c^2}\right)\left(dt^2 - 2R\Omega\, d\phi\, dt\right) - dr^2 - r^2 d\phi^2 - dZ^2 \quad (6)$$

2) where

$$\Omega' = \Omega \left(1 - \frac{v^2}{c^2}\right)^{-1} \quad -(7)$$

from eq. (6)

$$dt' = \left(1 - \frac{v^2}{c^2}\right)^{1/2} dt \quad -(8)$$

The phase diff:

$$\Delta\theta = \Omega' dt' - \Omega dt \quad -(9)$$

$$= \theta' - \theta$$

is called the Thomas precession.

$$\Delta\theta = \left(\left(1 - \frac{v^2}{c^2}\right)^{-1/2} - 1\right)\theta \quad -(10)$$

The Thomas precession is the change in angle as follows:

$$\theta' = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \theta \quad -(11)$$

the relativistic angular velocity is:

$$\Omega' = \frac{d\theta'}{dt'} \quad -(12)$$

$$= \left(1 - \frac{v^2}{c^2}\right)^{-1} \frac{d\theta}{dt} \quad -(13)$$

If Ω is the angular frequency at the Earth's equator:

$$\Omega = 7.29 \times 10^{-5} \text{ radians } s^{-1}$$

and if R is the mean radius of the Earth:

$$R = 6.37 \times 10^{6} \text{ m}$$

3) then

$$v = \Omega R = 464.4 \text{ m s}^{-1} \quad -(14)$$

For the propagation of light,

$$ds'^2 = 0, \quad -(15)$$

so in general:

$$c^2 dt^2 = dr^2 + r^2 (d\phi + \Omega dt)^2 + dZ^2. \quad -(16)$$

This result describes the propagation of electromagnetic radiation in a Minkowski frame rotating at Ω. If there were no rotation:

$$c^2 dt^2 = dr^2 + r^2 d\phi^2 + dZ^2 \quad -(17).$$

$$= dX^2 + dY^2 + dZ^2$$

Here:

$$X = r\cos\phi, \quad Y = r\sin\phi \quad -(18)$$

so:

$$\left.\begin{aligned} dX &= d(r\cos\phi) = \cos\phi\, dr - r\sin\phi\, d\phi \\ dY &= d(r\sin\phi) = \sin\phi\, dr + r\cos\phi\, d\phi \end{aligned}\right] -(19)$$

$$dX^2 + dY^2 = (\cos\phi\, dr - r\sin\phi\, d\phi)^2 + (\sin\phi\, dr + r\cos\phi\, d\phi)^2$$

$$= dr^2 + r^2 d\phi^2 \quad -(20)$$

Without loss of generality consider

$$r = \text{constant} \quad -(21)$$

so

$$dr = 0 \quad -(22)$$

and

$$dX^2 + dY^2 = r^2 d\phi^2. \quad -(23)$$

4) In this case:

$$c^2 dt^2 = r^2 d\phi^2 + dZ^2 \quad - (24)$$

$$= dX^2 + dY^2 + dZ^2 \quad - (25)$$

To simplify further consider:

$$dZ = 0 \quad - (26)$$

so that motion is considered in a plane, $X - Y$. So:

$$c^2 dt^2 = dX^2 + dY^2 = r^2 d\phi^2 \quad - (27)$$

The angular frequency of this motion is:

$$\omega = \frac{d\phi}{dt} = \pm \frac{c}{r} \quad - (28)$$

However, this angular frequency is that of the photon, so:

$$E = \hbar \omega \quad - (29)$$

and

$$\frac{c}{r} = \frac{E}{\hbar} \quad - (30)$$

The momentum of the photon is

$$p = \hbar \kappa \quad - (31)$$

where

$$\kappa = \frac{\omega}{c} \quad - (32)$$

so

$$E = \hbar \kappa c$$

and

$$\boxed{\kappa = \frac{1}{r}} \quad - (33)$$

Since r has been assumed to be constant, then

5) The metric (27) is for a constant wavenumber κ, i.e. for a monochromatic frequency ω of electromagnetic radiation. The concept that links together momentum p and wavenumber $\underline{\kappa}$, eq. (31), is the de Broglie wave particle duality. This concept leads to eq. (33), so eq. (28) is:

$$\boxed{\omega = \kappa c} \quad -(34)$$

If the Minkowski frame is now rotated:

$$\boxed{\frac{d\phi}{dt} = \omega \pm \Omega} \quad -(35)$$

This means that rotation shifts the frequency of electromagnetic radiation.

The quantum theory of absorption of radiation by an atom or molecule implies that the absorption takes place at precisely ω. So if ω is shifted by rotation, the effect can be observed in spectra of all kinds. The angular velocity Ω is different at different points on the Earth's surface.

So

$$\boxed{E = \hbar(\omega \pm \Omega)}$$

146(6): Gravitational Red Shift and de Sitter Precession or Geodetic Precession, Effect of Rotation on Spectra.

Two possible solutions of the orbital theorem of paper 111 are the Minkowski metric:

$$ds^2 = c^2 dt^2 - dr^2 - r^2 d\phi^2 - dZ^2 \quad -(1)$$

and the gravitational metric

$$ds^2 = \left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 - \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 - r^2 d\phi^2 - dZ^2 \quad -(2)$$

in cylindrical polar coordinates. The effect of gravitation is therefore

$$t = x t_0 \quad -(3)$$

$$r = r_0 / x \quad -(4)$$

where

$$x = \left(1 - \frac{2GM}{rc^2}\right)^{1/2} \quad -(5)$$

For the photon:

$$ds^2 = 0, \quad -(6)$$

and if we consider the plane:

$$dr = dZ = 0 \quad -(7)$$

then

$$\omega_0 = \frac{d\phi}{dt} = \frac{c}{r} \quad -(8)$$

from eq. (1). From eq. (2):

$$\omega = \frac{d\phi}{dt} = \frac{1}{x} \frac{c}{r} x \quad -(9)$$

so

$$\boxed{\frac{\omega_0}{\omega} = x^{-1}} \quad -(10)$$

2) which is the gravitational red shift. This means that the angular frequency of e/m radiation in the presence of the mass M is:

$$\omega = x\omega_0 = \left(1 - \frac{2GM}{c^2 r}\right)^{1/2} \omega_0 \quad - (11)$$

The frequency ω decreased, i.e. shifted towards the red part of the spectrum. This is a shift of the frequency of absorption of a spectral line, and so can be observed experimentally, as is well known.

It is known that for the photon:

$$\omega_0 = \kappa_0 c \quad - (12)$$

where κ_0 is the wavenumber. So:

$$\boxed{\begin{aligned} \kappa_0 &= \frac{1}{r} \quad &- (13) \\ \kappa &= x\kappa_0 \quad &- (14) \end{aligned}}$$

Eqs. (12) to (14) are expression of wave / particle dualism, the Fermat Principle and Hamilton Principle of least time and least action respectively. It is seen from eq. (13) that there is a duality between wavenumber and inverse position. This is a fundamental property of electromagnetic radiation and matter waves in general.

3) The gravitational red shift can be expressed in terms of energy as:

$$\frac{E_0}{E} = \frac{\hbar\omega_0}{\hbar\omega} = \left(1 - \frac{2GM}{c^2 r}\right)^{-1/2} \quad -(15)$$

so $$E = \left(1 - \frac{2GM}{c^2 r}\right)^{1/2} E_0 \quad -(16)$$

$$\sim \left(1 - \frac{GM}{c^2 r}\right) E_0 \text{ for } 2GM << c^2 r.$$

Now use the de Broglie photon mass equation:

$$E_0 = \hbar\omega_0 = mc^2 \quad -(17)$$

where m is the mass of the photon and ω_0 is the photon rest angular frequency. In the gravitational field:

$$E = mc^2 - \frac{GMm}{r} \quad -(18)$$

in the Newtonian approximation for the gravitational potential energy of interaction between the photon of mass m and a mass M. So:

$$\frac{E}{E_0} = \frac{mc^2 - GMm/r}{mc^2} \quad -(19)$$

$$= \left(1 - \frac{GM}{c^2 r}\right)$$

which is eq. (16). The Newtonian limit is therefore $2GM << c^2 r$.

4) The photon mass m is therefore implicit in the factor x:

$$x = \left(\frac{mc^2}{mc^2} \left(1 - \frac{2GM}{rc^2} \right) \right)^{1/2} \quad -(20)$$

but cancels out of the equation.

If a method could be devised of isolating m in these equations, an experiment to measure the photon mass could be devised. This has been an aim of physics for over a hundred years.

Rotation of the Metrics

The rotation of eq. (1) for the photon in the plane, eqs. (6) and (7), produces the Sagnac effect:

$$\frac{d\phi}{dt} = \omega \pm \Omega \quad -(21)$$

$$= \frac{c}{r} \pm \Omega .$$

where Ω is the angular frequency of rotation of the Sagnac platform. However

$$\omega = \kappa c \quad -(22)$$

so the Sagnac effect is a special case of the fact that any rotation at angular

5) frequency Ω affects the angular frequency of a wave.

So:

$$E = \hbar \omega \quad - (23)$$

is changed to

$$E = \hbar (\omega \pm \Omega) \quad - (24)$$

and:

$$\underline{p} = \hbar \underline{\kappa} \quad - (25)$$

is changed to

$$\underline{p} = \hbar (\underline{\kappa} + \underline{\kappa}_1) \quad - (26)$$

The rotation of metric (2) produces:

$$\frac{d\phi}{dt} = x\omega \pm \Omega. \quad - (27)$$

as in paper 145. Eq. (27) is the geodetic effect or de Sitter precession observed in the Sagnac effect. The gravitational red shift in eq. (11) can be considered as an example of the new general equation (24):

$$\omega = x\omega_0 \sim \omega_0 \left(1 - \frac{GM}{c^2 r}\right) \quad - (28)$$

$$= \omega_0 - \Omega$$

where

$$\boxed{\Omega = \frac{GM}{c^2 r}\,\omega_0} \quad - (29)$$

6) Eq. (29) is routinely observed in astronomy. So is an example of the law (24). The Sagnac effect is another example of the law (24). Therefore gravitation in the limit:

$$mG << c^2 r \quad -(30)$$

is the rotation of the Minkowski metric by:

$$d\phi' = d\phi + \Omega dt \quad -(31)$$

where Ω is given by eq. (29). This rotation modifies the Planck law to:

$$E = \hbar(\omega + \Omega) \quad -(32)$$

Conclusion

Any kind of rotation will shift spectral lines, not just gravitation or the Sagnac effect. For example, a spectrometer on a platform rotating at Ω will show absorption frequencies shifted from ω to $\omega \pm \Omega$, depending on the sense of rotation, clockwise or anti-clockwise.

146(7): Some Concepts of the Thomas Precession and the Dirac Equation

1) If we define the phase angle:

$$\theta = \omega t \quad -(1)$$

the Thomas precession is the phase shift:

$$\alpha = \theta(\gamma - 1) \quad -(2)$$

where

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(3)$$

is the Lorentz factor. So

$$\alpha = \omega' t' - \omega t. \quad -(4)$$

The proper time is defined as

$$\tau = \frac{t}{\gamma} = \left(1 - \frac{v^2}{c^2}\right)^{1/2} t \quad -(5)$$

and the simplest expression of the Thomas precession is

$$\theta' = \gamma \theta \quad -(6)$$

(www.aether.lbl.gov/www/classes/p139/homework/seven.pdf)

Therefore

$$\theta' = \Omega \tau = \gamma \theta = \gamma \omega t \quad -(7)$$

where

$$\Omega = \gamma^2 \omega \quad -(8)$$

2) The same result is obtained by rotating the Minkowski metric:

$$ds^2 = c^2 dt^2 - dr^2 - r^2 d\phi^2 - dZ^2 \quad -(9)$$

by

$$d\phi \rightarrow d\phi + \omega t \quad -(10)$$

2) to give:

$$ds'^2 = c^2 d\tau^2 - 2r^2 \frac{\Omega^2}{\gamma^2} d\phi\, dt - r^2 d\phi^2 - dZ^2 \quad -(11)$$

so rotation of the Minkowski metric at ω produces the proper time τ and the Lorentz factor γ.

3) If

$$\omega = \frac{d\phi}{dt} \quad -(12)$$

then

$$d\phi \rightarrow 2\, d\phi \quad -(13)$$

giving the Thomas factor of 2.

4) The origin of the Dirac equation can be thought of in several ways, but it is clear that the Dirac equation originates in eqn. (11), as does all of special relativity. One way of deriving the Dirac equation is from the relativistic momentum:

$$\underline{p} = m \frac{d\underline{r}}{d\tau} \quad -(14)$$

$$= \gamma m \frac{d\underline{r}}{dt} = \gamma m \underline{v}$$

where γ originates in eq. (10). Therefore γ and the Thomas factor 2 originate in the same equation.

From eq. (14):

$$\underline{F} = \frac{d\underline{p}}{dt} = \frac{d}{dt}\left(\gamma m \underline{v}\right) \quad -(15)$$

3) and:

$$T = W = \int \underline{F} \cdot \underline{v}\, dt = m \int_0^v v\, d(\gamma v) \quad -(16)$$

(Marion & Thornton, 3rd. ed., page 527). So:

$$T = \gamma m v^2 - m \int_0^v \frac{v\, dv}{(1 - v^2/c^2)^{1/2}}$$

$$= \gamma m v^2 + mc^2 \left(1 - \frac{v^2}{c^2}\right)^{1/2} \Big|_0^v$$

$$= \gamma m v^2 + mc^2 \left(1 - \frac{v^2}{c^2}\right) - mc^2$$

$$T = \gamma m c^2 - mc^2 \quad -(17)$$

$$\boxed{T = mc^2(\gamma - 1)} \quad -(18)$$

This is the relativistic kinetic energy: it is seen that

From eqs. (2) and (18):

$$\boxed{T = mc^2 \frac{\alpha}{\theta}} \quad -(19)$$

and for $\theta = 2\pi$

$$\boxed{T = \left(\frac{mc^2}{2\pi}\right) \alpha} \quad -(20)$$

or

$$T = E_0 \left(\frac{\alpha}{2\pi}\right) \quad -(21)$$

which shows that the relativistic kinetic energy of

4) any particle is due to the Thomas precession

The Dirac equation also originates in the Thomas precession. This is seen from the well known derivation of the Einstein energy equation from the relativistic momentum (14), (M&T p. 529). Start with:

$$p = \gamma m v \quad -(22)$$

so

$$p^2 c^2 = \gamma^2 m^2 c^4 \left(\frac{v^2}{c^2}\right) \quad -(23)$$

Use:

$$\frac{v^2}{c^2} = 1 - \frac{1}{\gamma^2} \quad -(24)$$

so

$$p^2 c^2 = \gamma^2 m^2 c^4 \left(1 - \frac{1}{\gamma^2}\right) = \gamma^2 m^2 c^4 - m^2 c^4 \quad -(25)$$

Finally define:

$$E = \gamma m c^2 \quad -(26)$$

so

$$\boxed{E^2 = p^2 c^2 + E_0^2} \quad -(27)$$

where

$$E_0 = mc^2 \quad -(28)$$

Note that

$$E = T + mc^2 = mc^2\left(\frac{\alpha}{\theta} + 1\right)$$

$$\boxed{E = mc^2\left(1 + \frac{\alpha}{\theta}\right)} \quad -(29)$$

) Here E is known as the total energy of a particle.

The relativistic total energy E and relativistic kinetic energy T both originate in the Thomas precession phase α.

The wave format of the Dirac equation is obtained from eq. (27) using

$$p^\mu = i\hbar\,\partial^\mu \quad — (30)$$

where

$$p_\mu = \left(\frac{E}{c}, \mathbf{p}\right) \quad — (31)$$

$$\partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\nabla\right) \quad — (32)$$

So:

$$E = i\hbar\frac{\partial}{\partial t}, \quad \mathbf{p} = -i\hbar\nabla \quad — (33)$$

The operators act on an eigenfunction ψ. So eq. (27) becomes:

$$-\hbar^2\frac{\partial^2\psi}{\partial t^2} = -\hbar^2 c^2\nabla^2\psi + m^2c^4\psi \quad — (34)$$

i.e.

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \left(\frac{mc}{\hbar}\right)^2\right)\psi = 0 \quad — (35)$$

i.e.

$$\boxed{(\Box + \kappa^2)\psi = 0} \quad — (36)$$

6) where

$$\kappa = \frac{mc}{\hbar} \quad -(37)$$

is the Compton wavelength. Eq. (36) is the wave form of the Dirac equation, which also originates in the Thomas precession.

5) The Dirac equation (36) factorizes into:

$$\left(i\gamma^\mu \partial_\mu - \kappa\right)\psi = 0 \quad -(38)$$

and can also be developed in terms of 2 x 2 matrices as in papers 129 and 130. Therefore eq. (38) originates in a rotation, i.e. in the Thomas precession. Eq. (10) corresponds to the rotation of the vector:

$$\underline{V} = V_x \underline{i} + V_y \underline{j} + V_z \underline{k} \quad -(39)$$

through an angle ϕ:

$$\begin{bmatrix} V_X' \\ V_Y' \\ V_Z' \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_X \\ V_Y \\ V_Z \end{bmatrix} \quad -(40)$$

and this is the rotational Lorentz transform. The latter is therefore the Thomas precession. The rotation generator from eq. (40) is:

$$J_Z = \frac{1}{i} \frac{dR_Z(\phi)}{d\phi}\bigg|_{\phi=0} \quad -(41)$$

7) where the rotation matrix is:

$$R_Z(\phi) = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad - (42)$$

so

$$J_Z = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad - (43)$$

This is an angular momentum generator within Φ. Therefore the angular momentum generator originates in the Thomas precession. We have:

$$[J_X, J_Y] = i J_Z \quad - (44)$$

et cyclicum

The rotation matrix (42) may be written as:

$$\exp(i J_Z \phi) = 1 + i J_Z \phi - J_Z^2 \phi^2 + \cdots \quad - (45)$$

so

$$\underline{V}' = \exp(i J_Z \phi) \underline{V} \quad - (46)$$

In general:

$$\underline{V}' = \exp(i \underline{J} \cdot \underline{n} \phi) \underline{V} \quad - (47)$$

$$= \exp(i \underline{J} \cdot \underline{\phi}) \underline{V}$$

These properties are of the rotation group.

If $\underline{V}$ is the position vector:

$$\underline{r} = X\underline{i} + Y\underline{j} + Z\underline{k} \quad -(48)$$

then eq. (47) is an O(3) transformation on $\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$.

This corresponds to an SU(2) transformation on the basic spinor:

$$\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \quad -(49)$$

The correspondence is

$$R = \exp(i\,\underline{J}\cdot\underline{\phi}) \quad -(50)$$

corresponds with:

$$U = \exp(i\,\underline{\sigma}\cdot\underline{\phi}/2) \quad -(51)$$

where

$$\left[\frac{\sigma_X}{2}, \frac{\sigma_Y}{2}\right] = i\frac{\sigma_Z}{2} \quad -(52)$$

et cyclicum.

Here, σ_X, σ_Y and σ_Z are the Pauli matrices.

$$\sigma_X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad -(53)$$

and

$$\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad -(54)$$

The Pauli matrices are basis elements of the SU(2) group. The Cartesian unit vectors $\underline{i}, \underline{j}, \underline{k}$ are basis elements of the O(3) group.

) The spinor rotates through half the angle through which the vector rotates. There is a link between this topological concept and eqn. (13), because ϕ in Thomas precession is increased to 2ϕ, the angle is twice as large in the rotating Minkowski spacetime. If the spinor is thought of as rotating through ϕ, the vector rotates through 2ϕ.

As shown in L. H. Ryder, "Quantum Field Theory" (CUP, 2nd ed.), pp. 41 ff., the Dirac equation (38) is derived from:

$$\phi^R = \exp\left(\frac{1}{2}\underline{\sigma}\cdot\underline{\phi}\right)\phi^R(0) \quad (55)$$

$$\phi^L = \exp\left(-\frac{1}{2}\underline{\sigma}\cdot\underline{\phi}\right)\phi^L(0) \quad (56)$$

where

$$\cosh\phi = \gamma \quad (57)$$

$$\sinh\phi = \frac{v}{c}\gamma \quad (58)$$

and

$$\phi(\underline{p}) = \begin{bmatrix}\phi^R \\ \phi^L\end{bmatrix} \quad (59)$$

$$\phi^R(0) = \phi^L(0), \quad (60)$$

with

$$\exp\left(\frac{1}{2}\underline{\sigma}\cdot\underline{\phi}\right) = \cosh\frac{\phi}{2} + \underline{\sigma}\cdot\underline{n}\,\sinh\frac{\phi}{2}. \quad (61)$$

Here ϕ^R and ϕ^L are the Pauli spinors.

10) Therefore it has been shown that the Dirac equation in its first order format (38) is a rotation of the Pauli spinors in SU(2). It has also been shown that the total energy E and kinetic energy T in special relativity originate in another type of rotation — that of the Minkowski metric in 4-D spacetime. The Eqs. (55) and (56) are also in 4-D spacetime. The link between the two types of rotation is found by:

$$i\gamma^\mu \partial_\mu \left(i\gamma^\nu \partial_\nu - \kappa \right)\psi = 0 \quad -(62)$$

i.e.

$$\left(\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu + \kappa^2 \right)\psi = 0 \quad -(63).$$

with

$$\Box = \gamma^\mu \gamma^\nu \partial_\mu \partial_\nu \quad -(64)$$

and

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 g^{\mu\nu} \quad -(65)$$

where the Minkowski metric is

$$g^{\mu\nu} = g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad (66)$$

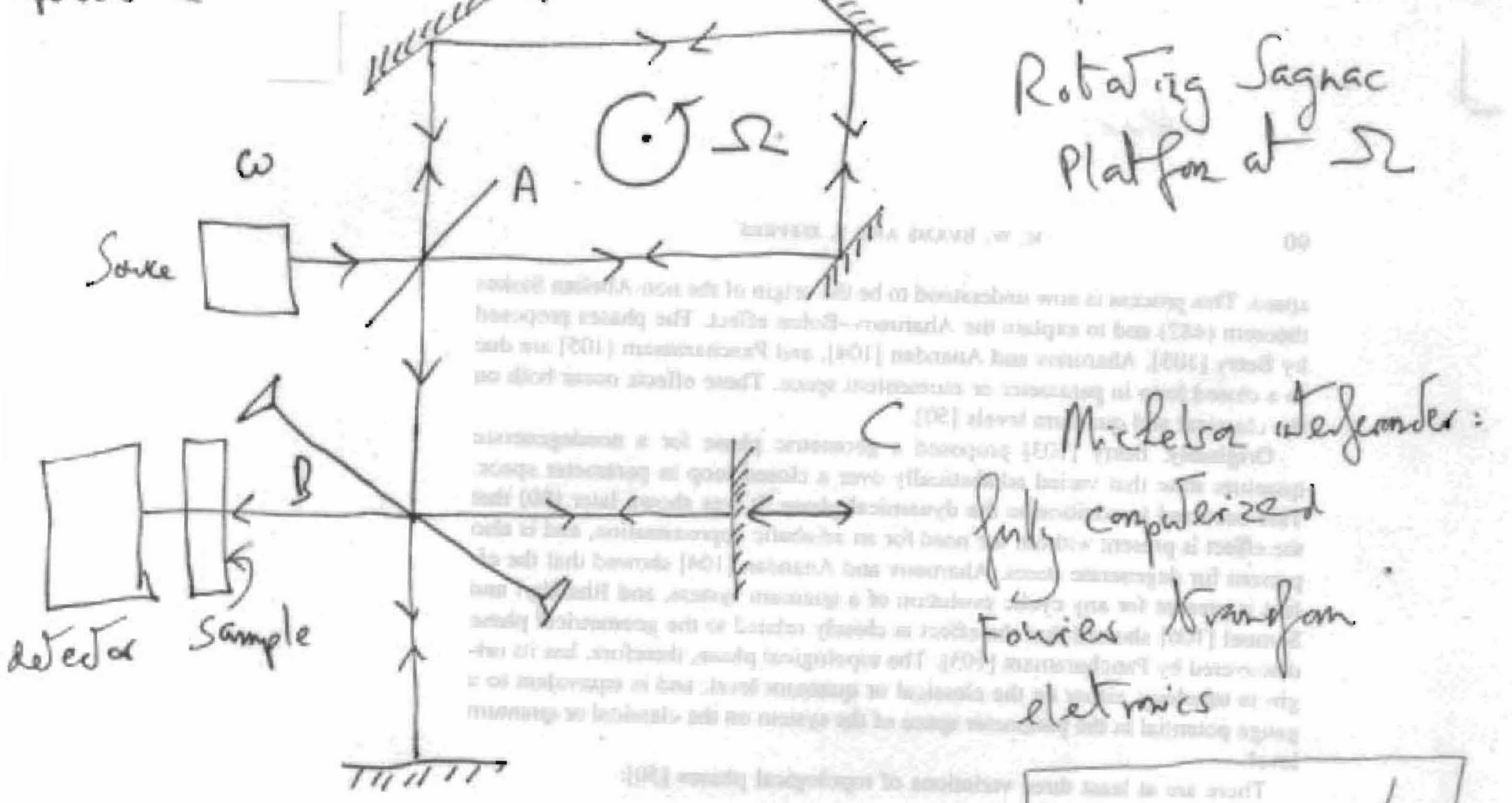
and the Dirac matrices are

$$\gamma^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \gamma^i = \begin{bmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{bmatrix} \quad -(67)$$

Rotating the metric (66) produces eqn. (63).

1) 146(8): Combined Sagnac / Michelson Interferometer

This is an interferometric system designed to test whether rotation Ω has an effect on an absorption spectrum.



$\boxed{\text{Tune } \omega = c/r}$

A = half silvered mirror

B = beam splitter

C = moving mirror

In the simplest case there is a monochromatic source at frequency ω and the sample absorbs at frequency ω. The rotation changes the source frequency to $\omega \pm \Omega$ and the absorption is changed. In a perfect experiment it disappears.

146(9): Some Further Observations on the Sagnac Effect.

Consider the Minkowski metric in cylindrical polar coordinates:

$$ds^2 = c^2 dt^2 - dr^2 - r^2 d\phi^2 - dZ^2 \quad -(1)$$

In the limit:

$$ds = dr = dZ = 0 \quad -(2)$$

$$r^2 d\phi^2 = c^2 dt^2 \quad -(3)$$

so

$$\omega_0 = \frac{d\phi}{dt} = \pm \frac{c}{r} \quad -(4)$$

which is a property of the photon because the lightlike condition has been used in eq. (2). Therefore:

$$\omega_0 = \frac{d\phi}{dt} = \frac{c}{r} = \kappa c \quad -(5)$$

in the quantum theory. The angular frequency ω_0 is that of the photon and the wavenumber of the photon has been identified as:

$$\kappa = \frac{1}{r} \quad -(6)$$

therefore the photon radius is:

$$r = \frac{1}{\kappa} \quad (7)$$

Therefore the photon is described by the metric (3):

$$d\phi^2 = \kappa^2 c^2 dt^2 = \omega_0^2 dt^2 \quad -(8)$$

and by the wave particle duality:

$$E = \hbar \omega_0 \quad -(9)$$

$$\underline{p} = \hbar \underline{\kappa} \quad -(10)$$

The units of ω_0 are radians per second so ϕ is the angle in radians. The frequency of the wave is

2)

$$f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}\frac{d\phi}{dt} \quad -(11)$$

If the metric is rotated such that:

$$d\phi \rightarrow d\phi \mp \Omega\, dt \quad -(12)$$

then:

$$c^2 dt^2 = r^2 \left(d\phi \mp \Omega\, dt\right)^2 \quad -(13)$$

$$d\phi \mp \Omega\, dt = \omega_0\, dt \quad -(14)$$

and

$$\frac{d\phi}{dt} = \omega_0 \pm \Omega \quad -(15)$$

$$\boxed{\frac{d\phi}{dt} = \kappa c \pm \Omega} \quad -(16)$$

The frequency of the light ω_0 is increased or decreased by Ω. This light shift is derived independently of r and Z, it depends only on the angle ϕ and the time t. It can be detected by the Sagnac / Michelson interferometer at any source frequency, so of note 146(8) a standard source such as a Philips HP 16 mercury source can be used.

If we define:

$$\Omega = \frac{v}{R} \quad -(17)$$

then if we take the positive sign in eq. (12):

3) $$d\phi \rightarrow d\phi + \Omega\, dt \quad - (18)$$

then

$$\left(1 - \frac{v^2}{c^2}\right) dt^2 = \frac{1}{\omega_0^2}\cdot\left(d\phi^2 + 2\Omega\, dt\, d\phi\right) \quad - (19)$$

from which:

$$\Omega + \frac{d\phi}{dt} = \pm\frac{1}{\gamma}\left(\gamma^2\Omega^2 + \omega_0^2\right)^{1/2} \quad - (20)$$

Here:

$$\left(\frac{\Omega}{\omega_0}\right)^2 = \frac{\gamma^2 - 1}{\gamma^2} \quad - (21)$$

$$= \left(\frac{\gamma + 1}{\gamma^2}\right)(\gamma - 1) \quad - (22).$$

The relativistic features of the phenomena are also understandable as in Jackson eq. (11.119):

$$\underline{\omega}_T = \left(\frac{\gamma^2}{\gamma + 1}\right)\frac{\underline{a}\times\underline{v}}{c^2} \quad - (23)$$

where $\underline{\omega}_T$ is the Thomas angular velocity, $\underline{v}$ a velocity and $\underline{a}$ is an acceleration, and c the speed of light.

Conclusion

We have verified that eq. (16) is a new phenomenon, rotation affects electromagnetic frequencies under all conditions.

146(10): Topological Description of the Sagnac and Tomita Chiao Effects

This is reviewed in the Omnia Opera of www.aias.us in "Advances in Classical Physics", vol. 119(3) Section X, pp. 93 ff. (2001). After one loop, the light in the Sagnac effect with platform at rest develops an electromagnetic phase shift:

$$\phi_S = \oint \underline{\kappa} \cdot d\underline{r} = \int \kappa^2 \, dA_r \quad -(1)$$

so

$$\psi \rightarrow \exp\left(i\left(\omega t - \underline{\kappa} \cdot \underline{r} + \phi_S\right)\right) \quad -(2)$$

After one loop the plane of plane polarized light is tilted. In the Tomita Chiao effect the plane is tilted after traversing a helical fibre optic cable. The latter is the same as the fibre optic gyro. The quantity κ in eq. (1) is

$$\kappa = \frac{1}{c} \frac{d\phi}{dt} \quad -(3)$$

where $d\phi / dt$ is found from the rotating Minkowski metric as in note 146(2):

$$dt = \pm \left(\frac{r}{c}\right)\left(d\phi + \omega \, dt\right) \quad -(4)$$

i.e.

$$\frac{d\phi}{dt} = \omega_0 \pm \omega \quad -(5)$$

where

$$\omega_0 = \frac{c}{r}. \quad -(6)$$

1) Therefore from eqs. (3) and (6):

$$\kappa = \frac{1}{r} \quad -(7)$$

which is wave particle duality.

From eq. (1), the Sagnac effect is:

$$\Delta\phi_S = \frac{1}{c^2}\int (\omega_0+\omega)^2 - (\omega_0-\omega)^2 \, dA_r \quad -(8)$$

$$= \left(\frac{4\omega A_r}{c^2}\right)\omega_0 \quad -(9)$$

$$:= \Delta t \, \omega_0$$

So

$$\boxed{\Delta t = \frac{4\omega A_r}{c^2}} \quad -(10)$$

as observed to $1 : 10^{25}$.

Therefore mechanical rotation at angular frequency ω affects the electromagnetic phase.

The phase:

$$\phi_S = \oint \underline{\kappa} \cdot d\underline{r} = \frac{1}{c^2}\int \omega_0^2 \, dA_r \quad -(11)$$

is electromagnetic through the wave particle duality (6). Therefore it is concluded that the mechanical rotation is a rotation of the

) electromagnetic phase:

$$e^{i\phi} \to e^{i\phi_s} e^{i\phi} = e^{i(\phi + \phi_s)} \quad -(12)$$

A mechanical angular frequency ω_0 can be added to an electromagnetic angular frequency. This has an effect on absorption spectra. In the Maxwell Heaviside theory these effects do not occur. They are basically phase effects that all derive from the ECE phase of paper 6. Examples are the Berry phase, the Aharonov Bohm effect, the topological phases, Sagnac effect, and Tomita Chiao effect. In paper 146, it is shown that they all derive from the Thomas precession, which is rotation of the Minkowski metric.

1) 146(11): Further Analysis of the Sagnac Effect

There have been many discussions about the Sagnac effect for about a century. The rotating metric method is fully relativistic from the outset, and produces the precisely correct result that the time taken for light to go around a circle, covering 360° or 2π radians, is:

$$t = \frac{2\pi}{\omega} = \frac{2\pi r}{c}. \quad — (1)$$

This is the circumference of the circle divided by the vacuum speed of light c. If the circular platform is rotated at an angular frequency Ω then:

$$t = \frac{2\pi}{\omega \pm \Omega} = \frac{2\pi r}{c \pm v} \quad — (2)$$

where

$$\omega = \frac{c}{r}, \quad \Omega = \frac{v}{r} \quad (3)$$

The result (2) is obtained by rotating the Minkowski metric of special relativity such that:

$$d\phi \rightarrow d\phi + \Omega\, dt, \quad — (4)$$

$$ds^2 = dr^2 = dZ^2 = 0. \quad — (5)$$

The difference in time for clockwise and anti-clockwise rotation is:

$$\Delta t = 2\pi r \left(\frac{1}{c-v} - \frac{1}{c+v} \right) \quad — (6)$$

so it seems that the speed of light in one direction is $c - v$ and $c + v$ in the other direction. If

) Special relativity is defined as one frame moving linearly at v with respect to another, the speed of light cannot be exceeded. So the Sagnac effect (6) seems to contradict this principle of special relativity. However, the latter applies only to linear motion, whereas the Sagnac effect involves <u>rotation</u>. From eq. (6)

$$\Delta t = \frac{4\pi r v}{(c-v)(c+v)} \quad - (7)$$

$$= \frac{4\pi r^2 \Omega}{(c-v)(c+v)} = \frac{4\Omega A_r}{(c-v)(c+v)} \quad - (8)$$

where

$$A_r = \pi r^2 \quad - (9)$$

If

$$v << c \quad - (10)$$

$$\Delta t \doteq \frac{4\Omega A_r}{c^2} \quad - (11)$$

The precise result is

$$\boxed{\Delta t = 2\pi r \left(\frac{1}{c-v} - \frac{1}{c+v} \right) = \frac{4\Omega A_r}{(c-v)(c+v)}} \quad - (12)$$

and has been verified to great accuracy experimentally.

The phase change due to eq. (12) is:

$$\Delta \phi = \omega \Delta t \quad - (13)$$

So:

3)

$$\Delta\phi = \phi_1 - \phi_2 \quad — (14)$$

where

$$\phi_1 = 2\pi r \left(\frac{\omega}{c-v}\right) \quad — (15)$$

$$\phi_2 = 2\pi r \left(\frac{\omega}{c+v}\right) \quad — (16)$$

This is a classical, relativistic, theory. However, it is a theory of light, because of the use of the null geodesic:

$$ds^2 = 0. \quad — (17)$$

More precisely, it is a theory of the photon. In a vacuum the latter travels on a null geodesic (17). This is loosely described as a "straight line". In such a case:

$$\left.\begin{aligned} E = \hbar\omega, \quad p = \hbar\kappa \\ \kappa = \frac{\omega}{c} \end{aligned}\right\} \quad — (18)$$

However in the Sagnac effect the photon is constrained to travel in a circle. In this case

$$\boxed{\kappa_1 = \frac{\omega}{c-v}, \quad \kappa_2 = \frac{\omega}{c+v}} \quad — (19)$$

and

$$\boxed{\Delta\phi = 2\pi r(\kappa_1 - \kappa_2)} \quad — (20)$$

From eq. (12):

$$\boxed{\Delta\phi = 4\left(\frac{\Omega}{\omega}\right)\kappa_1\kappa_2 A r} \quad — (21)$$

(+) Therefore:

$$\boxed{\Delta\phi = 2\pi r(\kappa_1-\kappa_2) = 4\left(\frac{\Omega}{\omega}\right)\kappa_1\kappa_2 A r} \quad -(22)$$

which is an example of

$$\Delta\phi = \oint \underline{\kappa}\cdot d\underline{r} = \int \kappa^2 A r \quad -(23)$$

where

$$\boxed{\kappa^2 = 4\left(\frac{\Omega}{\omega}\right)\kappa_1\kappa_2} \quad -(24)$$

Eq. (23) is a non-Abelian Stokes theorem, the line integral for κ is defined as:

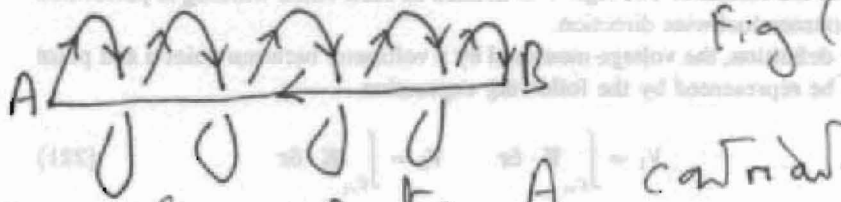


Fig (1)

and only the line from B to A contributes. The area Ar in eq (23) can be enclosed by any shape, as is verified experimentally to great precision.

Therefore the rotation of a Minkowski metric at Ω produces the following effect:

$$\omega \rightarrow \omega \pm \Omega \quad -(25)$$

$$\kappa = \frac{\omega}{c} \rightarrow \frac{\omega}{c \pm v} \quad -(26)$$

We define:

$$\boxed{\begin{aligned} \omega_r &= \omega \pm \Omega \qquad &(27)\\ \kappa_r &= \frac{\omega}{c \pm v} \qquad &(28)\end{aligned}}$$

These are related by:

$$\kappa_r = (\mu\epsilon)^{1/2}\,\omega_r = \frac{\omega_r}{v_r} \qquad (29)$$

(Jackson, third ed., eq. (7.4)). Therefore rotation of the Minkowski metric produces the refractive index:

$$n = \frac{c}{v_r} = c\,\frac{\kappa_r}{\omega_r} \qquad (30)$$

Either

$$n_+ = \left(\frac{\omega}{\omega+\Omega}\right)\left(\frac{c}{c+v}\right) \qquad (31)$$

or

$$n_- = \left(\frac{\omega}{\omega-\Omega}\right)\left(\frac{c}{c-v}\right) \qquad (32)$$

This is a classical, relativistic result. Rotation of a Minkowski spacetime produces a birefringence:

$$\Delta n = n_- - n_+ \qquad (33)$$

$$= \left(\frac{c}{c-v}\right)^2 - \left(\frac{c}{c+v}\right)^2$$

$$= \frac{4vc^3}{(c^2-v^2)^2} = 4vc\left(\frac{c}{c^2-v^2}\right)^2$$