Notes 58(3): Derivation of the Tetrad of the Schwarzschild Metric.

In spherical polar coordinates the tetrad of the Schwarzschild metric are described by:

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu \quad -(1)$$

$$g_{\mu\nu} = q^a_\mu q^b_\nu \eta_{ab}. \quad -(2)$$

The Minkowski metric in cartesian coordinates is implicitly described by the line element:

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2. \quad -(3)$$

In spherical polar coordinates this becomes:

$$ds^2 = -c^2 dt^2 + dr^2 + r^2 d\Omega^2 \quad -(4)$$

where:

$$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2. \quad -(5)$$

The line element squared in the Schwarzschild metric in spherical polar coordinates is:

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\Omega^2 \quad -(6)$$

Therefore in Cartesian coordinates in the Minkowski metric:

$$\eta_{ab} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad -(7)$$

and in spherical polar coordinates:

2)

$$\eta_{ab} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\phi \end{bmatrix} \quad -(8)$$

The Schwarzschild metric in spherical polar coordinates is:

$$g_{\mu\nu} = \begin{bmatrix} -\left(1-\frac{2GM}{c^2 r}\right) & 0 & 0 & 0 \\ 0 & \left(1-\frac{2GM}{c^2 r}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\phi \end{bmatrix} \quad -(9)$$

Therefore from a comparison of eqns (8) and (9) the tetrads of the Schwarzschild metric can be found from the elements of eqns (8) and (9) and the equation (2). The non-zero elements are:

$$\eta_{00} = -1,\ \eta_{11} = 1,\ \eta_{22} = r^2,\ \eta_{33} = r^2 \sin^2\phi \quad -(10)$$

and

$$g_{00} = -\left(1-\frac{2GM}{c^2 r}\right),\ g_{11} = \left(1-\frac{2GM}{c^2 r}\right)^{-1},\ g_{22} = r^2,\ g_{33} = r^2 \sin^2\phi \quad -(11)$$

where:

$$g_{00} = q^a_0 q^b_0 \eta_{ab}, \quad -(12)$$

$$g_{11} = q^a_1 q^b_1 \eta_{ab} \quad -(13)$$

$$g_{22} = q^a_2 q^b_2 \eta_{ab} \quad -(14)$$

$$g_{33} = q^a_3 q^b_3 \eta_{ab} \quad -(15)$$

3)

Considering only the diagonal elements of the tetrad eqs. (12) to (15) simplify to:

$$g_{00} = q^0_0 q^0_0 \eta_{00} \quad -(16)$$

$$g_{11} = q^1_1 q^1_1 \eta_{11} \quad -(17)$$

$$g_{22} = q^2_2 q^2_2 \eta_{22} \quad -(18)$$

$$g_{33} = q^3_3 q^3_3 \eta_{33} . \quad -(19)$$

Therefore:

$$\boxed{\begin{aligned} q^0_0 &= \left(1 - \frac{2GM}{c^2 r}\right)^{1/2} & -(20) \\ q^1_1 &= \left(1 - \frac{2GM}{c^2 r}\right)^{-1/2} & -(21) \\ q^2_2 &= 1 & -(22) \\ q^3_3 &= 1 . & -(23) \end{aligned}}$$

In the limit $r \to 0$ the Schwarzschild metric reduces to the Minkowski metric, i.e.

$$g_{00} \to \eta_{00} \text{ etc.} \quad -(24)$$

so eqs. (16) to (23) are compatible with this property.

The Schwarzschild scalar curvature R is obtained from:

$$\square q^0_0 = R_0 q^0_0 \quad -(25)$$

$$\square q^1_1 = R_1 q^1_1 \quad -(26)$$

4) In these equations:

$$r = (x^2 + y^2 + z^2)^{1/2} \quad -(27)$$

so eqns (25) and (26) become:

$$\nabla^2 q^0_0 = -R_0 q^0_0 \quad -(28)$$

$$\nabla^2 q^1_1 = -R_1 q^1_1. \quad -(29)$$

As the Minkowski metric is approached:

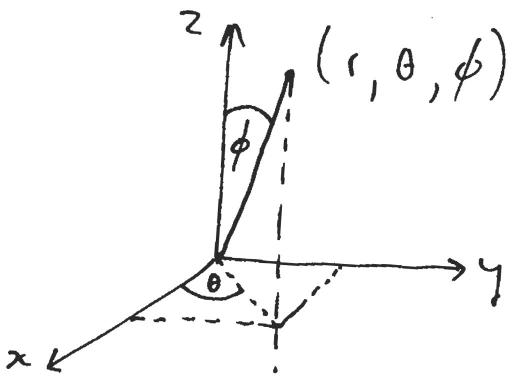
$$R_0 \rightarrow 0, \quad R_1 \rightarrow 0 \quad -(30)$$

i.e. we recover flat spacetime self-consistently.

Before proceeding to a discussion of the governing equations for the Schwarzschild methods some details are given of the spherical polar coordinate system as follows.

Spherical Polar Coordinates

$$\left.\begin{aligned} x &= r \sin\phi \cos\theta \\ y &= r \sin\phi \sin\theta \\ z &= r \cos\phi \end{aligned}\right\} \quad -(31)$$



$$x^2 + y^2 + z^2 = r^2 \sin^2\phi \cos^2\theta + r^2 \sin^2\phi \sin^2\theta + r^2 \cos^2\phi$$

$$= r^2 \quad -(32)$$

$$dx = -r \sin\phi \sin\theta \, d\theta + r\cos\phi \cos\theta \, d\phi + \sin\phi \cos\theta \, dr$$

$$dy = r \sin\phi \cos\theta \, d\theta + r\cos\phi \sin\theta \, d\phi + \sin\phi \sin\theta \, dr$$

$$dz = -r \sin\phi \, d\phi + \cos\phi \, dr$$

5) So:

$$ds^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\phi^2 + r^2 \sin^2\phi \, d\theta^2 \quad -(34)$$

The space-like metric elements are the squares of the weighting factors:

$$g_{11} = h_1^2, \; g_{22} = h_2^2, \; g_{33} = h_3^2. \quad -(35)$$

The weighting factors or scale factors in the spherical polar coordinates are:

$$h_1 = h_r = 1, \; h_2 = h_\phi = r, \; h_3 = h_\theta = r\sin\phi \quad -(36)$$

in Euclidean spacetime. In Minkowski spacetime we obtain the metric (8). The surface of a sphere is:

$$S = \int_0^{2\pi} d\theta \int_0^{\pi} r^2 \sin\phi \, d\phi = 4\pi r^2 \quad -(37)$$

and the volume of a sphere is:

$$V = \int_0^r 4\pi r^2 dr = \frac{4}{3}\pi r^3. \quad -(38)$$

The unit vectors are:

$$\underline{e}_r = \sin\phi\cos\theta \, \underline{i} + \sin\phi\sin\theta \, \underline{j} + \cos\phi \, \underline{k}$$

$$\underline{e}_\phi = \cos\phi\cos\theta \, \underline{i} + \cos\phi\sin\theta \, \underline{j} - \sin\phi \, \underline{k}$$

$$\underline{e}_\theta = -\sin\theta \, \underline{i} + \cos\theta \, \underline{j} \quad -(39)$$

So:

$$\underline{V} = V_r \underline{e}_r + V_\phi \underline{e}_\phi + V_\theta \underline{e}_\theta. \quad -(40)$$

The Governing Equations

The governing equations of EH theory are:

$$T^a = d \wedge q^a + \omega^a{}_b \wedge q^b = 0 \quad -(41)$$

$$R^a{}_b = d \wedge \omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \quad -(42)$$

$$R^a{}_b \wedge q^b = 0 \quad -(43)$$

$$D \wedge R^a{}_b = 0. \quad -(44)$$

The elements of the tetrad are diagonal as shown already. The non-vanishing elements of the Riemann tensor in the Schwarzschild metric are: $R^0{}_{101}$, $R^0{}_{202}$, $R^0{}_{303}$, $R^0{}_{212}$, $R^0{}_{313}$, $R^1{}_{212}$, $R^1{}_{313}$ and $R^2{}_{323}$. These are related to the Riemann form by:

$$R^a{}_{b\mu\nu} = q^a_\rho q^\sigma_b R^\rho{}_{\sigma\mu\nu}. \quad -(45)$$

Here $R^a{}_{b\mu\nu}$ is the Riemann form and $R^\rho{}_{\sigma\mu\nu}$ is the Riemann tensor.

In the presence of torsion, eqs. (41) to (44) are changed as follows:

$$T^a = d \wedge q^a + \omega^a{}_b \wedge q^b = R^a{}_b \wedge q^b. \quad -(46)$$

Therefore the familiar Ricci cyclic equation (43) of EH theory is no longer obeyed in the presence of torsion. The Ricci cyclic equation in tensor notation is:

$$R_{\sigma\mu\nu\rho} + R_{\sigma\rho\mu\nu} + R_{\sigma\nu\rho\mu} = 0, \quad -(47)$$

but this is no longer true in the presence of torsion. Therefore in the presence of torsion the Riemann

7) versa is no longer antisymmetric in its first two indices. However, it remains antisymmetric in its last two indices. This means that the EH field equation is no longer obeyed in the presence of torsion, so the Eddington experiment is changed.

Restricting attention for the time being to the EH field theory, the spin connection can be obtained from the Schwarzschild tetrads using eqn. (42):

$$d \wedge q^a + \omega^a{}_b \wedge q^b = 0. \quad -(48)$$

The non-zero tetrad elements are $q^0{}_0$, $q^1{}_1$, $q^2{}_2 = q^3{}_3 = 1$. The Riemann form and the spin connection are related by <u>the 2nd. Cartan equation</u>:

$$R^a{}_b = d \wedge \omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b . \quad -(49)$$

So the next stage of the calculation is to determine the spin connection of the Schwarzschild solution of the Einstein-Hilbert field equation. After that, the effect of torsion is calculated by adding a small perturbation to eqn. (48):

$$\left.\begin{array}{l} d \wedge q^a + \omega^a{}_b \wedge q^b = \delta T^a , \\ \delta T^a << T^a \end{array}\right\} -(50)$$