Notes 63(7): Hartree Undamped Oscillator, One Dimension.

Start with the Hartree-type resonance equation:

$$(Z-Z_i)^2\frac{d^2\phi}{dZ^2}+(Z-Z_i)\frac{d\phi}{dZ}-\phi=-(Z-Z_i)^2\frac{\rho}{\epsilon_0} \quad -(1)$$

where a negative sign for the spin connection has been used, i.e.:

$$\underline{E}=-\underline{\nabla}\phi-\underline{\omega}\phi \quad -(2)$$
$$=-(\underline{\nabla}+\underline{\omega})\phi$$

Now use:

$$Z-Z_i=(Z-Z_i)_0\, e^{i\kappa_i x} \quad -(3)$$

i.e.

$$i\kappa x=\log_e\left(\frac{(Z-Z_i)}{(Z-Z_i)_0}\right)$$

so

$$\frac{dx}{dZ}=-\frac{i}{\kappa_i}\frac{1}{Z-Z_i} \quad -(4)$$

$$\frac{d\phi}{dZ}=\frac{d\phi}{dx}\frac{dx}{dZ}=-\frac{i}{\kappa_i}\cdot\frac{1}{Z-Z_i}\frac{d\phi}{dx} \quad -(5)$$

$$\frac{d^2\phi}{dZ^2}=-\frac{i}{\kappa_i}\frac{d}{dZ}\left(\frac{1}{Z-Z_i}\frac{d\phi}{dx}\right) \quad -(6)$$

$$=\frac{i}{\kappa_i(Z-Z_i)^2}\frac{d\phi}{dx}-\frac{i}{\kappa_i(Z-Z_i)}\frac{d}{dZ}\frac{d\phi}{dx}$$

From (5):

$$\frac{d}{dZ}\frac{d\phi}{dx}=-\frac{i}{\kappa_i}\cdot\frac{1}{Z-Z_i}\frac{d^2\phi}{dx^2} \quad -(7)$$

2)

From (6) and (7):

$$(z-z_i)^2 \frac{d^2\phi}{dz^2} = \frac{i}{\kappa_i}\frac{d\phi}{dx} - \frac{1}{\kappa_i^2}\frac{d^2\phi}{dx^2} \quad -(8)$$

From (5):

$$(z-z_i)\frac{d\phi}{dz} = -\frac{i}{\kappa_i}\frac{d\phi}{dx} \quad -(9)$$

From (8) and (9) & (1)

$$\frac{d^2\phi}{dx^2} + \kappa_i^2\phi = (z-z_i)^2 \frac{\rho}{\epsilon_0} \quad -(10)$$

or

$$\frac{d^2\phi}{dz^2} + \kappa_i^2\phi = \overset{\kappa_i^2}{\cancel{(z-z_i)^2}} \frac{\rho}{\epsilon_0} \quad -(10a)$$

$$: = \overset{\kappa_i^2}{\cancel{(z-z_i)^2}} \frac{\rho_0}{\epsilon_0} \cos(\kappa(z-z_i))$$

Let

$$\kappa_i = \frac{1}{(z-z_i)_0} \quad -(11)$$

and:

$$\frac{d^2\phi}{dz^2} + \kappa_i^2\phi = e^{2i\kappa_i z}\frac{\rho_0}{\epsilon_0}\cos(\kappa_i(z-z_i))$$

i.e

$$\boxed{\begin{aligned}\frac{d^2\phi}{dz^2} + \kappa_i^2\phi &= \frac{\rho_0}{\epsilon_0}\mathrm{Re}\left(e^{2i\kappa_i z}\cos(\kappa_i(z-z_i))\right) \\ &= \frac{\rho_0}{\epsilon_0}\cos(2\kappa_i z)\cos(\kappa_i(z-z_i))\end{aligned}}$$