Notes 63(6) : Mathematical Proofs

To Prove

$$-\underline{\nabla}\left(\frac{1}{|\underline{r}-\underline{r}_i|}\right) = \frac{\underline{r}-\underline{r}_i}{|\underline{r}-\underline{r}_i|^3} \quad -(1)$$

Write eq. (1) as:

$$-\underline{\nabla}\left(|\underline{r}-\underline{a}|\right)^{-1} = \frac{\underline{r}-\underline{a}}{|\underline{r}-\underline{a}|^3} \quad -(2)$$

where:

$$|\underline{r}-\underline{a}| = \left((x-a_x)^2+(y-a_y)^2+(z-a_z)^2\right)^{1/2} \quad -(3)$$

and:

$$-\underline{\nabla}\left(\frac{1}{|\underline{r}-\underline{a}|}\right) = -\frac{\partial}{\partial x}\left(\frac{1}{|\underline{r}-\underline{a}|}\right)\underline{i} - \frac{\partial}{\partial y}\left(\frac{1}{|\underline{r}-\underline{a}|}\right)\underline{j} - \frac{\partial}{\partial z}\left(\frac{1}{|\underline{r}-\underline{a}|}\right)\underline{k} \quad -(4)$$

Consider terms such as:

$$\frac{\partial}{\partial x}\left((x-a_x)^2+(y-a_y)^2+(z-a_z)^2\right)^{-1/2} \quad -(5)$$

$$= -\frac{(x-a_x)}{\left((x-a_x)^2+(y-a_y)^2+(z-a_z)^2\right)^{3/2}}$$

to find eqs (2) and (1), Q.E.D.

In the resonance equation (21) of notes 63(4) there occurs:

2)

$$\underline{\nabla} \cdot \frac{(\underset{\sim}{r} - \underset{\sim}{r}_i)}{(|\underset{\sim}{r} - \underline{r}_i|)^2} := f(\underset{\sim}{r}) \quad -(6)$$

and this is worked out in the same way.

Write:

$$f(\underset{\sim}{r}) = \underline{\nabla} \cdot \frac{(\underset{\sim}{r} - \underline{a})}{|\underset{\sim}{r} - \underset{\sim}{a}|^2} \quad -(7)$$

where:

$$|\underset{\sim}{r} - \underset{\sim}{a}|^2 = (x - a_x)^2 + (y - a_y)^2 + (z - a_z)^2 \quad -(8)$$

and:

$$\underset{\sim}{r} - \underset{\sim}{a} = (x - a_x)\underline{i} + (y - a_y)\underline{j} + (z - a_z)\underline{k} \quad -(9)$$

so:

$$f(\underline{r}) = \frac{\partial}{\partial x}\frac{(x - a_x)}{|\underline{r} - \underline{a}|^2} + \frac{\partial}{\partial y}\frac{(y - a_y)}{|\underline{r} - \underline{a}|^2} + \frac{\partial}{\partial z}\frac{(z - a_z)}{|\underline{r} - \underline{a}|^2} \quad -(10)$$

$$= \frac{\partial}{\partial x}\frac{(x - a_x)}{(x - a_x)^2 + (y - a_y)^2 + (z - a_z)^2} + \cdots$$

Using the rules of differentiation:

3)

$$f(\underset{\sim}{r}) = \left((x-a_x)^2 + (y-a_y)^2 + (z-a_z)^2\right)^{-1}$$

$$- \frac{2(x-a_x)^2}{\left((x-a_x)^2 + (y-a_y)^2 + (z-a_z)^2\right)^2}$$

$$+ \dots$$

$$= \frac{1}{|\underset{\sim}{r} - \underline{a}|^2} - \frac{2}{|\underset{\sim}{r} - \underline{a}|^2}$$

$$= -\frac{1}{|\underset{\sim}{r} - \underline{a}|^2} \quad -(11)$$

$$\boxed{\underline{\nabla} \cdot \frac{(\underset{\sim}{r} - \underset{\sim}{r}_i)}{|\underset{\sim}{r} - \underset{\sim}{r}_i|^2} = -\frac{1}{|\underset{\sim}{r} - \underset{\sim}{r}_i|^2}}$$

$-(12)$

So the resonance equation is: $-(13)$

$$\boxed{\nabla^2 \phi - \frac{(\underset{\sim}{r} - \underset{\sim}{r}_i)}{|\underset{\sim}{r} - \underset{\sim}{r}_i|^2} \cdot \underline{\nabla} \phi + \frac{1}{|\underset{\sim}{r} - \underset{\sim}{r}_i|^2} \phi = -\frac{1}{\epsilon_0}\left(\sum_{i=1}^{n} q_i \, \delta(\underset{\sim}{r} - \underset{\sim}{r}_i)\right)}$$

4) which can be written as:

$$\boxed{|\underset{\sim}{r}-\underset{\sim}{r}_i|^2\nabla^2\phi-(\underset{\sim}{r}-\underset{\sim}{r}_i)\cdot\underline{\nabla}\phi+\phi = -\frac{|\underset{\sim}{r}-\underset{\sim}{r}_i|^2}{\epsilon_0}\sum_i q_i\,\delta(\underset{\sim}{r}-\underset{\sim}{r}_i) \quad -(14)}$$

In one dimension:

$$\underset{\sim}{r}-\underset{\sim}{r}_i = (z-z_i)\underline{k} \quad -(15)$$

$$|\underset{\sim}{r}-\underset{\sim}{r}_i| = z-z_i \quad -(16)$$

So eq. (14) becomes:

$$(z-z_i)^2\frac{\partial^2\phi}{\partial z^2}-(z-z_i)\frac{\partial\phi}{\partial z}+\phi = -\frac{(z-z_i)^2}{\epsilon_0}\sum_i q_i\,\delta(z-z_i) \quad -(17)$$

Finally reduce eqn. (17) with:

$$z-z_i = (z-z_i)_0\, e^{ikx} \quad -(18)$$

to a driven undamped oscillator.