2) : Electron - Positron Collision in ECE

Classical Special Relativity

Before collision and in the special relativistic limit:

$$p^\mu p_\mu = m^2 c^2 \text{ (electron)} \quad -(1)$$

$$p^\mu p_\mu = m^2 c^2 \text{ (positron)} \quad -(2)$$

$$p^\mu = \left(\frac{E_n}{c}, 0, 0, p_Z \right) \text{ (electron)} \quad -(3)$$

$$p^\mu = \left(\frac{E_n}{c}, 0, 0, -p_Z \right) \text{ (positron)} \quad -(4)$$

$$\text{Total energy} = 2E_n = mc^2 \quad -(5)$$

$$\text{Total momentum} = \underline{0} \quad -(6)$$

After collision:

$$p^\mu = \left(\frac{\hbar\omega}{c}, 0, 0, \hbar\kappa_Z \right) \text{ (photon one)} \quad -(7)$$

$$p^\mu = \left(\frac{\hbar\omega}{c}, 0, 0, -\hbar\kappa_Z \right) \text{ (photon two)} \quad -(8)$$

$$\text{Total energy} = 2\hbar\omega = 2mc^2 \quad -(9)$$

$$\text{Total momentum} = \underline{0} \quad -(10)$$

Total energy and linear momentum are conserved, rest energies are changed into photon energies. On the classical level there are no electron or positron spins. There are no virtual photons. This is a pure particle theory with no fields.

2)

The attraction between the electron and positron is now introduced using the minimal prescription:

$$p^\mu \rightarrow p^\mu + eA^\mu \quad (\text{electron}) \quad — (11)$$

$$p^\mu \rightarrow p^\mu + eA^\mu \quad (\text{positron}) \quad — (12)$$

where eA^μ represents an increase of energy momentum for both particles due to mutual attraction. So:

$$p^\mu = \left(\frac{E_n + e\phi}{c}, 0, 0, p_z + eA_z \right) \quad (\text{electron}) \quad — (13)$$

$$p^\mu = \left(\frac{E_n + e\phi}{c}, 0, 0, -(p_z + eA_z) \right) \quad (\text{positron}) \quad — (14)$$

$$\text{Total energy} = 2(E_n + e\phi) = mc^2 \quad — (15)$$

$$\text{Total momentum} = \underline{0} \quad — (16)$$

<u>Special Relativistic Quantum Mechanics</u>

The free electron is described by the Dirac equation:

$$\left(\Box + \left(\frac{mc}{\hbar} \right)^2 \right) \psi_e = 0 \quad — (17)$$

where:

$$\Box = \partial^\mu \partial_\mu \quad — (18)$$

and

$$\partial^\mu = \left(\frac{1}{c} \frac{\partial}{\partial t}, 0, 0, -\frac{\partial}{\partial Z} \right) \quad — (19)$$

$$\partial_\mu = \left(\frac{1}{c} \frac{\partial}{\partial t}, 0, 0, \frac{\partial}{\partial Z} \right) \quad — (20)$$

$$\Box = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial Z^2} \quad — (21)$$

3) Here ψ_e is the Dirac spinor of the electron, travelling in $+Z$. The free positron travels in $-Z$ and is described by:

$$\left(\Box + \left(\frac{mc}{\hbar} \right)^2 \right) \psi_p = 0 \quad -(22)$$

where ψ_p is the Dirac spinor of the positron. The phases of the spinors ψ_e and ψ_p are different, because e^- and e^+ are moving in opposite directions. In both cases, $\Box$ is given by eq. (21), so:

$$\left(\Box + \left(\frac{mc}{\hbar} \right)^2 \right) \psi_e \psi_p = 0. \quad -(23)$$

This has to be anti-symmetrized because of the Pauli exclusion principle.

The classical attraction between electron and positron changes the d'Alembertian as follows:

$$\Box = -\frac{1}{\hbar^2} p^\mu p_\mu$$

$$= -\frac{1}{\hbar^2} \left(p^\mu + eA^\mu \right) \left(p_\mu + eA_\mu \right)$$

$$= \Box - \frac{e}{\hbar^2} \left(A^\mu p_\mu + p^\mu A_\mu + eA^\mu A_\mu \right) \quad -(24)$$

where:

$$p^\mu = i\hbar \partial^\mu \quad -(25)$$

$$p_\mu = i\hbar \partial_\mu \quad -(26)$$

So:

$$\square \rightarrow \square - \frac{ie}{\hbar}\left(A^{\mu}\partial_{\mu} + \partial^{\mu}A_{\mu}\right) - \frac{e^2}{\hbar}A^{\mu}A_{\mu} \quad -(27)$$

So eq. (23) becomes:

$$\boxed{\left(\square - \frac{ie}{\hbar}\left(A^{\mu}\partial_{\mu} + \partial^{\mu}A_{\mu}\right) - \frac{e^2}{\hbar^2}A^{\mu}A_{\mu}\right)\psi_e\psi_p = 0} \quad -(2$$

This is the Dirac equation of positronium.

After Collision

There are two Proca equations for two photons moving in opposite direction:

$$\boxed{\left(\square + \left(\frac{m_\gamma c}{\hbar}\right)^2\right)A^a_{\mu} = 0} \quad -(29)$$

$$\square = p^{\mu}p_{\mu}, \quad p^{\mu} = \left(\frac{\hbar\omega}{c}, 0, 0, \hbar\kappa_z\right) \quad -(30)$$

and eqs. (29) and (30) for:

$$p^{\mu} = \left(\frac{\hbar\omega}{c}, 0, 0, -\hbar\kappa_z\right) \quad -(31)$$

Eqs. (28) and (29) have to be solved simultaneously in general. This is a numerical problem.

5)

Notes

The potential in eq. (28) refers to the classical Coulombic interaction, while that in eq. (29) refers to the quantized e/m field associated with the two γ rays. Eqs. (28) and (29) mean that the e^+ and e^- have transmuted into 2γ:

$$\underbrace{(e^+ + e^-)}_{\text{positronium}} = 2\gamma. \quad -(32)$$

If the photon mass is neglected eq. (29) is:

$$\Box A^a_\mu = 0. \quad -(33)$$

For one photon, moving along $+Z$:

$$\left.\begin{aligned} \underline{A}^{(1)} &= \frac{A^{(0)}}{\sqrt{2}}(\underline{i} - i\underline{j})e^{i\phi} \\ \phi &= \omega t - KZ \end{aligned}\right\} -(34)$$

For the other photon:

$$\phi = \omega t + KZ \quad -(35)$$

So the whole problem can be set up and solved without using virtual photons or renormalization.