APPENDIX 5, CHRISTOFFEL SYMBOLS AND RIEMANN ELEMENTS

In this appendix full details are given of the calculation of the non-vanishing Christoffel symbols and Riemann elements of the line element { 11 }:

$$ds^2 = -e^{2\alpha} dt^2 + e^{2\beta} dr^2 + r^2 d\Omega^2 \quad -(E1)$$

which can be considered as the spherically symmetric line element in four dimensions. This is written in spherical polar coordinates and α and β are in general functions of r and t. The metric elements are:

$$g_{00} = -e^{2\alpha}, \; g_{11} = e^{2\beta}, \; g_{22} = r^2, \; g_{33} = r^2 \sin^2\theta \quad -(E2)$$

and the inverse metric elements are:

$$g^{00} = -e^{-2\alpha}, \; g^{11} = e^{-2\beta}, \; g^{22} = r^{-2}, \; g^{33} = \frac{1}{r^2 \sin^2\theta} \quad -(E3)$$

in the spherical polar coordinate system (r , θ , ϕ). The Christoffel symbol in Riemann geometry is:

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} \right) \quad -(E4)$$

where summation is implied over repeated indices in the covariant - contravariant system. The calculations are given in full here for ease of reference. The non-vanishing Christoffel symbols are as follows.

1)
$$\Gamma^{0}_{00} = \frac{1}{2} g^{00} \left(\partial_0 g_{00} + \partial g_{00} - \partial_0 g_{00} \right)$$
$$= \frac{1}{2} g^{00} \partial_0 g_{00} \quad -(E5)$$

where:

$$\partial_0 g_{00} = -\partial_0\left(e^{2\alpha}\right) = -2\left(\partial_0\alpha\right)e^{2\alpha}. \quad -(E6)$$

Thus:

$$\Gamma^0_{00} = \partial_0\alpha \quad -(E7)$$

2) $$\Gamma^0_{01} = \frac{1}{2}g^{0\rho}\left(\partial_0 g_{1\rho} + \partial_1 g_{\rho 0} - \partial_\rho g_{01}\right)$$

$$= \frac{1}{2}g^{00}\partial_1 g_{00} = \partial_1\alpha. \quad -(E8)$$

3) $$\Gamma^1_{00} = \frac{1}{2}g^{1\rho}\left(\partial_0 g_{0\rho} + \partial_0 g_{\rho 0} - \partial_\rho g_{00}\right)$$

$$= -\frac{1}{2}g^{11}\partial_1 g_{00} = e^{(2\alpha - 2\beta)}\partial_1\alpha \quad -(E9)$$

4) $$\Gamma^0_{11} = \frac{1}{2}g^{0\rho}\left(\partial_1 g_{1\rho} + \partial_1 g_{\rho 1} - \partial_\rho g_{11}\right)$$

$$= -\frac{1}{2}g^{00}\partial_0 g_{11} = e^{(2\beta - 2\alpha)}\partial_0\beta. \quad -(E10)$$

5) $$\Gamma^1_{01} = \frac{1}{2}g^{1\rho}\left(\partial_0 g_{1\rho} + \partial_1 g_{\rho 0} - \partial_\rho g_{01}\right)$$

$$= \frac{1}{2}g^{11}\partial_0 g_{11} = \partial_0\beta. \quad -(E11)$$

6) $$\Gamma^1_{11} = \frac{1}{2}g^{1\rho}\left(\partial_1 g_{1\rho} + \partial_1 g_{\rho 1} - \partial_\rho g_{11}\right)$$

$$= \frac{1}{2}g^{11}\partial_1 g_{11} = \partial_1\beta. \quad -(E12)$$

7) $$\Gamma^2_{12} = \frac{1}{2} g^{2\rho}\left(\partial_1 g_{2\rho} + \partial_2 g_{\rho 1} - \partial_\rho g_{12}\right)$$
$$= \frac{1}{2} g^{22}\partial_1 g_{22} = \frac{1}{r}. \quad \text{— (E13)}$$

8) $$\Gamma^3_{13} = \frac{1}{2} g^{3\rho}\left(\partial_1 g_{3\rho} + \partial_3 g_{\rho 1} - \partial_\rho g_{13}\right)$$
$$= \frac{1}{2} g^{33}\partial_1 g_{33} = \frac{1}{r}. \quad \text{— (E14)}$$

9) $$\Gamma^1_{22} = -\frac{1}{2} g^{11}\partial_1 g_{22}$$
$$= -r e^{-2\beta}. \quad \text{— (E15)}$$

10) $$\Gamma^1_{33} = \frac{1}{2} g^{11}\partial_1 g_{33}$$
$$= -r e^{-2\beta}\sin^2\theta. \quad \text{— (E16)}$$

11) $$\Gamma^2_{33} = -\frac{1}{2} g^{22}\partial_2 g_{33}$$
$$= -\sin\theta\cos\theta. \quad \text{— (E17)}$$

12) $$\Gamma^3_{23} = \frac{1}{2} g^{33}\partial_2 g_{33}$$
$$= \frac{\cos\theta}{\sin\theta}. \quad \text{— (E18)}$$

The non-vanishing Riemann elements are calculated from the Christoffel symbols using the definition of the Riemann tensor:

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda} \Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda} \Gamma^{\lambda}_{\mu\sigma} \quad - (E19)$$

where summation is again implied over repeated indices. The relevant elements for this paper are calculated as follows.

1) $$R^{0}{}_{202} = \partial_0 \Gamma^{0}_{22} - \partial_2 \Gamma^{0}_{02} + \Gamma^{0}_{0\lambda} \Gamma^{\lambda}_{22} - \Gamma^{0}_{2\lambda} \Gamma^{\lambda}_{02} \quad - (E20)$$

and summing over repeated λ gives:

$$\Gamma^{0}_{0\lambda} \Gamma^{\lambda}_{22} = \Gamma^{0}_{00} \Gamma^{0}_{22} + \Gamma^{0}_{01} \Gamma^{1}_{22} + \cdots \quad - (E21)$$
$$\Gamma^{0}_{2\lambda} \Gamma^{\lambda}_{02} = \Gamma^{0}_{20} \Gamma^{0}_{02} + \Gamma^{0}_{21} \Gamma^{1}_{02} + \cdots$$

So:

$$R^{0}{}_{202} = \Gamma^{0}_{01} \Gamma^{1}_{22} = -r e^{-2\beta} \partial_1 \alpha \quad - (E22)$$

2) $$R^{0}{}_{303} = \Gamma^{0}_{01} \Gamma^{1}_{33} = -r e^{-2\beta} \sin^2\theta \, \partial_1 \alpha \quad - (E23)$$

3) The element $R^{0}{}_{101}$ is calculated as follows:

$$R^{0}{}_{101} = \partial_0 \Gamma^{0}_{11} - \partial_1 \Gamma^{0}_{01} + \Gamma^{0}_{0\lambda} \Gamma^{\lambda}_{11} - \Gamma^{0}_{1\lambda} \Gamma^{\lambda}_{01}$$
$$= \partial_0 \left(e^{2(\beta-\alpha)} \partial_0 \beta \right) - \partial_1 \left(\partial_1 \alpha \right)$$
$$+ \Gamma^{0}_{11} \left(\Gamma^{0}_{00} - \Gamma^{1}_{01} \right) + \Gamma^{0}_{01} \left(\Gamma^{1}_{11} - \Gamma^{0}_{01} \right)$$

$$= e^{2(\beta-\alpha)}\left(\partial_0(\partial_0\beta) + \partial_0\beta\left(\partial_0\alpha - \partial_0\beta\right) + \partial_1\alpha\left(\partial_1\beta - \partial_1\alpha\right) - \partial_1\left(\partial_1\alpha\right)\right). \quad -(E24)$$

It is to be noted that there is a sign error in Carroll's calculation { 11 }, where he gives

$$-\partial_0\beta\left(\partial_0\alpha - \partial_0\beta\right). \quad -(E25)$$

4) This element is:

$$R^1{}_{212} = \partial_1\Gamma^1{}_{22} + \Gamma^1{}_{11}\Gamma^1{}_{22} - \Gamma^1{}_{22}\Gamma^2{}_{12}$$
$$= r e^{-2\beta}\,\partial_1\beta. \quad -(E26)$$

5) This element is:

$$R^1{}_{313} = \partial_1\Gamma^1{}_{33} + \Gamma^1{}_{11}\Gamma^1{}_{33} - \Gamma^1{}_{33}\Gamma^3{}_{13}$$
$$= r e^{-2\beta}\sin^2\theta\,\partial_1\beta \quad -(E27)$$

6) This element is:

$$R^2{}_{323} = \partial_2\Gamma^2{}_{33} + \Gamma^2{}_{21}\Gamma^1{}_{33} - \Gamma^2{}_{33}\Gamma^3{}_{23}$$
$$= \left(1 - e^{-2\beta}\right)\sin^2\theta. \quad -(E28)$$

In the weak field, torsion free limit of this paper:

$$g_{00} = -\left(1 - \frac{2mG}{rc^2}\right),\quad g_{11} = \left(1 - \frac{2mG}{rc^2}\right)^{-1},$$
$$g_{22} = r^2,\quad g_{33} = r^2\sin^2\theta. \quad -(E29)$$

where the gravitational potential is:

$$\Phi = -\frac{MG}{r} \qquad -(E30)$$

and where G is Newton's constant, M is mass, c the speed of light, and r the radial coordinate.

In order to calculate the precise Coulomb law, indices must be raised with the metrics:

$$\begin{aligned} R^{0}{}_{1}{}^{10} &= -R^{0}{}_{1}{}^{01} = -g^{11} g^{00} R^{0}{}_{101} = R^{0}{}_{101} \\ R^{0}{}_{2}{}^{02} &= -g^{22} g^{00} R^{0}{}_{202}, \\ R^{0}{}_{3}{}^{03} &= -g^{33} g^{00} R^{0}{}_{303}. \end{aligned} \qquad -(E31)$$

The $R^{0}{}_{1}{}^{10}$ element is calculated firstly from Eq. (E24) with the simplifying properties {11}:

$$\partial_0 \beta = -\partial_0 \alpha = 0, \quad \alpha = -\beta. \qquad -(E32)$$

So:

$$R^{0}{}_{101} = -2(\partial_1 \alpha)^2 - \partial_1(\partial_1 \alpha) \qquad -(E33)$$

where {11}:

$$2r\,\partial_1 \alpha + 1 = e^{-2\alpha} \qquad -(E34)$$

So:

$$\partial_1 \alpha = \frac{1}{2r}\left(e^{-2\alpha} - 1\right). \qquad -(E35)$$

Working out the algebra it is found that:

$$R^{0}{}_{101} = \frac{1}{r^2}\left(\frac{2GM}{rc^2}\right)\left(1 - \frac{2GM}{rc^2}\right)^{-1} \qquad -(E36)$$

The indices are raised as follows:

$$R^{0}{}_{1}{}^{10} = R^{0}{}_{101} \qquad -(E37)$$

and the final result is:

$$R^{0}{}_{1}{}^{10} = \frac{1}{r^2}\left(\frac{2GM}{rc^2}\right)\left(1-\frac{2GM}{rc^2}\right)^{-1}. \qquad -(E38)$$

The $R^{0}{}_{2}{}^{20}$ element is:

$$R^{0}{}_{2}{}^{20} = \frac{1}{r^2}\cdot\frac{GM}{rc^2}\left(1-\frac{2GM}{rc^2}\right)^{-1}. \qquad -(E39)$$

Finally, the $R^{0}{}_{3}{}^{30}$ element is:

$$R^{0}{}_{3}{}^{30} = \frac{1}{r^2}\frac{GM}{rc^2}\left(1-\frac{2GM}{rc^2}\right)^{-1} \qquad -(E40)$$

so:

$$R^{0}{}_{1}{}^{10} + R^{0}{}_{2}{}^{20} + R^{0}{}_{3}{}^{30} = \frac{2}{r^2}\left(\frac{2GM}{rc^2}\right)\left(1-\frac{2GM}{rc^2}\right)^{-1}. \qquad -(E41)$$

Therefore the Coulomb law of ECE is:

$$\underline{\nabla}\cdot\underline{E} = -\frac{2\phi}{r^2}\left(\frac{2GM}{rc^2}\right)\left(1-\frac{2GM}{rc^2}\right)^{-1}. \qquad -(E42)$$