1) 66(1) (Vol 5, paper 1) : The Second Bianchi Identity and Noether Theorem in Cartan Geometry

Firstly in paper 66, the derivation of the Einstein Hilbert field equation is given following vol. 2, pp. 186 ff. It starts with the second Bianchi identity:

$$[[D_\lambda, D_\rho], D_\sigma] + [[D_\rho, D_\sigma], D_\lambda] + [[D_\sigma, D_\lambda], D_\rho] := 0 \quad -(1)$$

which can be written as:

$$D_\lambda R_{\rho\sigma\mu\nu} + D_\rho R_{\sigma\lambda\mu\nu} + D_\sigma R_{\lambda\rho\mu\nu} := 0. \quad -(2)$$

Eq. (2) is equivalent to:

$$D^\mu \left(R_{\rho\mu} - \frac{1}{2} R g_{\rho\mu} \right) = 0 \quad -(3)$$

as shown on pp. 186 ff of vol. 2. The Noether Theorem is:

$$D^\mu T_{\rho\mu} = 0 \quad -(4)$$

and expresses conservation of energy and momentum. Here $T_{\rho\mu}$ is the symmetric canonical energy-momentum density. Einstein inferred the famous field equation by using eqs. (3) and (4), so:

$$G_{\rho\mu} := R_{\rho\mu} - \frac{1}{2} R g_{\rho\mu} = k T_{\rho\mu}. \quad -(5)$$

Hilbert derived eq. (5) from the lagrangian.

It is shown in paper 66 that eq. (5) is a

2) special case of:

$$\boxed{R^a{}_b = k T^a{}_b,} \quad -(6)$$

where $\omega^a{}_b$ is the spin connection of Cartan geometry. The second Bianchi identity in Cartan geometry is:

$$D \wedge R^a{}_b := 0 \quad -(7)$$

so the most general form of the Noether theorem is:

$$D \wedge T^a{}_b = 0. \quad -(8)$$

The most general form of the EH field equation is:

$$D \wedge R^a{}_b = k D \wedge T^a{}_b = 0. \quad -(9)$$

This includes torsion as well as curvature. The field equation (5) only considers curvature and a symmetric Ricci tensor $R_{\rho\mu}$ and symmetric canonical energy-momentum density tensor $T_{\rho\mu}$. The most general EH field equation in Riemann geometrical notation is:

$$D_\lambda R_{\rho\sigma\mu\nu} + D_\rho R_{\sigma\lambda\mu\nu} + D_\sigma R_{\lambda\rho\mu\nu}$$
$$= k \left(D_\lambda T_{\rho\sigma\mu\nu} + D_\rho T_{\sigma\lambda\mu\nu} + D_\sigma T_{\lambda\rho\mu\nu} \right) = 0.$$

$$-(10)$$

Eq. (10) in tensor notation is the same as eq. (6) in form notation.

3.) The canonical energy-momentum density in eq. (6) is:

$$T^a{}_{b\mu\nu} = -T^a{}_{b\nu\mu} \quad -(11)$$

The canonical energy-momentum density in eq. (10) is

$$T_{\rho\sigma\mu\nu} = -T_{\rho\sigma\nu\mu}. \quad -(12)$$

In general, there is asymmetry in a and b in eq. (11) and asymmetry in ρ and σ in eq. (12). It is important to note that the Noether theorem in eq. (4) holds only for the special case:

$$T_{\mu\nu} = T_{\nu\mu}. \quad -(13)$$

The Ricci tensor is the special case:

$$R_{\mu\nu} = R_{\nu\mu} \quad -(14)$$

and the metric in eq. (5) is the special case:

$$g_{\mu\nu} = g_{\nu\mu}. \quad -(15)$$

These special cases are true if:

$$T^\kappa_{\mu\nu} = \Gamma^\kappa_{\mu\nu} - \Gamma^\kappa_{\nu\mu} = 0 \quad -(16)$$

i.e.

$$T^a = 0. \quad -(17)$$

4.) Only in these special cases is it possible to contract the Riemann tensor uniquely to the Ricci tensor. This loses a lot of information and makes the development of a generally covariant unified field ~~theory~~ impossible, because the torsion and e/m field are missing. The most general Bianchi identity is either (2) or (6), and includes both torsion and curvature. Eq (2) can be reduced to eq. (3) if and only if $R_{\rho\mu}$ and $g_{\rho\mu}$ are symmetric, (i.e. if and only if the torsion tensor is zero and if and only if the Christoffel connection has the property (16)). It is to be noted that the Noether theorem can be written as:

$$D^{\mu}\left(T_{\rho\mu} - \frac{1}{2} T g_{\rho\mu}\right) = 0 \quad -(18)$$

where

$$T = g^{\sigma\tau} T_{\sigma\tau}. \quad -(19)$$

This is because:

$$\begin{aligned} \frac{1}{2} T g_{\rho\mu} &= \frac{1}{2} g^{\rho\mu} T_{\rho\mu} g_{\rho\mu} \\ &= \frac{1}{2} g^{\sigma\tau} T_{\rho\mu} g_{\sigma\tau} \quad -(20) \\ &= 2 T_{\rho\mu} \end{aligned}$$

because:

$$g^{\sigma\tau} g_{\sigma\tau} = 4. \quad -(21)$$

Thus eq. (18) is:

$$D^{\mu}(T_{\rho\mu} - 2T_{\rho\mu})\,s = 0, \quad -(22)$$

$$D^{\mu}T_{\rho\mu} = D^{\mu}T_{\rho\mu} \quad -(22a)$$

<u>Q.E.D.</u>

It follows that the Noether Theorem in the usual form:

$$D^{\mu}T_{\rho\mu} = 0, \quad -(23)$$

$$T_{\rho\mu} = T_{\mu\rho} \quad -(23a)$$

is a special case of eq. (10), because in order to contract eq. (10) to eq. (23), eq. (23a) has to be used.

This checks that the most general form of the Noether Theorem is eq. (8), and the most general form of the field equation of gravitation is eq. (6). In fact eq. (6) is <u>the basic equation of the unified field.</u>

6) The second Bianchi identity is:

$$d \wedge R^a{}_b = j^a{}_b = R^a{}_c \wedge \omega^c{}_b - \omega^a{}_c \wedge R^c{}_b \qquad -(24)$$

$$d \wedge \tilde{R}^a{}_b = \tilde{j}^a{}_b = \tilde{R}^a{}_c \wedge \omega^c{}_b - \omega^a{}_c \wedge \tilde{R}^c{}_b \qquad -(25)$$

so:

$$\boxed{\begin{aligned} d \wedge T^a{}_b &= \frac{1}{k} j^a{}_b \\ d \wedge \tilde{T}^a{}_b &= \frac{1}{k} \tilde{j}^a{}_b \end{aligned}} \qquad \begin{aligned} -(26) \\ -(27) \end{aligned}$$

The second Cartan structure equation is:

$$R^a{}_b = d \wedge \omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \qquad -(28)$$

$$= k T^a{}_b \qquad -(28a)$$

so:

$$\boxed{T^a{}_b = \frac{1}{k}\left(d \wedge \omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b\right)} \qquad -(29)$$

Eqs. (26), (27) and (29) define the energy-momentum density form $T^a{}_b$ in terms of the spin connection form. These are true in the presence of both curvature and torsion.

This analysis suggests that there exists an energy-momentum density τ^a defined by:

$$\tau^a_{\mu\nu} = - \tau^a_{\nu\mu} \quad — (30)$$

and which is defined by the torsion T^a rather than the curvature $R^a{}_b$. The first Bianchi identity is:

$$D \wedge T^a = R^a{}_b \wedge q^b \quad — (31)$$

$$= k T^a{}_b \wedge q^b \quad — (32)$$

$$= d \wedge T^a + \omega^a{}_b \wedge T^b \quad — (33)$$

So this defines the torsion form in terms of the canonical energy-momentum density $T^a{}_b$:

$$d \wedge T^a + \omega^a{}_b \wedge T^b = k T^a{}_b \wedge q^b . \quad — (34)$$

So if we define:

$$T^a = k_1 \tau^a \quad — (35)$$

the constants k_1 and k are related by:

$$k_1 D \wedge \tau^a = k T^a{}_b \wedge q^b . \quad — (36)$$

Now <u>define</u> τ^a by choosing:

$$k_1 = k . \quad — (37)$$

8)

The electromagnetic field is therefore:

$$\boxed{F^a = A^{(0)} T^a = A^{(0)} k \tau^a} \quad -(38)$$

The units of τ^a are the units of $T^a{}_b$ multiplied by meters.

The first Cartan structure equation is:

$$T^a = D \wedge q^a = d \wedge q^a + \omega^a{}_b \wedge q^b \quad -(39)$$

so:

$$\boxed{\tau^a = \frac{1}{k}\left(d \wedge q^a + \omega^a{}_b \wedge q^b\right)} \quad -(40)$$

and:

$$d \wedge F^a = \mu_0 j^a \quad -(41)$$

$$d \wedge \tilde{F}^a = \mu_0 \tilde{j}^a \quad -(42)$$

where:

$$j^a = \frac{1}{k}\frac{A^{(0)}}{\mu_0}\left(T^a{}_b \wedge q^b - \omega^a{}_b \wedge \tau^b\right) \quad -(43)$$

$$\tilde{j}^a = \frac{1}{k}\frac{A^{(0)}}{\mu_0}\left(\tilde{T}^a{}_b \wedge q^b - \omega^a{}_b \wedge \tilde{\tau}^b\right) \quad -(44)$$

so the currents j^a and $\tilde{j}^a$ are energy - momentum currents.

9)

If there is no interaction between gravitation and electromagnetism:

$$T^a{}_b \wedge q^b = \omega^a{}_b \wedge \tau^b \quad -(45)$$

and:

$$j^a = 0. \quad -(46)$$

In this case we regain the non-resonant Gauss Law and Faraday law of induction:

$$d \wedge F^a = 0 \quad -(47)$$

for each polarization index a. If there is interaction between gravitation and electromagnetism

$$j^a \neq 0 \quad -(48)$$

and resonance is possible in these two laws.

If eq. (41) is true, then its Hodge dual $\tilde{j}^a$ is defined entirely by the matter field with no torsion:

$$\tilde{j}^a = \frac{1}{k} \frac{A^{(0)}}{\mu_0} \tilde{T}^a{}_b \wedge q^b \quad -(49)$$

i.e. eq. (49) is determined entirely by curvature (mass). In the electromagnetic field in this case ($j^a = 0$):

$$\tilde{T}^a{}_b \wedge q^b = \omega^a{}_b \wedge \tilde{\tau}^b \quad -(50)$$

or

$$\tilde{R}^a{}_b \wedge q^b = \omega^a{}_b \wedge \tilde{T}^b. \quad -(51)$$

In the absence of interaction between gravitation and electromagnetism:

$$\omega^a{}_b = -\frac{1}{2}\kappa\epsilon^a{}_{bc}\,q^c \quad -(52)$$

$$R^a{}_b = -\frac{1}{2}\kappa\epsilon^a{}_{bc}\,T^c \quad -(53)$$

so:

$$T^a{}_b = -\frac{1}{2}\kappa\epsilon^a{}_{bc}\,\tau^c. \quad -(54)$$

This means that in the absence of interaction, T^a and τ^a are confined to the electromagnetic field, and $R^a{}_b$ is confined to the matter field, i.e. to curvature by mass. In this case the $R^a{}_b$ of the field is the dual of T^c of the field as in eq. (53), but the $R^a{}_b$ of the mass is not the dual of a torsion. This is the standard model theory (MH theory):

$$d \wedge F = 0 \quad -(55)$$

$$d \wedge \tilde{F} = \mu_0 \tilde{J}. \quad -(56)$$