Check, Notes 92(4), Reduction to Coulomb Law

$$\frac{2}{r} + \omega_r = 2\beta \quad -(1)$$

$$\frac{1}{r^2}\left(2r\omega_r + r^2\frac{\partial \omega_r}{\partial r}\right) = \kappa_0^2 \quad -(2)$$

$$\omega_r = 2\beta - \frac{2}{r} \quad -(3)$$

$$\kappa_0^2 = \frac{2}{r}\left(2\beta - \frac{2}{r}\right) + \frac{\partial \omega_r}{\partial r} \quad -(4)$$

If $\beta = 0$

$$\omega_r = -\frac{2}{r}, \quad \frac{\partial \omega_r}{\partial r} = \frac{2}{r^2}, \quad -(5)$$

$$\kappa_0^2 = -\frac{2}{r^2} \quad -(6), \quad \kappa_0 = \kappa_0' + i\kappa_0'' \quad -(6a)$$

In general:

$$\omega_r = 2\left(\beta - \frac{1}{r}\right) \quad -(7)$$

$$\kappa_0^2 = \frac{4}{r}\left(\beta - \frac{1}{r}\right) + \frac{\partial \omega_r}{\partial r} \quad -(8)$$

and

$$\frac{\partial^2 \phi}{\partial r^2} + 2\beta\frac{\partial \phi}{\partial r} + \kappa_0^2\phi = -\frac{\rho}{\epsilon_0} \quad -(9)$$

Reduction to Coulomb Law

This occurs when:

2) This occurs when:

$$\beta = \frac{1}{r} \quad \text{— (10)}$$

Then:

$$\omega_r = 0 \quad \text{— (11)}$$

$$\kappa_0^2 = 0 \quad \text{— (12)}$$

Conclusions

1) When $\beta = \frac{1}{r}$, the Coulomb law is regained.

2) When $\beta = 0$, — (13)

$$\kappa_0'^2 - \kappa_0''^2 = -\frac{2}{r^2} \quad \text{— (14)}$$

Eq. (13) means that:

$\phi \to \infty$ at resonance

$$\frac{\partial^2 \phi}{\partial r^2} + \left(\kappa_0'^2 + \kappa_0''^2\right) \phi = -\frac{\rho}{\epsilon_0} \quad \text{— (15)}$$

where:

$$\kappa_0 = \kappa_0' + i \kappa_0'' \quad \text{— (16)}$$

$$\kappa_0^* = \kappa_0' - i \kappa_0'' \quad \text{— (17)}$$

$$\kappa_0 \kappa_0^* = \kappa_0'^2 + \kappa_0''^2 \; ; \; \kappa_0^2 = \kappa_0'^2 - \kappa_0''^2 = -\frac{2}{r^2} \quad \text{— (18)}$$