66(2): Second Bianchi Identity, Equivalence of Cartan and Riemann Geometry.

Cartan Geometry

$$d \wedge R^a{}_b + \omega^a{}_c \wedge R^c{}_b - R^a{}_c \wedge \omega^c{}_b := 0 \quad -(1)$$

Riemann Geometry

$$D_\rho R^\kappa{}_{\sigma\mu\nu} + D_\mu R^\kappa{}_{\sigma\nu\rho} + D_\nu R^\kappa{}_{\sigma\rho\mu} := 0 \quad -(2)$$

Equations (1) and (2) are fully equivalent in the presence of both curvature and torsion. Eq (1) can be written as:

$$\boxed{D \wedge R^a{}_b := 0} \quad -(3)$$

Proof

Firstly translate to tensor notation:

$$d \wedge R^a{}_b = \partial_\mu R^a{}_{b\nu\sigma} + \partial_\sigma R^a{}_{b\mu\nu} + \partial_\nu R^a{}_{b\sigma\mu}$$

$$R^a{}_c \wedge \omega^c{}_b = R^a{}_{c\mu\nu}\, \omega^c{}_{\sigma b} + R^a{}_{c\sigma\mu}\, \omega^c{}_{\nu b} + R^a{}_{c\nu\sigma}\, \omega^c{}_{\mu b}$$

$$\omega^a{}_c \wedge R^c{}_b = \omega^a_{\mu c} R^c{}_{b\nu\sigma} + \omega^a_{\sigma c} R^c{}_{b\mu\nu} + \omega^a_{\nu c} R^c{}_{b\sigma\mu} \quad -(4)$$

and evaluate:

$$\partial_\mu R^a{}_{b\nu\sigma} + \omega^a_{\mu c} R^c{}_{b\nu\sigma} - R^a{}_{c\nu\sigma}\, \omega^c_{\mu b} + \ldots\ldots := 0 \quad -(5)$$

2)

Use the definitions:

$$\left.\begin{aligned} R^a{}_{b\nu\sigma} &= q^a{}_\kappa q^\mu{}_b R^\kappa{}_{\mu\nu\sigma} \\ R^c{}_{b\nu\sigma} &= q^c{}_\kappa q^\mu{}_b R^\kappa{}_{\mu\nu\sigma} \\ R^a{}_{c\nu\sigma} &= q^a{}_\kappa q^\mu{}_c R^\kappa{}_{\mu\nu\sigma}, \end{aligned}\right\} \quad -(6)$$

and the tetrad postulates:

$$\partial_\mu q^a{}_\lambda + \omega^a{}_{\mu c} q^c{}_\lambda - \Gamma^\nu{}_{\mu\lambda} q^a{}_\nu = 0 \quad -(7)$$

$$\partial_\mu q^c{}_\lambda + \omega^c{}_{\mu b} q^b{}_\lambda - \Gamma^\nu{}_{\mu\lambda} q^c{}_\nu = 0 \quad -(8)$$

From (7):

$$\omega^a{}_{\mu c} = q^\lambda{}_c \left(\Gamma^\nu{}_{\mu\lambda} q^a{}_\nu - \partial_\mu q^a{}_\lambda \right). \quad -(9)$$

From (8):

$$\omega^c{}_{\mu b} = q^\lambda{}_b \left(\Gamma^\nu{}_{\mu\lambda} q^c{}_\nu - \partial_\mu q^c{}_\lambda \right). \quad -(10)$$

Differentiating:

$$\partial_\mu \left(q^a{}_\kappa q^\tau{}_b R^\kappa{}_{\tau\nu\sigma} \right) = \left(\partial_\mu q^a{}_\kappa \right) q^\tau{}_b R^\kappa{}_{\tau\nu\sigma} + q^a{}_\kappa \left(\partial_\mu q^\tau{}_b \right) R^\kappa{}_{\tau\nu\sigma} + q^a{}_\kappa q^\tau{}_b \partial_\mu R^\kappa{}_{\tau\nu\sigma} \quad -(11)$$

$$= \partial_\mu R^a{}_{b\nu\sigma}$$

3) The second and third terms in eq (5) are:

$$\omega_{\mu}{}^{a}{}_{c} R^{c}{}_{b\nu\sigma} = q^{\lambda}{}_{c}\left(\Gamma^{\nu}{}_{\mu\lambda} - \partial_\mu q^{a}{}_{\lambda}\right) q^{c}{}_{\alpha} q^{\beta}{}_{b} R^{\alpha}{}_{\beta\nu\sigma} \quad -(12)$$

and

$$-R^{a}{}_{c\nu\sigma}\,\omega_{\mu}{}^{c}{}_{b} = -q^{a}{}_{\kappa} q^{\mu}{}_{c} R^{\kappa}{}_{\mu\nu\sigma} q^{\lambda}{}_{b}\left(\Gamma^{\nu}{}_{\mu\lambda} q^{c}{}_{\nu} - \partial_\mu q^{c}{}_{\lambda}\right) \quad -(13)$$

Add eqs (11) to (13) and cancel terms as follows.

$$q^{\tau}{}_{b}\left(\partial_\mu q^{a}{}_{\kappa}\right) R^{\kappa}{}_{\tau\nu\sigma} - q^{\lambda}{}_{c}\left(\partial_\mu q^{a}{}_{\lambda}\right) q^{c}{}_{\alpha} q^{\beta}{}_{b} R^{\alpha}{}_{\beta\nu\sigma} = 0 \quad -(14)$$

using the dummy index transformations:

$$\alpha \to \kappa, \quad \beta \to \tau, \quad \lambda \to \kappa \quad -(15)$$

and

$$q^{\kappa}{}_{c}\, q^{c}{}_{\kappa} = 1; \quad -(16)$$

and:

$$q^{a}{}_{\kappa}\left(\partial_\mu q^{\tau}{}_{b}\right) R^{\kappa}{}_{\tau\nu\sigma} + q^{a}{}_{\kappa} q^{\mu}{}_{c} q^{\lambda}{}_{b} R^{\kappa}{}_{\mu\nu\sigma}\left(\partial_\mu q^{c}{}_{\lambda}\right) = 0 \quad -(17)$$

using:

$$\mu \to \tau \quad -(18)$$

and:

$$\partial_\mu q^{\tau}{}_{b} = -q^{\tau}{}_{c}\, q^{\lambda}{}_{b}\, \partial_\mu q^{c}{}_{\lambda} \quad -(19)$$
$$= -q^{\tau}{}_{c}\, q^{\nu}{}_{b}\, \partial_\mu q^{c}{}_{\nu}.$$

To prove eq. (19) use:

4)

$$q^{\tau}{}_{b} q^{b}{}_{\tau} = 1, \quad -(20)$$

so: $$\partial_\mu \left(q^{\tau}{}_{b} q^{b}{}_{\tau} \right) = q^{b}{}_{\tau} \partial_\mu q^{\tau}{}_{b} + q^{\tau}{}_{b} \partial_\mu q^{b}{}_{\tau}$$
$$= 0 \quad -(21)$$

and $$\partial_\mu q^{\tau}{}_{b} + q^{\tau}{}_{b} q^{\tau}{}_{b} \partial_\mu q^{b}{}_{\tau} = 0 \quad -(22)$$

i.e. $$\partial_\mu q^{\tau}{}_{b} = - q^{\tau}{}_{c} q^{\tau}{}_{b} \partial_\mu q^{c}{}_{\tau} \quad -(23)$$
$$= - q^{\tau}{}_{c} q^{\lambda}{}_{b} \partial_\mu q^{c}{}_{\lambda}$$

Q.E.D.

So the sum in eq. (5) is:

$$q^{a}{}_{\kappa} q^{\tau}{}_{b} \partial_\mu R^{\kappa}{}_{\tau\nu\sigma} + q^{\lambda}{}_{c} q^{c}{}_{\alpha} q^{\beta}{}_{b} q^{a}{}_{\nu} \Gamma^{\nu}_{\mu\lambda} R^{\alpha}{}_{\beta\nu\sigma}$$
$$- q^{a}{}_{\kappa} q^{\mu}{}_{c} q^{\lambda}{}_{b} q^{c}{}_{\nu} \Gamma^{\nu}_{\mu\lambda} R^{\kappa}{}_{\mu\nu\sigma} \quad -(24)$$
$$+ \ldots \quad := 0.$$

Using dummy index transformation this sum is:

$$\partial_\mu R^{\kappa}{}_{\tau\nu\sigma} + q^{\lambda}{}_{a} q^{a}{}_{\nu} \Gamma^{\nu}_{\mu\lambda} R^{\kappa}{}_{\tau\nu\sigma}$$
$$- q^{\lambda}{}_{b} q^{b}{}_{\nu} \Gamma^{\nu}_{\mu\lambda} R^{\kappa}{}_{\tau\nu\sigma} + \ldots := 0$$
$$-(25)$$

The covariant derivative of the Riemann Tensor is:

5)

$$D_\mu R^\kappa{}_{\tau\nu\sigma} = \partial_\mu R^\kappa{}_{\tau\nu\sigma} - \Gamma^\lambda_{\mu\nu} R^\kappa{}_{\tau\lambda\sigma} - \Gamma^\lambda_{\mu\sigma} R^\kappa{}_{\tau\nu\lambda}$$

$$-(26)$$

The first term in eq. (25) is given by the dummy index transformation:

$$\lambda \rightarrow \nu, \quad \nu \rightarrow \lambda \qquad -(27)$$

and becomes the second term on the RHS of eq (26).

Using the antisymmetry of the Riemann tensor in its last two indices the second term in eq (25) is:

$$- g^\lambda{}_a\, g^a{}_\nu\, \Gamma^\nu_{\mu\lambda} R^\kappa{}_{\tau\sigma\nu}$$

$$= - \Gamma^\lambda_{\mu\lambda} R^\kappa{}_{\tau\sigma\lambda} \qquad -(28)$$

If

$$\Gamma^\lambda_{\boxed{\mu\lambda}} := \Gamma^\lambda_{\boxed{\nu\mu}} \qquad -(29)$$

this term becomes: $- \Gamma^\lambda_{\nu\mu} R^\kappa{}_{\tau\sigma\lambda}$.

Using eqns (2) and (26) it is seen that:

$$\boxed{\partial_\mu R^\kappa{}_{\tau\nu\sigma} - \Gamma^\lambda_{\mu\nu} R^\kappa{}_{\tau\lambda\sigma}} - \Gamma^\lambda_{\mu\sigma} R^\kappa{}_{\tau\nu\lambda}$$

$$+ \partial_\sigma R^\kappa{}_{\tau\mu\nu} - \Gamma^\lambda_{\sigma\mu} R^\kappa{}_{\tau\lambda\nu} - \Gamma^\lambda_{\sigma\nu} R^\kappa{}_{\tau\mu\lambda}$$

$$+ \partial_\nu R^\kappa{}_{\tau\sigma\mu} - \Gamma^\lambda_{\nu\sigma} R^\kappa{}_{\tau\lambda\mu} \boxed{- \Gamma^\lambda_{\nu\mu} R^\kappa{}_{\tau\sigma\lambda}}$$

$$= 0 \qquad -(30)$$

so the three terms in eq. (5) produce the three dotted terms in eq (30). This completes the proof, Q.E.D.

Discussion

The Cartan equation (3) is much simpler than the Riemann equation (30), but contains the same information. The tetrad postulate is needed to demonstrate the equivalence of Riemann & Cartan geometry.