Notes 63(3): Review of Method

The original resonance equation is:

$$\frac{d^2\phi}{dZ^2} - \frac{A}{Z}\frac{d\phi}{dZ} + \frac{A\phi}{Z^2} = \frac{\rho_0}{\epsilon_0}\cos(\kappa Z)$$

Using:

$$Z = Z_0 e^{i\kappa x} \quad -(1)$$

$$-(2)$$

this becomes:

$$\frac{d^2\phi}{dx^2} - A\kappa^2\phi = -\kappa^2\frac{\rho_0}{\epsilon_0}\mathrm{Re}(Z^2\cos(\kappa Z))$$

Let $A = -1$, $x \rightarrow Z$, $\kappa = \frac{1}{Z_0}$ $-(3)$ $-(4)$

Then:

$$\frac{d^2\phi}{dZ^2} + \kappa^2\phi = -\frac{\rho_0}{\epsilon_0}\mathrm{Re}\left(e^{2i\kappa Z}\cos\left(e^{i\kappa Z}\right)\right)$$

$$:= \frac{\rho_0}{\epsilon_0}\cos(\kappa' Z) \quad -(5)$$

$$\boxed{\frac{d^2\phi}{dZ^2} + \kappa^2\phi = \frac{\rho_0}{\epsilon_0}\cos(\kappa' Z)}$$

$$\cos\kappa' Z := \mathrm{Re}\left(e^{2i\kappa Z}\cos\left(e^{i\kappa Z}\right)\right) \quad -(6)$$

a)
Real part of RHS in eq. (7) can be worked out with complex FORTRAN code. The analytical expression is somewhat complicated but straightforward as follows.

$$e^{2ikz} = \cos(2kz) + i\sin(2kz)$$

$$e^{ikz} = \cos(kz) + i\sin(kz)$$

$$\cos(A+B) = \cos A\cos B + \sin A \sin B$$

$$\cos(ix) = \cosh x$$

$$\sin(ix) = i\sinh x$$

So:

$$\mathrm{Re}\left(e^{2ikz}\cos\left(e^{ikz}\right)\right)$$

$$= \cos(2kz)\cos(\cos(kz))\cosh(\sin(kz))$$

$$- \sin(2kz)\sin(\cos(kz))\sinh(\sin(kz)) \quad — (8)$$

— (9)

From eq. (6)

$$\boxed{\phi = \frac{\rho_0}{\epsilon_0}\frac{\cos(k'z)}{(k^2 - k'^2)}}$$

Resonance at $\boxed{k = k'}$ — (10)