1)

Calculation for $2p_z$

$$\varphi_0(2p_z) = \frac{1}{4}\frac{r}{a^{5/2}} e^{-r/(2a)} \quad -(1)$$

$$\frac{d\varphi_0}{dt} = \frac{e^{-r/(2a)}}{4a^{5/2}} - \frac{1}{8a^{7/2}} e^{-r/(2a)} \quad -(2)$$

$$\boxed{\frac{d\varphi_0}{dt} = \frac{1}{4a^{5/2}}\left(1-\frac{r}{2a}\right)e^{-r/(2a)}} \quad -(3)$$

$$\frac{d^2\varphi_0}{dt^2} = -\frac{1}{8a^{7/2}} e^{-r/(2a)} - \frac{1}{8a^{7/2}}\left(1-\frac{r}{2a}\right)e^{-r/(2a)}$$

$$= -\frac{1}{4a^{7/2}} e^{-r/(2a)} + \frac{r}{16a^{9/2}} e^{-r/(2a)}$$

$$\boxed{\frac{d^2\varphi_0}{dt^2} = \frac{1}{4a^{7/2}} e^{-r/(2a)}\left(\frac{r}{4a}-1\right)} \quad -(4)$$

$$\nabla^2\varphi_0 = \frac{e^{-r/(2a)}}{4a^{7/2}}\left(\frac{r}{4a}-1\right) + \frac{1 \cdot e^{-r/(2a)}}{2ra^{5/2}}\left(1-\frac{r}{2a}\right)$$

$$\boxed{\nabla^2\varphi_0 = \frac{e^{-r/(2a)}}{2a^{5/2}}\left(\frac{1}{2a}\left(\frac{r}{4a}-1\right) + \frac{1}{r}\left(1-\frac{r}{2a}\right)\right)}$$

$$-(5)$$

2)

$$= -\frac{4mc}{\hbar}\cdot\frac{1}{4}\frac{r}{a^{5/2}}e^{-r/(2a)}\frac{1}{r_{vac}}$$

Therefore:

$$\frac{1}{2}\left(\frac{1}{2a}\left(\frac{r}{4a}-1\right)+\frac{1}{r}\left(1-\frac{r}{2a}\right)\right) = -\frac{mc}{\hbar}\frac{r}{r_{vac}}$$

$$\frac{1}{r_{vac}} = -\frac{\hbar}{2mcr}\left(\frac{1}{2a}\left(\frac{r}{4a}-1\right)+\frac{1}{r}\left(1-\frac{r}{2a}\right)\right)$$

$$\boxed{\frac{1}{r_{vac}} = \frac{\hbar}{2mca}\left(\frac{1}{2r}\left(1-\frac{r}{4a}\right)+\frac{a}{r^2}\left(\frac{r}{2a}-1\right)\right)} \qquad \text{— (7)}$$

At $r = 2a$:

$$\boxed{\frac{1}{r_{vac}} = \frac{\hbar}{2mca}\left(\frac{1}{4a}\right) = \frac{0.0180}{a}}$$

Summary Sheet

Relative Wavenumber Shift

$$\frac{\Delta\tilde{\nu}}{\tilde{\nu}} = \frac{\alpha}{2\pi} \cdot \frac{a}{r_{vac}} \quad -(1)$$

1s Orbital

$$\frac{1}{r_{vac}}(1s) = \frac{\hbar}{2mca} \cdot \frac{1}{2\pi^{1/2}} \left(\frac{1}{r} - \frac{1}{2a} \right) -(2)$$

2s Orbital

$$\frac{1}{r_{vac}}(2s) = \frac{\hbar}{2mca} \cdot \frac{1}{2\pi^{1/2}} \frac{\left(\frac{2}{r} - \frac{5}{4a} + \frac{r}{8a^2} \right)}{\left(2 - \frac{r}{a} \right)} -(3)$$

$2p_z$ Orbital

At $\cos\theta = 1$:

$$\frac{1}{r_{vac}}(2p_z) = \frac{\hbar}{2mca} \cdot \frac{1}{2\pi^{1/2}} \cdot \left(\frac{1}{2r}\left(1 - \frac{r}{4a}\right) + \frac{a}{r^2}\left(\frac{r}{2a} - 1\right)\right)$$

$-(4)$

Suggest fitting r from $2s - 2p$.

$$\frac{\Delta\tilde{\nu}}{\tilde{\nu}} = \alpha \left(\frac{\hbar}{mc}\right) \cdot \frac{1}{4\pi^{3/2}} \left(\left(\frac{2}{r} - \frac{5}{4a} + \frac{r}{8a^2} \right) \Big/ \left(2 - \frac{r}{a} \right) - \left(\frac{1}{2r}\left(1 - \frac{r}{4a}\right) + \frac{a}{r^2}\left(\frac{r}{2a} - 1\right) \right) \right) -(5)$$