1)

56(6): Equations of the Gravitational AB Effect

The gravitational AB effect is defined in regions where:

$$\boxed{\begin{aligned} R = D \wedge \omega &= 0 \quad &-(1) \\ \omega &\neq 0 \quad &-(2) \end{aligned}}$$

From eq. (1):

$$d \wedge \omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = 0. \quad -(3)$$

The spin connection is no longer antisymmetric:

$$\omega^a{}_b \neq -\omega^b{}_a \quad -(4)$$

and is no longer the dual of the tetrad:

$$\omega^a{}_b \neq -\frac{\kappa}{2} \epsilon^a{}_{bc} q^c \quad -(5)$$

If it is assumed that:

$$\omega^a{}_b = \omega^b{}_a \quad -(6)$$

~~then~~:

$$d \wedge \omega^0{}_0 + \omega^0{}_c \wedge \omega^c{}_0 = 0 \quad -(7)$$

etc.

If there is no interaction between gravitation and electromagnetism ~~then~~:

$$T^a = 0 \quad -(8)$$

So:

$$d \wedge q^a + \omega^a{}_b \wedge q^b = 0 \quad -(9)$$

Therefore the spin connection must obey eqs. (3)

2) and (9). The tetrad obeys the tetrad postulate.

$$D_\nu q^a_\mu = 0. \quad -(10)$$

Eq (10) relates the spin connection and ~~the~~ the Christoffel connection:

$$\Gamma^\kappa_{\mu\nu} = \Gamma^\kappa_{\nu\mu} \quad -(11)$$

for central gravitation. The metric for central gravitation is:

$$g_{\mu\nu} = g_{\nu\mu} = q^a_\mu q^b_\nu \eta_{ab}. \quad -(12)$$

The tetrad also obeys the ECE Lemma:

$$\square q^a_\mu = R q^a_\mu \quad -(13)$$

and wave equation:

$$(\square + kT) q^a_\mu = 0. \quad -(14)$$

<u>Symmetry of the Spin Connection</u>

Eq (10) is:

$$D_\mu q^a_\lambda = \partial_\mu q^a_\lambda + \omega^a_{\mu b} q^b_\lambda - \Gamma^\nu_{\mu\lambda} q^a_\nu = 0 \quad -(15)$$

Interchange μ and λ:

$$D_\lambda q^a_\mu = \partial_\lambda q^a_\mu + \omega^a_{\lambda b} q^b_\mu - \Gamma^\nu_{\lambda\mu} q^a_\nu = 0$$

$$-(16)$$

3) For central gravitation:

$$\Gamma^{\nu}_{\mu\lambda} = \Gamma^{\nu}_{\lambda\mu} \quad -(17)$$

Subtracting eq. (16) from (15):

$$\partial_\mu q^a_\lambda + \omega^a_{\mu b} q^b_\lambda - \left(\partial_\lambda q^a_\mu + \omega^a_{\lambda b} q^b_\mu\right) = 0. \quad -(18)$$

For central gravitation the torsion is zero, so:

$$T^a_{\mu\lambda} = \partial_\mu q^a_\lambda + \omega^a_{\mu b} q^b_\lambda = 0, \quad -(19)$$

$$T^a_{\lambda\mu} = \partial_\lambda q^a_\mu + \omega^a_{\lambda b} q^b_\mu = 0. \quad -(20)$$

Eqs (18) to (20) must be symmetric under interchange of μ and λ, so:

$$\omega^a_{\mu b} q^b_\lambda = \omega^a_{\lambda b} q^b_\mu \quad -(21)$$

i.e.

$$\boxed{\omega^a_{\mu\lambda} = \omega^a_{\lambda\mu}.} \quad -(22)$$

Interchange the indices a and b:

$$\boxed{\omega^b_{\mu\lambda} = \omega^b_{\lambda\mu}}. \quad -(23)$$

Eqs. (22) and (23) are the same, so $\omega^a_{\mu b}$ is symmetric:

$$\boxed{\omega^a_{\mu b} = \omega_\mu{}^b{}_a} \quad -(24)$$

4) The equations for the diagonal elements of $\omega^a_{\mu b}$ are:

$$\left.\begin{aligned} d \wedge \omega^0{}_0 + \omega^0{}_c \wedge \omega^c{}_0 &= 0 \\ d \wedge \omega^1{}_1 + \omega^1{}_c \wedge \omega^c{}_1 &= 0 \\ d \wedge \omega^2{}_2 + \omega^2{}_c \wedge \omega^c{}_2 &= 0 \\ d \wedge \omega^3{}_3 + \omega^3{}_c \wedge \omega^c{}_3 &= 0 \end{aligned}\right\} \quad (25)$$

$$c = 0, 1, 2, 3.$$

There is also a set of equations for the off-diagonal elements defined by:

$$\left.\begin{aligned} \omega^a_{\mu b} &= \omega^b_{\mu a}, \\ a &\neq b. \end{aligned}\right\} \quad (26)$$

Under these conditions the Riemann form is zero, but the spin connection is not zero. These conditions define <u>the gravitational AB effect</u>.