Notes 63(21) : Complete Analytical Solution of the Resonant Coulomb Law.

1

The eqn. is:

$$\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} - \frac{1}{r^2}\phi = -\frac{\rho}{\epsilon_0} \quad -(1)$$

Using:

$$\kappa r = \exp(i\kappa R) \quad -(2)$$

eq. (1) becomes:

$$\frac{d^2\phi}{dR^2} + \kappa^2\phi = \frac{\rho(0)}{\epsilon_0}\,\mathrm{Real}\left(e^{2i\kappa R}\cos\left(e^{i\kappa R}\right)\right) \quad -(3)$$

Assume that the solution is:

$$\phi = \frac{A\rho(0)}{\epsilon_0}\,\mathrm{Real}\left(e^{2i\kappa R}\cos\left(e^{i\kappa R}\right)\right) \quad -(4)$$

where A is to be determined. Substituting eq. (4) in eq. (3) gives:

$$A = \frac{\mathrm{Real}\left(e^{2i\kappa R}\cos\left(e^{i\kappa R}\right)\right)}{\kappa^2\left(1 - 4e^{2i\kappa R}\cos\left(e^{i\kappa R}\right) + 5e^{3i\kappa R}\sin\left(e^{i\kappa R}\right) + e^{4i\kappa R}\cos\left(e^{i\kappa R}\right)\right)}$$

$$= \frac{\kappa^2 r^2\cos(\kappa r)}{\kappa^2\left(1 - 4\kappa^2 r^2\cos(\kappa r) + 5\kappa^3 r^3\sin(\kappa r) + \kappa^4 r^4\cos(\kappa r)\right)} \quad -(5)$$

So:

$$\boxed{\phi = \frac{\rho(0)}{\epsilon_0}\left(\frac{\kappa^2 r^4\cos^2(\kappa r)}{1 + \kappa^4 r^4\cos(\kappa r) + 5\kappa^3 r^3\sin(\kappa r) - 4\kappa^2 r^2\cos(\kappa r)}\right)}$$

Units : $C\,m^{-3}\,m^2\,J\,C^{-2}\,m = J\,C^{-1} = $ volts

(2)

Resonance Condition

This is: — (6)

$$1 + \kappa^4 r^4 \cos(\kappa r) + 5\kappa^3 r^3 \sin(\kappa r) = 4\kappa^2 r^2 \cos(\kappa r)$$

When:

$$\kappa r = 2\pi n \quad -(7)$$

$$1 + x^2 = 4x, \text{ where } x = \kappa^2 r^2 \quad -(8)$$

This has roots: $x = \kappa^2 r^2 = 2 \pm \sqrt{3}$ — (9)

$$= 2 \pm 1 \cdot 732$$

Resonance occurs at:

$$\boxed{\kappa^2 r^2 = 3 \cdot 732 \text{ and } 0 \cdot 268}$$

Comments

There are two peaks as already observed numerically by Dr. Horst Eckardt.

Appendix: Differentiation Details

If $\phi = e^{2i\kappa R} \cos(e^{i\kappa R})$

$$\frac{d\phi}{dR} = 2i\kappa e^{2i\kappa R} \cos(e^{i\kappa R}) - i\kappa e^{i\kappa R} \sin(e^{i\kappa R}) e^{2i\kappa R}$$

$$\frac{d^2\phi}{dR^2} = -4\kappa^2 e^{2i\kappa R} \cos(e^{i\kappa R}) + 5\kappa^2 e^{3i\kappa R} \sin(e^{i\kappa R}) + \kappa^2 e^{4i\kappa R} \cos(e^{i\kappa R})$$