Notes for Paper 52, Part 5

The Fundamental Wave Equations in Various Notations

In Cartan's notation:

$$d \wedge (d \wedge A) + d \wedge (\omega \wedge A) = \mu_0 j \quad -(1)$$

which in standard notation of differential geometry is:

$$d \wedge (d \wedge A^a) + d \wedge (\omega^a{}_b \wedge A^b) = \mu_0 j^a \quad -(2)$$

i.e.

$$d \wedge F^a = \mu_0 j^a \quad -(3)$$

where

$$F^a = d \wedge A^a + \omega^a{}_b \wedge A^b. \quad -(4)$$

In tensor notation:

$$F^a_{\mu\nu} = -F^a_{\nu\mu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \omega^a_{\mu b} A^b_\nu - \omega^a_{\nu b} A^b_\mu \quad -(5)$$

which in vector notation is:

$$\underline{B}^a = \underline{\nabla} \times \underline{A}^a + \underline{\omega}^a{}_b \times \underline{A}^b \quad -(6)$$

$$\underline{E}^a = -\frac{\partial \underline{A}^a}{\partial t} - c \underline{\nabla} A^a_0 - c\,\omega^a{}_{0b} \underline{A}^b - c\,\underline{\omega}^a{}_b A^b_0 \quad -(7)$$

where:

$$A^a_\mu = (A^a_0, -\underline{A}^a) \quad -(8)$$

$$\omega^a_{\mu b} = (\omega^a{}_{0b}, -\underline{\omega}^a{}_b) \quad -(9)$$

2) So the wave equations in vector notation are obtained by substituting eqns. (6) and (7) into the four field equations (paper 50, eqns. (84) to (87)):

$$\underline{\nabla} \cdot \underline{B}^a = \mu_0 \tilde{j}^{a0} \qquad -(10)$$

$$\underline{\nabla} \times \underline{E}^a + \frac{\partial \underline{B}^a}{\partial t} = \mu_0 \underline{\tilde{J}}^a \qquad -(11)$$

$$\underline{\nabla} \cdot \underline{E}^a = \mu_0 c \tilde{J}^{a0} \qquad -(12)$$

$$\underline{\nabla} \times \underline{B}^a - \frac{1}{c^2} \frac{\partial \underline{E}^a}{\partial t} = \frac{\mu_0}{c} \underline{\tilde{J}}^a \qquad -(13)$$

<u>Examples</u>

Using (6) in (10):

$$\boxed{\underline{\nabla} \cdot \left(\underline{\nabla} \times \underline{A}^a + \underline{\omega}^a{}_b \times \underline{A}^b \right) = \mu_0 \tilde{j}^{a0}} \qquad -(14)$$

Using (7) in (12): $\qquad -(15)$

$$\boxed{\begin{aligned} &\underline{\nabla} \cdot \underline{\nabla} A^a_0 + \frac{1}{c} \underline{\nabla} \cdot \frac{\partial \underline{A}^a}{\partial t} + \omega^a{}_{0b} \underline{\nabla} \cdot \underline{A}^b \\ &\qquad + \left(\underline{\nabla} \cdot \underline{\omega}^a{}_b \right) A^b_0 = -\mu_0 \tilde{J}^{a0} \end{aligned}}$$

3) There are two other wave equations which can be constructed. In one dimension, eqn. (15) is:

$$\frac{\partial^2 A_0^a}{\partial Z^2} + \frac{1}{c}\frac{\partial}{\partial Z}\left(\frac{\partial A_Z^a}{\partial t}\right) + \omega^a{}_{0b}\frac{\partial A_Z^b}{\partial Z} + \left(\frac{\partial \omega^a{}_{Zb}}{\partial Z}\right) A_0^b$$

$$= -\mu_0 \tilde{J}^{a0} \qquad -(16)$$

and this is a linear inhomogeneous equation which gives resonances, Q.E.D.