1)

149(1): Equivalence of ECE Metric and Gravitational Metric for the Processing Ellipse.

In the XY plane defined by:

$$dZ^2 = 0 \quad -(1)$$

The ECE metric is:

$$d\tau^2 = dt^2 - \frac{1}{c^2}\left(dr^2 + r^2 d\phi^2\right) \quad -(2)$$

and the gravitational metric is:

$$d\tau^2 = x\,dt^2 - \frac{1}{c^2}\left(\frac{dr^2}{x} + r^2 d\phi^2\right) \quad -(3)$$

where

$$x = 1 - \frac{r_0}{r} = 1 - \frac{2mG}{c^2 r} \quad -(4)$$

The two metrics are the same if:

$$dt^2 - \frac{1}{c^2}dr^2 = x\,dt^2 - \frac{1}{xc^2}dr^2 \quad -(5)$$

i.e. $$(1-x)\,dt^2 = \frac{1}{c^2}dr^2\left(1 - \frac{1}{x}\right) \quad -(6)$$

$$\frac{1}{c^2}\left(\frac{dr}{dt}\right)^2 = -\left(1 - \frac{2mG}{c^2 r}\right) \quad -(7)$$

or $$\frac{1}{c^2}\left(\frac{dr}{dt}\right)^2 + 1 = \frac{2mG}{c^2 r}, \quad -(8)$$

where $$\left(\frac{dr}{dt}\right)^2 = \left(\frac{a^2}{1+a^2}\right)v^2, \quad -(9)$$

$$a = \frac{y\epsilon r}{\alpha}\sin(y\phi) \quad -(10)$$

$$v = |\underline{v}| = \left|\frac{d\underline{r}}{dt}\right| \quad -(11)$$

$$d\underline{r}\cdot d\underline{r} = dr^2 + r^2 d\phi^2 \quad -(12)$$

Therefore:

$$\frac{1}{c^2}\left(\frac{a^2}{1+a^2}\right)v^2 + 1 = \frac{2MG}{c^2 r} \quad \text{— (13)}$$

i.e. $$\boxed{\frac{1}{2}\left(v^2\left(\frac{a^2}{1+a^2}\right) + c^2\right) = \frac{MG}{r} = -V} \quad \text{— (14)}$$

Multiply both sides by the mass of the test particle m, to find:

$$\frac{1}{2}\left(\frac{a^2}{1+a^2}\right)mv^2 + \frac{1}{2}mc^2 = \frac{mMG}{r} = -U \quad \text{— (15)}$$

where $$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\phi}{dt}\right)^2 \quad \text{— (16)}$$

Here U is the potential energy:

$$U = mV \quad \text{— (17)}$$

where V is the gravitational potential.

Denote the kinetic energy by:

$$T = \frac{1}{2}\left(\frac{a^2}{1+a^2}\right)mv^2 + \frac{1}{2}mc^2 \quad \text{— (18)}$$

to find that $$\boxed{H = T + U = 0} \quad \text{— (19)}$$

where H is the hamiltonian.

This means that the total energy of the orbit is zero.

By transforming the gravitational metric (3) into the ECE Minkowski metric (2) the gravitational potential energy has been transformed into pure kinetic energy, defined by eq. (18). Thus eq. (2) is a pure kinetic description of the orbit, which is observed experimentally to be the precessing ellipse:

$$r = \frac{\alpha}{1 + \epsilon \cos(y\phi)} \quad -(20)$$

In the limit:

$$y \rightarrow 1 \quad -(21)$$

the ellipse stops precessing, and the perihelion stops advancing. In the limit

$$\epsilon \rightarrow 0 \quad -(22)$$

the orbit approaches the circular orbit, i.e.

$$a \rightarrow 0 \quad -(23)$$

and

$$U + \frac{1}{2} mc^2 \rightarrow 0. \quad -(24)$$

This means that in the limit of a circular orbit:

$$\frac{MG}{r} \rightarrow \frac{1}{2} c^2, \quad -(25)$$

$$\boxed{r \rightarrow \frac{2MG}{c^2} = r_0} \quad -(26)$$

In the standard literature r_0 is incorrectly referred to as the Schwarzschild radius. However, contemporary scholarship shows conclusively that

4) Schwarzschild did not infer the metric that is incorrectly attributed to him. In ECE physics this is known as the gravitational metric, eq. (3). Both eqs. (2) and (3) are solutions of the ECE Orbital Theorem of UFT 111.

Results

For the precessing elliptical orbit, (the relativistic Keplerian orbit):

$$H = T + U = 0 \quad -(27)$$

where

$$T = \frac{m}{2}\left(\frac{a^2}{1+a^2}\right)\left(\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\phi}{dt}\right)^2\right) + \frac{1}{2}mc^2, \quad -(28)$$

$$U = -\frac{mMG}{r}. \quad (29)$$

The total linear velocity in a plane (XY) is:

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\phi}{dt}\right)^2 \quad -(30)$$

and

$$a = \frac{y\epsilon r}{\alpha}\sin(y\phi) \quad -(31)$$

Numerical Calculation

Use: Mass of earth $m = 5.98 \times 10^{24}$ kgm

Mass of sun $M = 1.99 \times 10^{30}$ kgm

$G = 6.67 \times 10^{-11}\ m^3 kg^{-1} s^{-2}$

5) Earth to sun distance $r = 1.50 \times 10^{11}$ m

Speed of light $c \sim 3 \times 10^{8}\ m\,s^{-1}$

Orbital velocity v of earth around sun $= 2.98 \times 10^{4}\ m s^{-1}$

Use these data in the equation

$$\frac{1}{2}\left(\frac{a^2}{1+a^2}\right) m v^2 + \frac{1}{2} m c^2 = \frac{m M G}{r} \quad -(32)$$

Assume: $\frac{a^2}{1+a^2} \sim 1 \quad -(33)$

in a rough approximation. Thus, if r is the earth to sun distance to be calculated

$$\frac{mMG}{r} = 7.937 \times 10^{44} / r \text{ joules}$$

$$= 2.691 \times 10^{41} \text{ joules}$$

and $\frac{1}{2} m c^2 =$

$$2.655 \times 10^{33} \text{ joules.}$$

$$\frac{1}{2} m v^2 =$$

So:

$$2.655 \times 10^{33}\left(\frac{a^2}{1+a^2}\right) + 2.691 \times 10^{41} = \frac{7.937 \times 10^{44}}{r} \quad -(34)$$

So using eq. (33)

$$r \sim \frac{2 m G}{c^2} = 2.95 \times 10^{3}\ m \quad -(35)$$

to an excellent approximation.

Eq. (32) is:

)

$$\frac{mMG}{r} - \frac{1}{2}mc^2 = \frac{1}{2}\left(\frac{a^2}{1+a^2}\right)mv^2 \quad -(36)$$

where

$$f(r,\phi) = \frac{a^2}{1+a^2} = \frac{\left(\frac{y\epsilon r}{\alpha}\sin(y\phi)\right)^2}{1+\left(\frac{y\epsilon r}{\alpha}\sin(y\phi)\right)^2} \quad -(37)$$

It is seen that for $r \sim 2.95 \times 10^3 \ m$ $-(38)$

then $f(r,\phi) \sim 1$ $-(39)$

self consistently.

<u>Graphical Work</u>

It may be interesting to graph eq. (37) for various ϵ, α and y of the precessing ellipse.

7) Overall Conclusion

The experimentally observed orbit is eq. (20), and is described straightforwardly by the metric (2) of Minkowski spacetime using $dr/d\phi$ from eq. (20). There are problems with the received opinion (3) because when

$$r_0 = r \quad (40)$$

there is a singularity. The latter has been severely misinterpreted as "Big Bang". Data show that the latter does not exist.

This note shows that the much vaunted but deeply flawed eq. (3) is equivalent to the observational eq. (2) only if

$$r \leq r_0. \quad (41)$$

This analysis reveals another severe limitation of eq. (3). Finally, it is well known now that the Einstein field equation uses an incorrect connection. Schwarzschild did not derive eq. (3), which is merely a meaningless solution of an incorrect equation. It produces the observed function (20) only in the limit (41). So eq. (3) is not generally applicable. Eq. (2) on the other hand is generally applicable to all observed orbits.

149(2): Equations of Motion from the Minkowski Metric, ECE Equation of Orbits.

The Minkowski metric in cylindrical polar coordinates is:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - d\underline{r} \cdot d\underline{r} \quad -(1)$$

where

$$\left.\begin{aligned} d\underline{r} \cdot d\underline{r} &= dr^2 + r^2 d\phi^2 + dZ^2 \\ &= dX^2 + dY^2 + dZ^2 \end{aligned}\right\} -(2)$$

The total linear velocity is defined as:

$$\underline{v} = \frac{d\underline{r}}{dt} \quad -(3)$$

Therefore:

$$c^2 d\tau^2 = c^2 dt^2 - v^2 dt^2 \quad -(4)$$

$$= (c^2 - v^2) dt^2 \quad -(5)$$

so

$$d\tau^2 = \left(1 - \frac{v^2}{c^2}\right) dt^2 \quad -(6)$$

The infinitesimal of proper time is:

$$d\tau = \left(1 - \frac{v^2}{c^2}\right)^{1/2} dt \quad -(7)$$

and

$$dt = \gamma\, d\tau \quad -(8)$$

where

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(9)$$

If we restrict attention to the XY plane:

$$dZ = 0. \quad -(10)$$

The rest energy is defined as:

$$E_0 = mc^2 = m\left(\frac{dS}{d\tau}\right)^2 = m g_{\mu\nu} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \quad -(11)$$

where S is the action.

Thus:

2)

$$E_0 = mc^2\left(\frac{dt}{d\tau}\right)^2 - m\left(\frac{dr}{d\tau}\right)^2 - mr^2\left(\frac{d\phi}{d\tau}\right)^2 \quad -(12)$$

The Lagrange equation is:

$$\frac{d}{d\tau}\left(\frac{\partial E_0}{\partial \dot{x}^\mu}\right) = \frac{\partial E_0}{\partial x^\mu} = 0 \quad -(13)$$

so

$$\frac{d}{d\tau}\left(mr^2\frac{d\phi}{d\tau}\right) = 0 \quad -(14)$$

$$\frac{d}{d\tau}\left(m\frac{dr}{d\tau}\right) = 0 \quad -(15)$$

$$\frac{d}{d\tau}\left(mc^2\frac{dt}{d\tau}\right) = 0. \quad -(16)$$

The constants of motion from eqns (14) to (16) are:

$$E = mc^2\frac{dt}{d\tau} = \gamma mc^2 \quad -(17)$$

$$L = mr^2\frac{d\phi}{d\tau} = \gamma mr^2\frac{d\phi}{dt} \quad -(18)$$

$$p = m\frac{dr}{d\tau} = \gamma m\frac{dr}{dt} \quad -(19)$$

So:

$$\frac{p^2}{m} = \frac{E^2}{mc^2} - mc^2 - \frac{L^2}{mr^2} \quad -(20)$$

$$= (\gamma^2 - 1)mc^2 - \frac{L^2}{mr^2}$$

$$\boxed{(\gamma+1)T = (\gamma+1)(\gamma-1)mc^2 = \frac{p^2}{m} + \frac{L^2}{mr^2}} \quad -(21)$$

where

$$E = T + E_0 \quad -(22)$$

Here E is the total energy, T is the relativistic energy, L is the relativistic angular momentum, and p is the relativistic momentum. The result is purely kinetic, and there is no concept of potential energy or attractive

5) force. Eq. (21) is equivalent to the description:

$$\underline{dr} \cdot \underline{dr} = c^2 (dt^2 - d\tau^2) \quad - (23)$$

$$= v^2 dt^2$$

i.e.

$$\boxed{mv^2 = m\left(\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\phi}{dt}\right)^2\right)} \quad - (24)$$

where

$$\frac{d\phi}{dt} = \frac{d\phi}{dr}\frac{dr}{dt} \quad - (25)$$

so

$$\boxed{mv^2 = m\left(\frac{dr}{dt}\right)^2\left(1 + r^2\left(\frac{d\phi}{dr}\right)^2\right)} \quad - (26)$$

For a precessing elliptical orbit, it is found by experimental observation that:

$$\frac{d\phi}{dr} = \frac{\alpha}{y \epsilon r \sin(y\phi)} \quad - (27)$$

so

$$\boxed{mv^2 = m\left(\frac{dr}{dt}\right)\left(1 + \left(\frac{\alpha}{y\epsilon r \sin(y\phi)}\right)^2\right)} \quad - (28)$$

In the non-relativistic limit:

$$(\gamma^2 - 1)mc^2 \rightarrow \left(\left(1 - \frac{v^2}{c^2}\right)^{-1} - 1\right)mc^2$$

$$\sim mv^2 \quad - (29)$$

So the non relativistic limit of eq. (21) is

4)

$$mv^2 = \frac{p^2}{m} + \frac{L^2}{mr^2} \quad -(30)$$

So

$$\boxed{T = \frac{1}{2}mv^2 = \frac{p^2}{2m} + \frac{L^2}{2mr^2}} \quad -(31)$$

In this limit $\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \rightarrow 1$ -(32)

and

$$L \rightarrow mr^2 \frac{d\phi}{dt}, \quad -(33)$$

$$p \rightarrow m \frac{dr}{dt} \quad -(34)$$

In the conventional non-relativistic treatment of orbits (e.g. Marion and Thornton) the second term on the right hand side of eq. (31) is called "the centrifugal potential energy". However, it is part of the kinetic energy. The conventional Newtonian treatment is therefore flawed as is well known. The ECE treatment fully describes the orbit without using the concepts of "centrifugal force" or the derived concept of potential energy":

$$F_c := -\frac{\partial U_c}{\partial r} = \frac{L^2}{mr^3} = mr\left(\frac{d\phi}{dt}\right)^2$$

-(35)

5) Eq. (20) is:

$$m\left(\frac{dr}{d\tau}\right)^2 = \frac{p^2}{m} = \frac{E^2}{mc^2} - mc^2 - \frac{L^2}{mr^2} \quad -(36)$$

and can be compared with the well known result from the gravitational metric:

$$m\left(\frac{dr}{d\tau}\right)^2 = \frac{E^2}{mc^2} - mc^2 - \frac{L^2}{mr^2} + \frac{2mMG}{r} + \frac{ML^2G}{c^2mr^3} \quad -(37)$$

Both eqs. (36) and (37) describe the same precessing ellipse defined by eq. (27), but eq. (36) is preferred by Ockham's Razor, it is simpler and much clearer. So the last two terms of eq. (37) can be removed by using the limit:

$$r \rightarrow \infty, \quad -(38)$$

in which case eq. (37) reduces to the purely kinetic eq. (36). Conventionally the last two terms of eq. (37) are the relativistically corrected potential "attraction". The concept of attraction in the received opinion is balanced by the artificial energy of attraction" identification (35) as a "potential and incorrect " whereas eq. (31) shows clearly all is kinetic energy.
energy" is kinetic energy.

For a circular orbit:

$$dr = 0 \quad -(39)$$

So $$(\gamma^2 - 1) mc^2 = \frac{L^2}{mr^2} \quad -(40)$$

In the limit:

$$v << c \quad -(41)$$

eq. (40) becomes:

$$\boxed{\frac{1}{2} mv^2 = \frac{L^2}{2mr^2}} \quad -(42)$$

i.e <u>the two components of the kinetic energy are the same in a circular orbit.</u>

Note carefully that in this description, the orbit (of any observable type) is always described by the simplest possible solution of the Orbital Theorem, the Minkowski metric. <u>There</u> is no <u>concept of attractive and repulsive force</u>, and no <u>concept of potential energy</u>. The orbit is the <u>geodesic</u> or metric. Light orbits in a null geodesic:

$$ds^2 = 0 \quad -(43)$$

i.e

$$v = c, \quad -(44)$$

so <u>the orbit of light</u> from eq. (26) is:

$$\boxed{m \left(\frac{dr}{dt}\right)^2 \left(1 + r^2 \left(\frac{d\phi}{dr}\right)^2\right) = mc^2 = E_0} \quad -(45)$$

7) In the received opinion, originating with Kepler's idea of "force", eq. (43) is:

$$\underline{F} = \underline{0} = m\underline{g} + mr\left(\frac{d\phi}{dt}\right)^2 \underline{k} \quad -(46)$$

so the attractive, Newtonian, inward $m\underline{g}$ is balanced exactly by the outward, non-Newtonian, repulsive $F_c \underline{k}$. Newton's first law is that m travels in "a straight line" unless acted upon by "force". In the ECE treatment m travels in an orbit which has kinetic energy only. The "straight line" of Newton is replaced by the orbit, which is the metric. The Newtonian metric is:

$$\left.\begin{aligned} ds^2 &= dX^2 + dY^2 + dZ^2 \\ &= dr^2 + r^2 d\phi^2 + dZ^2 \end{aligned}\right\} -(47)$$

and as is well known, exists only in three dimensional space, not spacetime.

If the quantity $d\phi / dt$ vanishes, then in eq. (38), in the non-relativistic limit:

$$mv^2 = m\left(\frac{dr}{dt}\right)^2 \quad -(48)$$

and the particle m has a velocity:

$$\underline{v} = \frac{d\underline{r}}{dt} \quad -(49)$$

and an acceleration from the Lagrange equation.

8) In the general orbit, the total kinetic energy in the limit $v <<< c$ is:

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m \left(\frac{dr}{dt} \right)^2 + \frac{L^2}{2 m r^2} \quad -(50)$$

so

$$\frac{dT}{dt} = m \frac{d^2 r}{dt^2} + \frac{d}{dt} \left(\frac{L^2}{2 m r^2} \right)$$

$$\rightarrow m \frac{d^2 r}{dt^2} \quad -(51)$$

as

$$L \rightarrow 0, \quad -(52)$$

so the particle m falls towards M at with an acceleration:

$$\boxed{\frac{d^2 r}{dt^2} = \frac{1}{m} \frac{dT}{dt}} \quad -(53)$$

In the received opinion (second Newton law) the change of kinetic energy with time is described as "force", which is defined in the Newton second law as mass multiplied by acceleration. In the ECE description, which is simpler and clearer, it is defined purely by the metric itself.

49(3): The Conventional Description of Orbits from the Gravitational Metric.

The gravitational metric is:

$$ds^2 = c^2 d\tau^2 = \left(1 - \frac{r_s}{r}\right) c^2 dt^2 - \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 - r^2 d\phi^2 - dZ^2 \quad -(1)$$

in cylindrical polar coordinates, where:

$$r_s = \frac{2GM}{c^2 r} \quad -(2)$$

It is claimed incorrectly that it was derived by Schwarzschild, from the Einstein field equation. The latter is now known to be incorrect. However, eq (1) is formally a possible solution of the ECE Orbital Theorem, one of an infinite set of possible solutions of the Orbital Theorem. So formally the following method derives from the ECE orbital Theorem. In the received opinion it is described as "the relativistic Kepler problem".

Define:

$$T := \frac{1}{2} m c^2 = \frac{1}{2} m \left(\frac{dS}{d\tau}\right)^2 = \frac{m}{2} g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \quad -(3)$$

where S is the action and:

$$c^2 d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu \quad -(4)$$

So:

$$g_{00} = 1 - \frac{r_s}{r}, \; g_{11} = -\left(1 - \frac{r_s}{r}\right)^{-1}, \; g_{22} = -1, \; g_{33} = -1, \quad -(5)$$

$$dx^0 = c\,dt, \; dx^1 = dr, \; dx^2 = r\,d\phi, \; dx^3 = dZ. \quad -(6)$$

So:

$$\frac{2T}{m} = c^2 = g_{00}\left(\frac{\partial x^0}{\partial \tau}\right)^2 + g_{11}\left(\frac{\partial x^1}{\partial \tau}\right)^2 + g_{22}\left(\frac{\partial x^2}{\partial \tau}\right)^2 + g_{33}\left(\frac{\partial x^3}{\partial \tau}\right)^2 \quad -(7)$$

$$= \left(1 - \frac{r_s}{r}\right) c^2 \left(\frac{dt}{d\tau}\right)^2 - \left(1 - \frac{r_s}{r}\right)^{-1} \left(\frac{\partial r}{\partial \tau}\right)^2 - r^2 \left(\frac{\partial \phi}{\partial \tau}\right)^2 - \left(\frac{\partial Z}{\partial t}\right)^2$$

The Lagrange equation is:

$$\frac{d}{d\tau}\left(\frac{\partial T}{\partial \dot{x}^\mu}\right) = \frac{\partial T}{\partial x^\mu} \quad -(8)$$

So

$$\frac{d}{d\tau}\left(mr^2\frac{d\phi}{d\tau}\right) = 0 \quad -(9)$$

$$\frac{d}{d\tau}\left(m\left(1-\frac{r_s}{r}\right)\frac{dt}{d\tau}\right) = 0 \quad -(10)$$

$$\frac{d}{d\tau}\left(m\left(1-\frac{r_s}{r}\right)^{-1}\frac{dr}{d\tau}\right) = 0 \quad -(11)$$

Eqns (9) to (11) give the constants of motion:

$$E = mc^2\left(1-\frac{r_s}{r}\right)\frac{dt}{d\tau} \quad -(12)$$

$$L = mr^2\frac{d\phi}{d\tau} \quad -(13)$$

$$p = m\left(1-\frac{r_s}{r}\right)^{-1}\frac{dr}{d\tau} \quad -(14)$$

These are the total energy E, the momentum p and the angular momentum L.

In the plane:

$$dZ = 0 \quad -(15)$$

we have:

$$\left(1-\frac{r_s}{r}\right)T = mc^2\left(1-\frac{r_s}{r}\right)^2\left(\frac{dt}{d\tau}\right)^2 - \frac{m}{2}\left(\frac{dr}{d\tau}\right)^2 - m\left(1-\frac{r_s}{r}\right)r^2\left(\frac{d\phi}{d\tau}\right)^2$$

$$= \frac{E^2}{mc^2} - m\left(\frac{dr}{d\tau}\right)^2 - \left(1-\frac{r_s}{r}\right)\frac{L^2}{mr^2} \quad -(16)$$

Therefore:

$$m\left(\frac{dr}{d\tau}\right)^2 = \frac{E^2}{mc^2} - \left(1-\frac{r_s}{r}\right)T - \left(1-\frac{r_s}{r}\right)\frac{L^2}{mr^2}$$

3) i.e.

$$\frac{1}{2}m\left(\frac{dr}{d\tau}\right)^2 = \frac{1}{2}\frac{E^2}{mc^2} - \frac{1}{2}\left(1-\frac{r_s}{r}\right)\left(mc^2 - \frac{L^2}{mr^2}\right)$$

$$= \left(\frac{E^2}{2mc^2} - \frac{1}{2}mc^2\right) + \left(\frac{mMG}{r} - \frac{L^2}{2mr^2} + \frac{GML^2}{mc^2r^3}\right) \quad -(17)$$

This is eq. (30) of note 149(2). It gives the same processing elliptical orbit as the ECE Minkowski theory:

$$m\left(\frac{dr}{d\tau}\right)^2 = (\gamma-1)mc^2 - \frac{L^2}{mr^2} \quad -(18)$$

or

$$\frac{1}{2}m\left(\frac{dr}{d\tau}\right)^2 = \frac{1}{2}(\gamma-1)mc^2 - \frac{L^2}{2mr^2}. \quad -(19)$$

This is because the Minkowski and gravitational metrics are both solutions of the same equation, the ECE Orbital Theorem of UFT ~~the~~ 111 for spherically symmetric spacetime. So the two solutions (17) and (19) contain the same information - the processing elliptical orbit. However, eq. (19) can be generalized to any orbit, using observation, ~~and~~ but eq. (17) cannot, it always gives a processing elliptical orbit. Eq. (17) is one that uses the Newton constant G, which does not appear in eq. (19). Eq. (17) reduces to eq. (19) when:

$$r \rightarrow \infty \quad -(20)$$

$$r_s \rightarrow 0 \quad -(21)$$

so

$$\frac{r_s}{r} \rightarrow 0$$

4) and

$$E \rightarrow \gamma mc^2 = mc^2 \frac{dt}{d\tau} \quad -(22)$$

$$p \rightarrow \gamma m \frac{dr}{dt} = m \frac{dr}{d\tau} \quad -(23)$$

As shown in note 149(2), the concepts of force and potential energy are not needed. Neither the first nor the second law of Newton are used.

The Einsteinian description is not used because the connection must be antisymmetric, not symmetric as used by Einstein. Dark matter is not used. The concepts in these notes are radically new, but the calculations are relatively straightforward.

1) 149(4): The Equation of Orbits

Start with the fully relativistic definition:

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\phi}{dt}\right)^2 \quad -(1)$$

and use the fact that

$$p^2 = \gamma^2 m^2 \left(\frac{dr}{dt}\right)^2 \quad -(2)$$

is a constant of motion:

$$\gamma^2\left(\frac{dr}{dt}\right)^2 = \left(\frac{p}{m}\right)^2 := A \quad -(3)$$

So

$$\gamma^2 v^2 = A + \gamma^2 r^2 \left(\frac{d\phi}{dt}\right)^2 \quad -(4)$$

$$= A + \gamma^2 r^2 \left(\frac{d\phi}{dr}\right)^2 \left(\frac{dr}{dt}\right)^2$$

$$\boxed{\gamma^2 v^2 = A\left(1 + r^2\left(\frac{d\phi}{dr}\right)^2\right)} \quad -(5)$$

The equation of orbits is therefore:

$$1 + r^2\left(\frac{d\phi}{dr}\right)^2 = \frac{\gamma^2 v^2}{A} \quad -(6)$$

i.e.

$$\boxed{\frac{d\phi}{dr} = \frac{1}{r}\left(\frac{\gamma^2 v^2}{A} - 1\right)^{1/2}} \quad -(7)$$

Precessing Ellipse

By observation:

2) $$\frac{d\phi}{dr} = \frac{d}{y\epsilon \sin(y\phi)} \cdot \frac{1}{r} \quad -(8)$$

$$:= \frac{b}{r}$$

so $$b = \left(\frac{\gamma^2 v^2}{A} - 1\right)^{1/2} \quad -(9)$$

<u>Whirlpool Galaxy</u>

$$\frac{dr}{d\phi} = b_1 r \quad -(10)$$

where $$r(t) = r \exp\left(b_1 \phi(t)\right) \quad -(11)$$

so $$\frac{1}{b_1} = \left(\frac{\gamma^2 v^2}{A} - 1\right)^{1/2} \quad -(12).$$

i.e $$1 + \frac{1}{b_1^2} = \frac{\gamma^2}{A} v^2 \quad -(13)$$

$$v = \frac{1}{\gamma}\left(A\left(1 + \frac{1}{b_1^2}\right)\right)^{1/2} \quad -(14)$$

In the limit $\gamma \to 1$, $\underline{v = \text{constant}}$ $-(15)$

Similarly, for the precessing ellipse:

$$v = \frac{1}{\gamma}\left(A\left(1 + b^2\right)\right)^{1/2} \quad -(16)$$

$$\underline{v = \frac{A^{1/2}}{\gamma}\left(1 + \left(\frac{d}{y\epsilon \sin(y\phi)}\right)^2\right)^{1/2}} \quad -(16)$$

3) For the ellipse:

$$\left.\begin{array}{l}\gamma \to 1 \\ y \to 1\end{array}\right\} \quad -(17)$$

so

$$v^2 = A\left(1 + r^2\left(\frac{d\phi}{dr}\right)^2\right) \quad -(18)$$

$$= \left(\frac{2\pi}{\tau}\right)^2 a^3\left(\frac{2}{r} - \frac{1}{a}\right) \quad -(19)$$

from Kepler's equation of the ellipse:

$$\frac{(x + a\epsilon)^2}{a^2} + \frac{y^2}{b_2^2} = 1 \quad -(20)$$

and where $2\pi t/\tau$ is the mean anomaly.

1) 149(5), Another Form of the Equation of Orbits.

In note 149(2) it was shown that the purely kinetic description of an orbit is, from the Minkowski metric:

$$(\gamma^2 - 1) m c^2 = \frac{p^2}{m} + \frac{L^2}{m r^2}, \quad -(1)$$

where:

$$p = m \frac{dr}{d\tau} = \gamma m \frac{dr}{dt}, \quad -(2)$$

$$L = m r^2 \frac{d\phi}{d\tau} = \gamma m r^2 \frac{d\phi}{dt} \quad -(3)$$

In the non-relativistic limit, eq. (1) becomes:

$$T = \frac{1}{2} m v^2 = \frac{m}{2}\left(\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\phi}{dt}\right)^2\right) - (4)$$

$$\boxed{T = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\phi}^2\right)} \quad -(5)$$

In the purely kinetic theory of orbits there is no potential energy, all is kinetic energy, therefore the hamiltonian and lagrangian are:

$$H = \mathcal{L} = T \quad -(6)$$

How is it possible to describe an orbit without the traditional:

$$U = -\frac{mMG}{r} \quad -(7)$$

$$F = -\frac{\partial U}{\partial r} = -\frac{mMG}{r^2} \quad -(8)$$

of the inverse square law?

To answer this question set up the lagrangian:

$$\mathcal{L} = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\phi}^2\right) \quad -(9)$$

instead of the traditional:

$$\mathcal{L} = T - U \quad -(10)$$

The two ~~the~~ Lagrange equations are:

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \quad - (11)$$

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} \quad - (12)$$

with the Lagrangian (9). Eq. (11) gives:

$$L = m r^2 \dot{\phi} = \text{constant} \quad - (13)$$

and eq. (12) gives:

$$m \ddot{r} = m r \dot{\phi}^2 \quad - (14)$$

$$\boxed{m \ddot{r} = \frac{L^2}{m r^3}} \quad - (15)$$

In the traditional point of view "the centrifugal force of repulsion" is:

$$F_c = \frac{L^2}{m r^3} \quad - (16)$$

and "the Newtonian force of attraction" is:

$$F_A = m \ddot{r} \quad - (17)$$

so

$$F_A = F_C \quad - (18)$$

if there is no potential energy. It is important to note that eq. (15) is a complete description of the stable orbit in the plane XY. This can be an orbit of any kind.

3) Eq. (5) is:

$$T = \frac{p^2}{2m} + \frac{L^2}{2mr^2} \quad -(19)$$

where in the non-relativistic limit:

$$v << c, \quad -(20)$$

we have

$$p = mv \quad -(21)$$

$$L = mr^2 \dot{\phi}. \quad -(22)$$

Therefore

$$\frac{L^2}{2mr^2} = T - \frac{p^2}{2m} \quad -(23)$$

so in eq. (15):

$$\boxed{m\ddot{r} = \frac{L^2}{mr^3} = \frac{2}{r}\left(T - \frac{p^2}{2m}\right) = mr\dot{\phi}^2} \quad -(24)$$

In eq. (24):

$$\dot{\phi}^2 = \left(\frac{d\phi}{dt}\right)^2 = \left(\frac{d\phi}{dr}\right)^2\left(\frac{dr}{dt}\right)^2 \quad -(25)$$

i.e

$$\left(\frac{d\phi}{dr}\right)^2 = \dot{\phi}^2 / \dot{r}^2 \quad -(26)$$

$$= \frac{2}{m(r\dot{r})^2}\left(T - \frac{p^2}{2m}\right)$$

$$\frac{d\phi}{dr} = \frac{1}{r\dot{r}}\left(\frac{2}{m}\left(T - \frac{p^2}{2m}\right)\right)^{1/2} \quad -(27)$$

or

$$\boxed{\frac{d\phi}{dr} = \frac{L}{mr^2\dot{r}}} \quad -(28)$$

4) In the non-relativistic limit the orbit is observed in astronomy to be an ellipse:

$$r = \frac{\alpha}{1+\epsilon\cos\phi} \quad -(29)$$

so

$$\frac{dr}{d\phi} = \left(\frac{\epsilon\sin\phi}{\alpha}\right) r^2 \quad -(30)$$

From eqs. (28) and (30):

$$\frac{d\phi}{dr} = \frac{L}{mr^2 r} = \left(\frac{\alpha}{\epsilon\sin\phi}\right)\cdot\frac{1}{r^2} \quad -(31)$$

so

$$L = m\left(\frac{\alpha}{\epsilon\sin\phi}\right)\frac{dr}{dt} = mr^2\frac{d\phi}{dt}. \quad -(32)$$

Now use

$$\frac{dr}{dt} = \frac{dr}{d\phi}\frac{d\phi}{dt} \quad -(33)$$

in eq. (32), which becomes, self-consistently,

$$\frac{dr}{d\phi} = \frac{\epsilon\sin\phi}{\alpha}\, r^2 \quad -(34)$$

Eq. (28) is:

$$\frac{dr}{d\phi} = \left(\frac{m}{L}\right) r^2 \left(\frac{dr}{dt}\right). \quad -(35)$$

The areal velocity of any orbit is:

$$\frac{dA}{dt} = \frac{L}{2m} = \frac{1}{2} r^2 \dot{\phi} \quad -(36)$$

so

$$\frac{dr}{d\phi} = \frac{1}{2}\frac{dr}{dt}\frac{dt}{dA}\, r^2 = \frac{1}{2}\frac{dr}{dA}\, r^2 \quad -(37)$$

and for any orbit.

5)

$$\boxed{\frac{dA}{dr} = \frac{r^2}{2}\frac{d\phi}{dr}} \quad - (38)$$

This is another form of the general equation of orbits.

For an ellipse:

$$\frac{dA}{dr} = \frac{r^2}{2}\left(\frac{\alpha}{\epsilon \sin\phi}\right)\frac{1}{r^2} \quad - (39)$$

$$= \frac{1}{2}\left(\frac{\alpha}{\epsilon \sin\phi}\right)$$

So

$$\boxed{\sin\phi = \frac{\alpha}{2\epsilon}\frac{dr}{dA}} \quad - (40)$$

Summary

1) The purely kinetic description of all orbits in the XY plane is:

$$\boxed{\begin{aligned} m\ddot{r} &= \frac{L^2}{mr^3} && - (41) \\ \frac{dr}{d\phi} &= \frac{mr^2}{L}\left(\frac{dr}{dt}\right) && - (42) \\ \frac{dA}{dr} &= \frac{r^2}{2}\frac{d\phi}{dr} && - (43) \\ H - \mathcal{L} &= T && - (44) \end{aligned}}$$

2) The conventional description is:

$$F(r) = -\frac{\partial U}{\partial r} = m\ddot{r} - \frac{L^2}{mr^3} \quad - (45)$$

$$H = T + U \quad - (46)$$

$$\mathcal{L} = T - U \quad - (47)$$

$$U = -\frac{mMG}{r} \quad - (48)$$

6) Therefore in the conventional description:

$$m\ddot{r} = \frac{L^2}{mr^3} - \frac{\partial U}{\partial r} \quad -(49)$$

$$m\ddot{r} = \frac{L^2}{mr^3} - \frac{mMG}{r^2} \quad -(50)$$

$$= 0$$

The conventional equivalence principle is, therefore:

$$\boxed{F = mg = m\ddot{r} - \frac{L^2}{mr^3} = -\frac{mMG}{r^2}} \quad -(51)$$

In the purely kinetic view:

$$m\ddot{r} = \frac{L^2}{mr^3} - \frac{mMG}{r^2} \quad -(52)$$

i.e.

$$\boxed{\frac{d^2 r}{dt^2} = -\frac{MG}{r^2}} \quad -(53)$$

Eq. (53) is the purely kinetic interpretation of the observation of the inverse square law:

$$\frac{d^2 r}{dt^2} = \frac{d}{dt}\left(\frac{1}{2}m\left(\frac{dr}{dt}\right)^2\right) = -\frac{mG}{r^2} \quad -(54)$$

i.e.

$$\boxed{\frac{dT_N}{dt} = -\frac{MG}{r^2}} \quad -(55)$$

where

$$T_N = \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 \quad -(56)$$

is the purely central, or Newtonian, kinetic energy.

Interpretation

Eq. (41) means that in any orbit in the XY plane,

$$\boxed{\frac{dT_N}{dt} = \frac{L^2}{mr^3}} = F_2 \quad -(57)$$

The change of central kinetic energy with time is counter-balanced by a kinetic $L^2/(mr^3)$, deriving from angular momentum. Eq. (57) is a property of spacetime itself, as always in relativity.

The conventional description is:

$$\boxed{\frac{dT_N}{dt} = \frac{L^2}{mr^3} - \frac{\partial U}{\partial r} = 0} \quad -(58)$$

and artificially introduces the concepts of:

$$F_A = -\frac{\partial U}{\partial r} \quad -(59)$$

i.e the concepts of force and potential energy.

The kinetic description is simpler and preferred by Ockham's Razor. Note that the introduction of U into the lagrangian makes no difference to eq. (11) or (43), i.e to:

$$dA = \frac{1}{2} r^2 d\phi \quad -(60)$$

$$\frac{dA}{dt} = \frac{L}{2m} \quad -(61)$$

which is Kepler's second law (1609). In other words the absence of potential U makes no

8)

difference to Kepler's second law, which was derived from the elliptical orbit of Mars in 1609. Kepler's second law is true for all types of orbits, and is independent of any choice of u. Therefore all orbits are given by:

$$\boxed{\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\phi}{dt}} \quad \text{—} (62)$$

and the concepts of force and potential energy are not needed.

This is another statement of the ECE Principle of Orbits.

1) 149(6): Effect of Inverse Square Term

As shown in previous notes the Minkowski metric:

$$ds^2 = c^2 dt^2 - dr^2 - r^2 d\phi^2 \quad -(1)$$

gives:

$$(\gamma^2 - 1) mc^2 = \frac{p_1^2}{2m} + \frac{L^2}{mr^2} \quad -(2)$$

where

$$T = (\gamma - 1) mc^2 \quad -(3)$$

is the relativistic kinetic energy. The relativistic momentum is

$$\underline{p} = \gamma m \frac{d\underline{r}}{dt} \quad -(4)$$

where

$$d\underline{r} \cdot d\underline{r} = dr^2 + r^2 d\phi^2 \quad -(5)$$

In eq. (2):

$$p_1 := \gamma m \frac{dr}{dt} \quad -(6)$$

$$L := \gamma m r^2 \frac{d\phi}{dt} \quad -(7)$$

and are constants of motion.

If it is assumed that r and ϕ are related by the conical section equation:

$$r = \frac{\alpha}{1 + \epsilon \cos\phi} \quad -(8)$$

then

$$\frac{dr}{d\phi} = \left(\frac{\epsilon}{\alpha} \sin\phi\right) r^2 \quad -(9)$$

Eq. (9) introduces a constraint on the free Minkowski metric. For the circle:

$$\epsilon = 0 \quad -(10)$$

so: $dr = 0$ — (11)

and the metric becomes:

$$r^2 d\phi^2 \quad -(12)$$

$$d\underline{r} \cdot d\underline{r} = r^2 d\phi^2 \quad -(13)$$

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - r^2 d\phi^2 \quad -(14)$$

i.e. $r^2 d\phi^2 = c^2 (dt^2 - d\tau^2)$

$$= v^2 dt^2$$

i.e. $$\frac{d\phi}{dt} = \frac{v}{r} = \omega \quad -(15)$$

$$v = \omega r. \quad -(16)$$

In the free metric (1) there is no relation between dr and $d\phi$. Therefore an orbit of any type constrains the free Minkowski metric.

For the ellipse (9):

$$\boxed{p_1 := \gamma m \frac{dr}{dt} = \left(\frac{\epsilon}{\alpha} \sin\phi \right) L} \quad -(17)$$

so $$\frac{\epsilon}{\alpha} \sin\phi = \frac{p_1}{L} \quad -(18)$$

is a constant of motion. The orbital equation is:

$$\boxed{\frac{d\phi}{dr} = \frac{1}{r^2} \left(\frac{L}{p_1} \right)} \quad -(19)$$

The non-relativistic limit of eq. (2) is

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m \left(\left(\frac{dr}{dt} \right)^2 + r^2 \left(\frac{d\phi}{dt} \right)^2 \right) \quad -(20)$$

This kinetic energy is obtained from the work:

$$W_{12} = \int_1^2 \underline{F} \cdot d\underline{r} = T_2 - T_1 \quad -(21)$$

in which

$$d\left(\frac{1}{2} m v^2\right) = \underline{F} \cdot d\underline{r} \quad -(22)$$

where $\underline{F}$ is the force. Usually, the potential energy is introduced as:

$$W_{12} = \int_1^2 \underline{F} \cdot d\underline{r} = T_2 - T_1 = U_1 - U_2 \quad -(23)$$

So the hamiltonian is:

$$H = T_1 + U_1 = T_2 + U_2 \quad -(24)$$

If

$$U_2 = T_1 = 0 \quad -(25)$$

then

$$U_1 = T_2 \quad -(26)$$

The hamiltonian and lagrangian are:

$$H = T + T_2 \quad -(27)$$

$$\mathcal{L} = T - T_2. \quad -(28)$$

If the inverse square attraction is introduced:

$$T = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\phi}^2\right) \quad -(29)$$

$$T_2 = U_1 = -\frac{k}{r} \quad -(30)$$

where

$$k = m M G. \quad -(31)$$

Note carefully that this is a purely kinetic theory. Eq. (23) shows clearly that the idea of "potential energy" is interchangeable with kinetic energy.

The Lagrangian is therefore:

$$\mathcal{L} = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\phi}^2 \right) - \frac{k}{r} \quad -(32)$$

The Lagrange equation is:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{r}} \quad -(33)$$

giving:

$$m \left(\ddot{r} - r \dot{\phi}^2 \right) = - \frac{k}{r^2} \quad -(34)$$

which is the same as:

$$\frac{d^2 u}{d\phi^2} + u = - \frac{m}{L^2} \frac{1}{u^2} F(u) \quad -(35)$$

where

$$F(u) = - \frac{\partial T_2}{\partial r_2} \quad -(36)$$

$$\left.\begin{aligned} L &= m r^2 \dot{\phi} \\ &= \text{constant of motion} \end{aligned}\right\} \quad -(37)$$

The solution of eq. (35), i.e. the Lagrange equation of motion (32), is the ellipse (8).

This is seen from the fact that:

$$F(u) = - \frac{k}{r^2} = - k u^2 \quad -(38)$$

so

$$\frac{d^2 u}{d\phi^2} + u = \frac{mk}{L^2}, \quad -(39)$$

$$u = \frac{1}{\alpha} \left(1 + \epsilon \cos \phi \right) \quad -(40)$$

5) So

$$\frac{d^2u}{d\phi^2} = -\frac{\epsilon}{\alpha}\cos\phi \quad -(41)$$

and

$$\frac{d^2u}{d\phi^2} + u = \frac{1}{\alpha} = \frac{mk}{L^2} \quad -(42)$$

i.e

$$\alpha = \frac{L^2}{mk} \quad -(43)$$

Discussion

The object :

$$T_2 = U_1 = -\frac{k}{r} \quad -(44)$$

constraints the free Minkowski metric to produce an elliptical or conical section function of r on ϕ, i.e it relates dr and $d\phi$ by eq. (9). In the free metric this relation does not exist. So in the free metric :

$$\frac{k}{r} \to 0, \; r \to \infty \quad -(45)$$

producing

$$\frac{d^2u}{d\phi^2} + u = 0 \quad -(46)$$

and

$$\alpha \to \infty \quad -(47)$$

in the conical section equation (8). In the free metric :

$$H = \mathcal{L} = T \quad -(48)$$

and

$$\frac{1}{2} m\ddot{r} = \frac{1}{2} m r \dot{\phi}^2 \quad -(49)$$

i.e

$$F_N = F_C \quad -(50)$$

6) where

$$F_N = m\ddot{r} \quad -(51)$$

is the central, or Newtonian, force, and F_c is the centrifugal force.

Otherwise:

$$m\ddot{r} = mr\dot{\phi}^2 - \frac{k}{r^2} \quad -(51)$$

i.e.

$$\boxed{F_N = F_c + F_2} \quad -(52)$$

where

$$F_2 = -\frac{\partial T_2}{\partial r} \quad -(53)$$

is the change of kinetic energy with distance. Therefore the balance of forces in the elliptical orbit is:

$$\boxed{F_c = F_N + \frac{\partial T_2}{\partial r} = F_N - F_2} \quad -(54)$$

where

$$\boxed{-\frac{\partial T_2}{\partial r} = F_2 = \frac{mMG}{r^2}} \quad -(55)$$

is determined by observation. There is no theoretical way of deriving eq. (55), and in a whirlpool galaxy, this equation no longer holds at all.

Note that both F_N and F_2 are inwards towards M, and F_c is outwards.

1)

149(7): Relation of Minkowski and Gravitational Metrics.

As shown in note 149(2) the Minkowski metric:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - d\underline{r} \cdot d\underline{r} \quad -(1)$$

is equivalent to the Einstein energy equation:

$$E^2 = c^2 p^2 + m^2 c^4 \quad -(2)$$

where

$$p^2 = p_1^2 + \frac{L^2}{mr^2} \quad -(3)$$

In this equation the following constants of motion are defined:

$$E = mc^2 \frac{dt}{d\tau} = \gamma mc^2 \quad -(4)$$

$$p_1 = m\frac{dr}{d\tau} = \gamma m \frac{dr}{dt} \quad -(5)$$

$$L = mr^2 \frac{d\phi}{d\tau} = \gamma mr^2 \frac{d\phi}{dt} \quad -(6)$$

and

$$d\underline{r} \cdot d\underline{r} = dr^2 + r^2 d\phi^2 + dZ^2. \quad -(7)$$

The total energy from eq. (2) is:

$$E = \gamma mc^2 \quad -(8)$$

and the relativistic kinetic energy is:

$$T = (\gamma - 1) mc^2. \quad -(9)$$

The relativistic linear momentum is:

$$p_1 = |\underline{p_1}| = \gamma m \frac{dr}{dt} \quad -(10)$$

and the the total relativistic linear momentum is:

$$p = \gamma m v \quad -(11)$$

$$= \gamma m \left(p_1^2 + \frac{L^2}{mr^2} \right)^{1/2} \quad -(12)$$

2) In the non-relativistic limit:

$$v << c \quad - (13)$$

eq. (9) becomes:

$$T = \frac{1}{2} m v^2 \quad - (14)$$

The introduction of an inverse square law of attraction is equivalent to defining the Hamiltonian:

$$H = (\gamma - 1) m c^2 + U \quad - (15)$$

where

$$U = - \frac{m M G}{r} \quad - (16)$$

A Hamiltonian of the type (15) was used by Sommerfeld in a slightly different context in his semi-classical description of the hydrogen atom. It was shown by Sommerfeld that a Hamiltonian of type (15) produces a precessing ellipse. Therefore there is no need for the Einstein field equation or the wrongly named Schwarzschild metric. Eq. (15) is:

$$H = T + U \quad - (17)$$

so the effect of U is:

$$\boxed{T \rightarrow T - \frac{k}{r}} \quad - (18)$$

in the Minkowski metric (1). It is shown as follows that the Hamiltonian (17) is obtained from the metric:

$$ds^2 = c^2 d\tau^2 = \left(1 - \frac{r_0}{r}\right) c^2 dt^2 - d\underline{r} \cdot d\underline{r} \quad - (19)$$

where

$$r_0 = \frac{2 m G}{c^2} \quad - (20)$$

5)

The constants of motion of this metric are:

$$E = mc^2\left(1-\frac{r_0}{r}\right)\frac{dt}{d\tau} = \gamma mc^2\left(1-\frac{r_0}{r}\right) \quad -(21)$$

$$p_1 = m\frac{dr}{d\tau} = \gamma m\frac{dr}{dt} \quad -(22)$$

$$L = mr^2\frac{d\phi}{d\tau} = \gamma mr^2\frac{d\phi}{dt} \quad -(23)$$

With these definitions:

$$\frac{1}{2}\left(1-\frac{r_0}{r}\right)\frac{p_1^2}{m} = \frac{E^2}{2mc^2} - \frac{1}{2}\left(1-\frac{r_0}{r}\right)\left(mc^2+\frac{L^2}{mr^2}\right) \quad -(24)$$

i.e.

$$\boxed{\frac{1}{2}\left(1-\frac{r_0}{r}\right)m\left(\frac{dr}{d\tau}\right)^2 = \frac{E^2}{2mc^2} - \frac{1}{2}\left(1-\frac{r_0}{r}\right)\left(mc^2-\frac{L^2}{mr^2}\right)} \quad -(25)$$

For comparison, the standard gravitational metric is:

$$\frac{1}{2}m\left(\frac{dr}{d\tau}\right)^2 = \frac{E^2}{2mc^2} - \frac{1}{2}\left(1-\frac{r_0}{r}\right)\left(mc^2-\frac{L^2}{mr^2}\right) \quad -(26)$$

In both metrics the right hand side is: $\quad -(27)$

$$RHS = \left(\frac{E^2}{2mc^2}-\frac{1}{2}mc^2\right) + \left(\frac{mMG}{r} - \frac{L^2}{2mr^2} + \frac{GML^2}{mc^2r^3}\right)$$

This gives a precessing ellipse as is well known.

Therefore the metric (19) gives a precessing ellipse by:

$$\boxed{dt^2 \rightarrow \left(1-\frac{r_0}{r}\right)dt^2} \quad -(28)$$

observed experimentally as the gravitational red shift.

4)
In the limit:

$$r_0 << r \quad -(29)$$

eq. (27) becomes:

$$RHS \to \frac{1}{2} mc^2 (\gamma^2 - 1) + \frac{mMG}{r} - \frac{L^2}{2mr^2}$$

$$-(30)$$

In the limit:

$$v << c \quad -(31)$$

this becomes:

$$RHS \to mc^2 (\gamma - 1) + \frac{mMG}{r} - \frac{L^2}{2mr^2}$$

$$= \frac{1}{2} mv^2 + \frac{mMG}{r} + \frac{L^2}{2mr^2} \quad -(32)$$

As

$$r \to \infty \quad -(33)$$

eq. (25) therefore becomes:

$$\frac{1}{2} mv^2 = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) \quad -(34)$$

The Lagrangian used by Sommerfeld is:

$$\mathcal{L} = mc^2 (\gamma - 1) + \frac{mMG}{r} \quad -(35)$$

and the Hamiltonian used by Sommerfeld is:

$$H = mc^2 (\gamma - 1) - \frac{mMG}{r} \quad -(36)$$

These are approximations of the complete

5)

Lagrangian:

$$\mathcal{L} = \left(\frac{E^2}{2mc^2} - \frac{1}{2}mc^2\right) + \left(\frac{mMG}{r} + \frac{GML^2}{mc^2 r^3}\right) \quad -(37)$$

and the complete hamiltonian:

$$H = \left(\frac{E^2}{2mc^2} - \frac{1}{2}mc^2\right) - \left(\frac{mMG}{r} + \frac{GML^2}{mc^2 r^3}\right) \quad -(38)$$

from the metric (19). This metric is:

$$\boxed{ds^2 = c^2 d\tau^2 = c^2 dt^2 - d\mathbf{r}\cdot d\mathbf{r} - \left(\frac{2MG}{c^2 r}\right) dt^2} \quad -(39)$$

This metric gives a precessing ellipse:

$$r = \frac{d}{1 + \epsilon\cos(y\phi)} \quad -(40)$$

and in the Sommerfeld approximation, also gives the Dirac equation of the hydrogen atom. Using eq. (40) and the principle of orbits, the metric (39) is:

$$\boxed{ds^2 = c^2 d\tau^2 = c^2 dt^2 - (1 + r^2 x^2)\, r^2 d\phi^2} \quad -(41)$$

where

$$x = \frac{y\epsilon}{d}\sin(y\phi) \quad -(42)$$

Eq. (41) is a Minkowski metric with:

$$\boxed{dr = x r^2 d\phi} \quad -(43)$$

6) Conclusion

The usual gravitational metric:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 \left(1 - \frac{r_0}{r}\right) - \left(1 - \frac{r_0}{r}\right)^{-1} dr^2 - r^2 d\phi^2 - dZ^2 \quad -(44)$$

is not needed for the description of a precessing ellipse. The most economical solution of the problem is the metric (41), which is a Minkowski metric with the functional relation (43) of the precessing ellipse. The metrics (41) and (39) are the same. In the Sommerfeld approximation (36) they give the same solution for the orbital of the electron in a H atom as the Dirac equation.

1) 149(8): The Kinetic Nature of General Relativity

The lagrangian in general relativity is purely kinetic:

$$\mathcal{L} = T = \frac{1}{2} m c^2 = \frac{1}{2} m g_{\mu\nu} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \quad -(1)$$

In the new metric of note 149(7):

$$\mathcal{L} = \frac{1}{2} m c^2 = \frac{1}{2} m c^2 \left(1 - \frac{r_0}{r}\right)\left(\frac{dt}{d\tau}\right)^2 - \frac{m}{2}\left(\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\phi}{d\tau}\right)^2\right) \quad -(2)$$

i.e. $$\frac{1}{2} m c^2 \left(\left(1 - \frac{r_0}{r}\right)\left(\frac{dt}{d\tau}\right)^2 - 1\right) = \frac{m}{2}\left(\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\phi}{d\tau}\right)^2\right) \quad -(3)$$

In the non-relativistic limit this means that

$$\boxed{\frac{1}{2} m v^2 \rightarrow \frac{1}{2} m v^2 - \frac{mMG}{r}} \quad -(4)$$

because

$$\frac{m}{2}\left(\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\phi}{d\tau}\right)^2\right) \rightarrow \frac{1}{2} m v^2 \quad -(5)$$

and

$$\frac{1}{2} m c^2 \left(\left(1 - \frac{r_0}{r}\right)\left(\frac{dt}{d\tau}\right)^2 - 1\right)$$

$$= \frac{1}{2} m c^2 (\gamma^2 - 1) - \frac{1}{2} m c^2 \frac{r_0}{r}\left(\frac{dt}{d\tau}\right)^2$$

$$\rightarrow \frac{1}{2} m v^2 - \frac{mMG}{r} \quad -(6)$$

This is a clear way of showing that in a [...]trical analysis, there is no concept of potential [...]nergy, because the lagrangian is T.

The usual analysis is obtained by multiplying eq. (2) by $\left(1-\frac{r_0}{r}\right)$:

$$\frac{1}{2}mc^2\left(1-\frac{r_0}{r}\right) = \frac{1}{2}mc^2\left(1-\frac{r_0}{r}\right)^2\left(\frac{dt}{d\tau}\right)^2 - \left(1-\frac{r_0}{r}\right)\frac{m}{2}\left(\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\phi}{d\tau}\right)^2\right) \quad -(7)$$

and re-arranging:

$$\frac{1}{2}\left(1-\frac{r_0}{r}\right)m\left(\frac{dr}{d\tau}\right)^2 = \frac{1}{2}mc^2\left(1-\frac{r_0}{r}\right)^2\left(\frac{dt}{d\tau}\right)^2 - \frac{1}{2}mc^2\left(1-\frac{r_0}{r}\right) - \left(1-\frac{r_0}{r}\right)^2\frac{m}{2}r^2\left(\frac{d\phi}{d\tau}\right)^2$$

$$= \frac{E^2}{2mc^2} - \frac{1}{2}mc^2 + \left(\frac{mMG}{r} - \frac{L^2}{2mr^2} + \frac{GML^2}{mc^2r^3}\right) \quad -(8)$$

as in note 149(7). Eq. (8) means:

$$\mathcal{L} = T = \frac{1}{2}mv^2 \rightarrow \frac{1}{2}mv^2 + \frac{mMG}{r} \quad -(9)$$

in the non-relativistic limit, and this again is purely kinetic.

Inverse square attraction in the metric of note 149(7) means:

$$\boxed{dt^2 \rightarrow \left(1-\frac{2GM}{c^2r}\right)dt^2} \quad -(10)$$