Paper 63(1): Reduction of the Resonance Equation Using the Euler Method

The resonance equation is:

$$\frac{d^2\phi}{dZ^2} - \frac{1}{Z}\frac{d\phi}{dZ} + \frac{\phi}{Z^2} = \frac{f_0}{E_0}\cos(KZ) \quad -(1)$$

this is a second order differential equation with variable coefficients $-1/Z$ and $1/Z^2$. This is reduced as follows to an undamped oscillator equation that has analytical solutions.

Let:

$$Z = Z_0 e^{iK_0 x} \quad -(2)$$

then

$$\frac{d\phi}{dZ} = \frac{d\phi}{dx}\frac{dx}{dZ} \quad -(3)$$

where

$$\frac{dx}{dZ} = -\frac{i}{K_0}\frac{1}{Z}. \quad -(4)$$

From eq (3):

$$\frac{d^2\phi}{dZ^2} = -\frac{i}{K_0}\frac{d}{dZ}\left(\frac{1}{Z}\frac{d\phi}{dx}\right)$$

$$= \frac{i}{K_0 Z^2}\frac{d\phi}{dx} - \frac{i}{K_0 Z}\frac{d}{dZ}\left(\frac{d\phi}{dx}\right) \quad -(5)$$

2) From eq. (4):

$$\frac{d}{dZ}\left(\frac{d\phi}{dx}\right) = -\frac{i}{k_0 Z}\frac{d^2\phi}{dx^2} \quad \text{— (6)}$$

so

$$Z^2\frac{d^2\phi}{dZ^2} = \frac{i}{k_0}\frac{d\phi}{dx} - \frac{1}{k_0^2}\frac{d^2\phi}{dx^2} \quad \text{— (7)}$$

Using eqs. (4) and (7) in eq. (1):

$$\frac{d^2\phi}{dx^2} - 2ik_0\frac{d\phi}{dx} + k_0^2\phi = -\frac{f_0}{\epsilon_0}k_0^2 Z_0^2 e^{2ik_0x} \times \cos\left(kZ_0 e^{ik_0x}\right) \quad \text{— (8)}$$

Let: $k_0 = 1/Z_0$ — (9)

and compare real parts:

$$\frac{d^2\phi}{dx^2} + \frac{1}{Z_0^2}\phi = -\frac{f_0}{\epsilon_0}\,\mathrm{Re}\left(e^{2ik_0x}\cos\left(kZ_0 e^{ik_0x}\right)\right) \quad \text{— (10)}$$

Using $\cos(ix) = \cosh x$ — (11)

the real part of the right hand side of eq. (10) is a bounded quantity that can be expressed as:

3)

$$\frac{d^2\phi}{dx^2} + \kappa_0^2\phi = \frac{\rho_0}{\epsilon_0}\cos(\kappa' x) \quad -(12)$$

where:

$$\cos(\kappa' x) := -\cos(2\kappa_0 x)\left(\cos\left(\kappa Z_0\left(\cos\left(\kappa_0 x + \cosh(\sin(\kappa_0 x)\right)\right)\right)\right) \quad -(13)$$

Finally, changing variables in eq. (12):

$$\boxed{\frac{d^2\phi}{dZ^2} + \frac{1}{Z_0^2}\phi = \frac{\rho_0}{\epsilon_0}\cos(\kappa' Z)} \quad -(14)$$

This is an undamped oscillator equation.

Frequency and kinetic energy resonances occur from eq. (4) at:

$$\boxed{\kappa_R = \kappa_E = \frac{1}{Z_0}} \quad -(15)$$

at which:

$$\boxed{\phi = \frac{\rho_0}{\epsilon_0} \cdot \frac{\cos(\kappa' Z)}{\left(\frac{1}{Z_0^2} - \kappa'^2\right)}} \quad -(16)$$