# Notes 60(2): Dirac and Schrödinger Equations from the ECE Wave Equation.

The ECE Wave Equation is:

$$(\Box + kT)\, q^a_\mu = 0 \quad -(1)$$

where:

$$R = -kT = q^\lambda_a \partial^\mu \left( \Gamma^\nu_{\mu\lambda} q^a_\nu - \omega^a_{\mu b} q^b_\lambda \right) \quad -(2)$$

Here $k$ is Einstein's constant, $T$ is the index reduced canonical energy-momentum density, $q^a_\mu$ is the tetrad, $\Gamma^\nu_{\mu\lambda}$ is the general connection of Riemann geometry, and $\omega^a_{\mu b}$ is the spin connection of Cartan geometry.

Using Einstein's equivalence principle, eq. (1) must reduce to equations of special relativity when there is no gravitational field present. In order to recover the free particle Dirac equation:

$$kT = \left(\frac{mc}{\hbar}\right)^2 = \frac{1}{\lambda_c^2} \quad -(3)$$

where $m$ is the mass of the free particle (e.g. an electron), $c$ is the speed of light and $\hbar$ the reduced Planck constant. Here $\lambda_c$ is the Compton wavelength. The Dirac equation is therefore:

$$\left(\Box + \frac{m^2c^2}{\hbar^2}\right) q^a_\mu = 0. \quad -(4)$$

2) The Dirac spinor is therefore the tetrad $q^a_\mu$ in the limit of special relativity. Therefore the tetrad must be defined by:

$$V^a = q^a_\mu V^\mu \quad -(5)$$

i.e. (paper 38):

$$\begin{bmatrix} V^R \\ V^L \end{bmatrix} = \begin{bmatrix} q^R_1 & q^R_2 \\ q^L_1 & q^L_2 \end{bmatrix} \begin{bmatrix} V^1 \\ V^2 \end{bmatrix} \quad -(6)$$

in SU(2) representation space. The Dirac spinor is, in the usual field / particle theory notation:

$$\phi = \begin{bmatrix} \xi^R \\ \xi^L \end{bmatrix} \quad -(7)$$

where the Pauli spinors are:

$$\xi^R = \begin{bmatrix} q^R_1 \\ q^R_2 \end{bmatrix}, \quad \xi^L = \begin{bmatrix} q^L_1 \\ q^L_2 \end{bmatrix}. \quad -(8)$$

So the Dirac equation is:

$$\left( \Box + \frac{m^2 c^2}{\hbar^2} \right) \phi = 0 \quad -(9)$$

which can be expressed as:

$$\left( i \gamma^\mu \partial_\mu - \frac{mc}{\hbar} \right) \phi = 0 \quad -(10)$$

where $\gamma^\mu$ is the Dirac matrix. In vector notation eq. (10) is:

$$(E + c\underline{\sigma}\cdot\underline{p})\,\xi^L(\underline{p}) = mc^2\,\xi^R(\underline{p}) \quad -(11)$$

$$(E - c\underline{\sigma}\cdot\underline{p})\,\xi^R(\underline{p}) = mc^2\,\xi^L(\underline{p}) \quad -(12)$$

where:

$$\xi^L(\underline{0}) = \xi^R(\underline{0}) \quad -(13)$$

Here $E$ is total energy, $\underline{p}$ is momentum, and $\underline{\sigma}$ is the Pauli matrix.

The Schrödinger equation is the non-relativistic limit of the Dirac equation. However, the former is written in O(3) rep. space and the latter is SU(2) rep. space, introducing the half-integral spin and Fermi Dirac statistics. So the link between the two equations is not trivial. The Dirac eqn. (9) is a Klein-Gordon type equation for each of $\psi^R_1$, $\psi^R_2$, $\psi^L_1$ and $\psi^L_2$, so can be related to the Einstein equation of special relativity using the operator equivalence:

$$p^\mu = \left(\frac{E}{c}, \underline{p}\right) = i\hbar\partial^\mu = i\hbar\left(\frac{1}{c}\frac{\partial}{\partial t}, -\underline{\nabla}\right)$$

$$-(14)$$

4)

In this way eq. (9) becomes:

$$E^2 = c^2 p^2 + m^2 c^4 \quad -(15)$$

where

$$E = \gamma m c^2, \quad E_0 = m c^2, \quad \gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}.$$

Here $E$ is the total energy, $E_0$ is the rest energy, and

$$\underline{p} = \gamma m \underline{v} \quad -(16)$$

is the relativistic momentum. From eq. (16) it is seen that:

$$p^2 c^2 = \gamma^2 m^2 c^4 \frac{v^2}{c^2} = \gamma^2 m^2 c^4 \left(1 - \frac{1}{\gamma^2}\right) = \gamma^2 m^2 c^4 - m^2 c^4 \quad -(17)$$

The relativistic kinetic energy is obtained from eq. (16), and is:

$$T = mc^2(\gamma - 1) \quad -(18)$$

$$= mc^2\left(1 - \frac{v^2}{c^2}\right)^{-1/2} - mc^2$$

$$\rightarrow mc^2\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \ldots\right) - mc^2$$

$$= \frac{1}{2} m v^2 = \frac{p^2}{2m} \quad -(19)$$

when $v << c$.

Eq. (19) is the Newtonian kinetic energy and in eq. (19) $p$ is the Newtonian momentum:

$$p = mv \quad -(20)$$

5) The Schrödinger equation is obtained from:

$$T = \frac{p^2}{2m} \quad -(21)$$

using eq. (14). So:

$$-\frac{\hbar^2}{2m}\nabla^2\phi = E\phi = i\hbar\frac{\partial\phi}{\partial t} \quad -(22)$$

The hamiltonian operator is defined as:

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 \quad -(23)$$

so:

$$\boxed{\hat{H}\phi = E\phi} \quad -(24)$$

However, the Schrödinger equation (24) has no sense of helicity or half-integral spin, unlike the special relativistic Dirac equation. The latter is the limit of the generally covariant ECE wave equation of unified field theory, eq. (1). The usual route of introducing half-integral spin into the Schrödinger equation is to replace the orbital angular momentum $\underline{L}$ by $\underline{L} + 2\underline{S}$.

The hydrogen atom is described from eq. (24) by adding a potential energy to the kinetic energy in eq. (21), so:

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + V \quad -(25)$$

6) where:

$$V = -\frac{e^2}{4\pi\epsilon_0}\frac{1}{r}, \quad -(26)$$

So:

$$-\frac{\hbar^2}{2m}\nabla^2\phi - \frac{e^2}{4\pi\epsilon_0 r}\phi = E\phi. \quad -(27)$$

A resonance Coulomb law can be used to replace the Coulombic term in eq. (27). More rigorously the resonance Coulomb law should be used with eq. (4) and in a fully rigorous theory with eq. (1).

The d'Alembertian operator in eq. (4) is:

$$\square = \partial^\mu\partial_\mu = -p^\mu p_\mu/\hbar^2 \quad -(28)$$

$$= -\frac{1}{\hbar^2}\left(\frac{E^2}{c^2} - p^2\right) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2.$$

So: $$\left(-\frac{1}{\hbar^2}\frac{E^2}{c^2} - \nabla^2 + \left(\frac{mc}{\hbar}\right)^2\right) q^a_\mu = 0$$

$$\boxed{-\hbar^2\nabla^2 q^a_\mu = \left(\frac{E^2}{c^2} - m^2c^2\right) q^a_\mu}$$

$-(29)$

This is the relativistic form of eq. (24). Multiplying eq. (29) by $c^2$ and using eq. (15) it is

7) found that eq. (29) is:

$$-c^2\hbar^2\nabla^2 q_\mu^a = c^2 p^2 q_\mu^a$$

i.e.

$$-\hbar^2\nabla^2 q_\mu^a = p^2 q_\mu^a \quad -(30)$$

where:

$$p^2 = \frac{E^2}{c^2} - m^2c^2. \quad -(31)$$

To describe the H atom with the Dirac equation a Coulombic term is usually added to eq. (29). Thus:

$$\boxed{-\frac{\hbar^2\nabla^2}{2m} q_\mu^a - \frac{e^2}{4\pi\epsilon_0 r} q_\mu^a = \frac{1}{2m}\left(\frac{E^2}{c^2} - m^2c^2\right) q_\mu^a}$$

$$-(32)$$

In ECE theory a resonance Coulomb law can be ~~add~~ used in eq. (32). The most rigorous method is to use the resonance Coulomb law in eq. (1). Solutions in all cases must be numerical.