1)

Notes 58(7): Tetrad Method in the Calculation of Light Deflection without Torsion, and with Torsion.

The Schwarzschild metric (SM) is:

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right)c^2 dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\phi^2 + r^2 \sin^2\phi \, d\theta^2 \quad -(1)$$

The coordinate system being used is sketched in Fig (1). Eq (1) is written in the correct S.I. units.

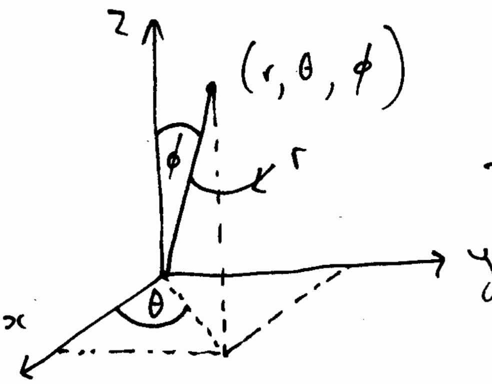


Fig (1)

Light travels along null paths,

$$ds^2 = 0. \quad -(2)$$

Restrict consideration to a single plane through the centre of mass:

$$\theta = 0. \quad -(3)$$

It follows that:

$$c^2 dt^2 = \left(1 - \frac{2GM}{c^2 r}\right)^{-2} dr^2 + r^2 \left(1 - \frac{2GM}{c^2 r}\right)^{-1} d\phi^2 \quad -(4)$$

In the Minkowski metric:

$$\left(1 - \frac{2GM}{c^2 r}\right)^{-1} \rightarrow 1 \quad -(5)$$

2) The metric corresponding to eq. (4) is:

$$g_{\mu\nu} = \begin{bmatrix} \left(1 - \frac{2GM}{c^2 r}\right)^{-2} & 0 \\ 0 & r^2 \left(1 - \frac{2GM}{c^2 r}\right)^{-1} \end{bmatrix} \quad - (6$$

while the Minkowski metric is:

$$\eta_{ab} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix} . \quad - (7)$$

So using:

$$g_{\mu\nu} = q_\mu^{\ a} q_\nu^{\ b} \eta_{ab} \quad - (8)$$

we obtain the two tetrad elements:

$$\boxed{\begin{aligned} q_{rr} &= \left(1 - \frac{2GM}{c^2 r}\right)^{-1} \quad &- (9) \\ q_{\phi\phi} &= \left(1 - \frac{2GM}{c^2 r}\right)^{-1/2} \quad &- (10) \end{aligned}}$$

If

$$2GM << c^2 r \quad - (11)$$

$$q_{rr} \rightarrow 1 + \frac{2GM}{c^2 r} + \ldots \quad - (12)$$

$$q_{\phi\phi} \rightarrow 1 + \frac{GM}{c^2 r} + \ldots \quad - (13)$$

3)

The element q_{rr} means that r is not a straight line, it is the curve:

$$x(r) = x^{(0)} q_{rr} \quad \text{— (14)}$$

where $x^{(0)}$ is a proportionality. So $x(r)$ is an orbit.

It is found that:

$$c^2 \frac{\partial q_{rr}}{\partial r} = -\frac{2GM}{r^2}. \quad \text{— (15)}$$

The Newtonian force between a photon of mass m and the sun of mass M is:

$$F = -\frac{GmM}{r^2}. \quad \text{— (16)}$$

The force from eq. (15) is:

$$\boxed{F = -mc^2 \frac{\partial q_{rr}}{\partial r} = -\frac{2GmM}{r^2}} \quad \text{— (17)}$$

This is twice the Newtonian force (16), and is the photon rest energy:

$$E_0 = mc^2 = \hbar\omega_0 \quad \text{— (18)}$$

multiplied by $\partial q_{rr} / \partial r$. Eq. (18) is the Planck / Einstein / de Broglie equation. So the Newtonian inverse square law is only half the answer, as found experimentally.

4) The force (17) is:

$$F = mg = -mc^2 \frac{\partial q_{rr}}{\partial r} \quad \text{--- (19)}$$

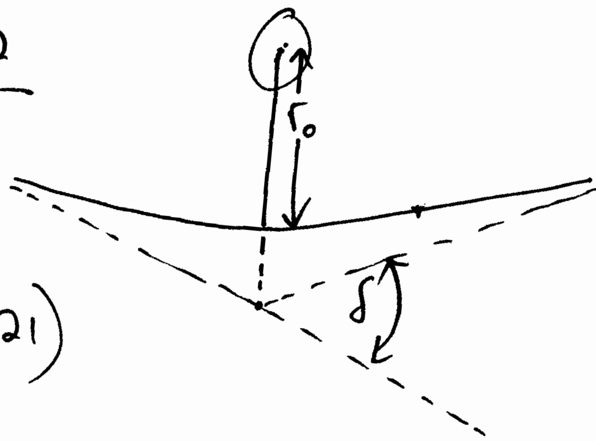
The acceleration due to gravity is:

$$\boxed{g = -c^2 \frac{\partial q_{rr}}{\partial r}} \quad \text{--- (20)}$$

using the equivalence of gravitational and inertial mass.

The angle of deflection is twice that expected from the Newtonian orbit:

Fig 2



$$\delta(\text{Newton}) = \frac{2MG}{c^2 r_0} \quad \text{--- (21)}$$

where r_0 is the distance of closest approach. So:

$$\boxed{\delta(\text{Schwarzschild}) = \frac{4MG}{c^2 r_0}} \,. \quad \text{--- (22)}$$

The calculation has been carried out in the correct S.I. units. So this is our baseline, the T = 0 case.

5)

The Effect of Cartan Torsion

The Sachs? geometry is:

$$R \wedge q = 0, \quad T = 0 \qquad -(23)$$

which is the Einstein Hilbert geometry, in which the Riemann tensor is antisymmetric in its first two indices and also its last two indices. In the presence of torsion:

$$R \wedge q = d \wedge T + \omega \wedge T \qquad -(24)$$

$$\neq 0$$

so the Riemann tensor is no longer antisymmetric in its first two indices. This means that the Einstein Hilbert equation is no longer true. However, the second Bianchi identity of Cartan geometry is still true:

$$D \wedge R := 0. \qquad -(25)$$

In the EH geometry:

$$T = d \wedge q + \omega \wedge q = 0 \qquad -(26)$$

so:

$$d \wedge q = -\omega \wedge q. \qquad -(27)$$

In the presence of torsion:

$$\boxed{d \wedge q = -\omega \wedge q + T} \qquad -(28)$$

So the mathematical task is to find the elements of the spin connection from eqs. (12), (13) and (26) then perturb with a small δT:

$$d \wedge q \rightarrow -\omega \wedge q + \delta T \quad -(29)$$

In the first approximation the spin connection can be assumed to be changed only slightly, so:

$$\boxed{d \wedge q \rightarrow -\omega_0 \wedge q + \delta T} \quad -(30)$$

where ω_0 is the baseline spin connection from eq. (26). We may proceed very roughly by inputting a δT by inspection. It is clear that the tetrad element (12) will be changed by torsion. The angle of deflection (22) and the acceleration g will also change.

The complete problem is defined by the Cartan structure equations and Bianchi identities of Cartan geometry:

$$T = D \wedge q \quad -(31)$$

$$R = D \wedge \omega \quad -(32)$$

$$D \wedge T = R \wedge q \quad -(33)$$

$$D \wedge R = 0 \quad -(34)$$