6(b). Basic Definition of Angular Momentum as a Tetrad.

The angular momentum was defined in paper SS as a tetrad:

$$J^a_{\ \mu} = J^{(0)} q^a_{\ \mu} \quad -(1)$$

which implies that there exist vectors W^a and V^μ:

$$W^a = J^a_{\ \mu} V^\mu \quad -(2)$$

i.e.

$$\begin{bmatrix} W^1 \\ W^2 \\ W^3 \end{bmatrix} = \begin{bmatrix} J^1_{\ 1} & J^1_{\ 2} & J^1_{\ 3} \\ J^2_{\ 1} & J^2_{\ 2} & J^2_{\ 3} \\ J^3_{\ 1} & J^3_{\ 2} & J^3_{\ 3} \end{bmatrix} \begin{bmatrix} V^1 \\ V^2 \\ V^3 \end{bmatrix} \quad -(3)$$

if we restrict consideration to three space dimensions.

Thus:

$$W^1 = J^1_{\ 1} V^1 + J^1_{\ 2} V^2 + J^1_{\ 3} V^3$$
$$W^2 = J^2_{\ 1} V^1 + J^2_{\ 2} V^2 + J^2_{\ 3} V^3$$
$$W^3 = J^3_{\ 1} V^1 + J^3_{\ 2} V^2 + J^3_{\ 3} V^3$$
$$-(4)$$

It is known that angular momentum must be an antisymmetric tensor so:

$$\left.\begin{aligned} W^1 &= J^1_{\ 2} V^2 + J^1_{\ 3} V^3 \\ W^2 &= J^2_{\ 1} V^1 + J^2_{\ 3} V^3 \\ W^3 &= J^3_{\ 1} V^1 + J^3_{\ 2} V^2 \end{aligned}\right\} \quad -(5)$$

In 3-D space we can write this as:

2)

$$\left.\begin{aligned} W_1 &= J_{12} V_2 + J_{13} V_3 \\ W_2 &= J_{21} V_1 + J_{23} V_3 \\ W_3 &= J_{31} V_1 + J_{32} V_2 \end{aligned}\right\} \quad -(6)$$

where:

$$J_{ij} = \frac{1}{2} \epsilon_{ijk} J_k . \quad -(7)$$

So:

$$\left.\begin{aligned} W_1 &= J_3 V_2 - J_2 V_3 \\ W_2 &= J_3 V_1 - J_1 V_3 \\ W_3 &= J_1 V_2 - J_2 V_1 \end{aligned}\right\} \quad -(8)$$

It is seen that:

$$\underline{W} = \underline{J} \times \underline{V} \quad -(9)$$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ J_1 & J_2 & J_3 \\ V_1 & V_2 & V_3 \end{vmatrix}$$

Now identify:

$$\underline{W} = \underline{p} , \quad \underline{V} = \underline{r} / r^2 \quad -(10)$$

so:

$$\underline{p} = \underline{J} \times \underline{r} / r^2 \quad -(11)$$

i.e.

$$\boxed{\underline{J} = \underline{r} \times \underline{p}} \quad -(12)$$

3)

From paper 55, :

$$\underline{N}^a \text{ (torque)} = c J^{(0)} \underline{T}^a \text{ (torsion)} \quad -(13)$$

and

$$\boxed{N^a = c \left(d \wedge J^a + \omega^a{}_b \wedge J^b \right)} \quad -(14)$$

This is the definition of torque (in joules) in a spacetime which in general has non-zero torsion and curvature.

In paper 76, a constant torsion model is used:

$$N^a = \text{constant} \quad -(15)$$

$$d \wedge N^a = 0 \quad -(16)$$

with

$$\omega^a{}_b = -\frac{1}{2} \kappa \epsilon^a{}_{bc} q^c \quad -(17)$$

Adopting the complex circular basis:

$$\underline{N}^{(1)*} = c \left(\underline{\nabla} \times \underline{J}^{(1)*} - i \kappa \underline{q}^{(2)} \times \underline{J}^{(3)} \right)$$

et cyclicum $\quad -(18)$

and the constant torque is:

4)

$$\left.\begin{aligned} \underline{N}^{(3)*} &= -i\kappa c \underline{q}^{(1)} \times \underline{J}^{(2)} \\ &= c\kappa J^{(0)} \underline{k} \\ &= c\kappa m r v \underline{k} \end{aligned}\right\} \quad -(19)$$

Thus:

$$\boxed{\underline{N} = c m v \underline{k}} \quad -(20)$$

because:

$$\kappa r = 1. \quad -(21)$$

<u>This is the precise analogy of $\underline{B}^{(3)}$ theory in electrodynamics.</u>

As shown in notes 76(5),

$$\left.\begin{aligned} \theta &= v_0 \tau / r \\ r &= v_0 \tau / \theta \end{aligned}\right\} \quad -(22)$$

from the angular momentum:

$$\omega = \frac{d\theta}{dt} = \frac{v}{r} \quad -(23)$$

with $v = v_0 =$ constant. Eq (22) is the equation of the <u>hyperbolic spiral</u>.