98(4): Rotation as a General Coordinate Transformation

The general coordinate transformation in general relativity generates the vector V_ν from the vector V_μ as follows:

$$V_\nu = \left(\frac{\partial x^\mu}{\partial x^\nu} \right) V_\mu \quad -(1)$$

$$:= \epsilon^\mu{}_\nu V_\mu . \quad -(2)$$

The Lorentz transformation is a special case of $\epsilon^\mu{}_\nu$ and rotation is a special case of the Lorentz transformation. This is because the rotation group is a sub-group of the Lorentz group and the latter is a sub-group of the Einstein group.

Therefore in general:

$$\boxed{\epsilon^\mu{}_\nu = \frac{\partial x^\mu}{\partial x^\nu}} . \quad -(3)$$

This generalizes the $\epsilon^\mu{}_\nu$ used in Ryder's chapter three to a spacetime which in general has torsion and curvature. Rotation is defined in this spacetime by:

$$\epsilon^{\rho\sigma} = -\epsilon^{\sigma\rho} = g^{\rho\kappa} \epsilon^\sigma{}_\kappa \quad -(4)$$

where the Kronecker delta is defined by:

$$\delta^\mu{}_\sigma = g^{\mu\nu} g_{\nu\sigma} . \quad -(5)$$

It is to be noted that the general coordinate transform (1) also applies to the partial derivatives

2)

$$\partial_\nu = \left(\frac{\partial x^\mu}{\partial x^\nu} \right) \partial_\mu \quad -(6)$$

and that the set of partial derivatives $\{\partial_\mu\}$ at a point P in the base manifold defines a basis set (Carroll, chapter 2).

In Ryder chapter three it is stated that

if: $$x^\mu = \epsilon^\mu{}_\nu x^\nu = X^\mu{}_{\rho\sigma} \epsilon^{\rho\sigma} \quad -(7)$$

then: $$X^\mu{}_{\rho\sigma} = \frac{1}{2} \left(\delta^\mu{}_\rho x_\sigma - \delta^\mu{}_\sigma x_\rho \right) \quad -(8)$$

in a Minkowski spacetime. This is the point at which the rank three tensor $X^\mu{}_{\rho\sigma}$ is introduced into the theory of ~~rank three~~ rotations, and leads to the definition of the rank three angular momentum density tensor. In Cartan's differential geometry, the Cartan torsion is also a rank three tensor, so its origin is rotational. The gamma connection also has three indices but is not a tensor. The Cartan torsion is the tensor:

$$T^\mu{}_{\rho\sigma} = \Gamma^\mu{}_{\rho\sigma} - \Gamma^\mu{}_{\sigma\rho} \quad -(9)$$

and the difference of two general gamma connections is a tensor (see Carroll for more details). Therefore a proof of eqs. (7) and (8) is important. Since Ryder does not give a proof the following "generalization" is suggested

3) First generalize Ryder's equation (7) to any vector in a space-time with torsion and curvature:

$$\boxed{V^\mu = V^\mu{}_{\rho\sigma}\,\epsilon^{\rho\sigma}} \quad -(10)$$

where:

$$\epsilon^{\rho\sigma} = -\epsilon^{\sigma\rho} \quad -(11)$$

$$V^\mu{}_{\rho\sigma} = -V^\mu{}_{\sigma\rho} \quad -(12)$$

This means that any vector V^μ can be expressed as a rank three tensor of type (12). In 3-D Euclidean space this result is well known, any axial vector can be expressed as a rank two antisymmetric tensor:

$$V_i = \frac{1}{2}\epsilon_{ijk}V_{jk} \quad -(13)$$

where ϵ_{ijk} is the rank three totally anti-symmetric unit tensor in 3-D Euclidean space.

In general, eq. (10) is:

$$\frac{1}{2}V^\mu = V^\mu{}_{01}\epsilon^{01} + V^\mu{}_{02}\epsilon^{02} + V^\mu{}_{03}\epsilon^{03} + V^\mu{}_{12}\epsilon^{12} + V^\mu{}_{13}\epsilon^{13} + V^\mu{}_{23}\epsilon^{23}. \quad -(14)$$

This can be expressed as a 4-D matrix with a structure similar to the electromagnetic field matrix:

4)

$$\bar{V}^{\mu}{}_{\rho\sigma} = \begin{bmatrix} 0 & \bar{V}^{\mu}{}_{01} & \bar{V}^{\mu}{}_{02} & \bar{V}^{\mu}{}_{03} \\ \bar{V}^{\mu}{}_{10} & 0 & \bar{V}^{\mu}{}_{12} & \bar{V}^{\mu}{}_{13} \\ \bar{V}^{\mu}{}_{20} & \bar{V}^{\mu}{}_{21} & 0 & \bar{V}^{\mu}{}_{23} \\ \bar{V}^{\mu}{}_{30} & \bar{V}^{\mu}{}_{31} & \bar{V}_{32} & 0 \end{bmatrix} \quad -(15)$$

Rotation About the 3 Axis $(Z = 3)$, Axial Vectors

This is the special case:

$$\mu = 3, \quad \epsilon^{12} = -\epsilon^{21} = 1 \quad -(16)$$

all other $\epsilon^{\mu\nu} = 0$, so:

$$\boxed{\bar{V}^{3} = 2\bar{V}^{3}{}_{12}.} \quad -(17)$$

This is the magnetic field relation used in paper 93 apart from the factor 2. For a fixed 3 it is the same as eq (13).

Polar Vector Relations

These are:

$$\boxed{\frac{1}{2}\bar{V}^{i} = \bar{V}^{i}{}_{0i}}, \quad i = 1, 2, 3 \quad -(18)$$

Apart from a factor 2 they are the electric field relations used in paper 93.

We now define the rank three tensor:

$$\boxed{\bar{V}^{\mu}{}_{\rho\sigma} = \bar{V}^{\mu}\epsilon_{\rho\sigma}} \quad -(19)$$

where in general:

$$\epsilon_{\rho\sigma} = g_{\rho\kappa}\,\epsilon^{\kappa}{}_{\sigma} \quad -(20)$$

and

$$\epsilon^{\kappa}{}_{\sigma} = \frac{\partial x^{\kappa}}{\partial x^{\sigma}}. \quad -(21)$$

The matrix $\epsilon^{\kappa}{}_{\sigma}$ is similar to the definition used for the tetrad in differential geometry:

$$V^{a} = q^{a}{}_{\mu} V^{\mu}, \quad -(22)$$

so:

$$\boxed{q^{\kappa}{}_{\sigma} = \frac{\partial x^{\kappa}}{\partial x^{\sigma}}} \quad -(23).$$

is a special case where:

$$\epsilon^{\mu}{}_{\nu} = q^{\mu}{}_{\nu} = q^{a}{}_{\nu}\, q^{\mu}{}_{a} \quad -(24)$$

The Cartan torsion form is:

$$T^{a}{}_{\mu\nu} = d\wedge q^{a}{}_{\nu} + \omega^{a}{}_{\mu b}\wedge q^{b}{}_{\nu} \quad -(25)$$

$$= \partial_\mu q^{a}{}_{\nu} - \partial_\nu q^{a}{}_{\mu} + \omega^{a}{}_{\mu b}\, q^{b}{}_{\nu} - \omega^{a}{}_{\nu b}\, q^{b}{}_{\mu} \quad -(26)$$

and the Cartan torsion tensor is

$$T^{\kappa}{}_{\mu\nu} = \Gamma^{\kappa}{}_{\mu\nu} - \Gamma^{\kappa}{}_{\nu\mu} \quad -(27)$$

b) They are related by:

$$T^a_{\mu\nu} = q^a_\kappa T^\kappa_{\mu\nu} \quad — (28)$$

and the tetrad postulate:

$$D_\mu q^a_\nu = 0. \quad — (29)$$

Comparison of the structures of eqns. (28) and (19) shows that the Cartan torsion is:

$$T^a_{\rho\sigma} = \partial_\rho \epsilon^a_\sigma - \partial_\sigma \epsilon^a_\rho + \omega^a_{\rho b} \epsilon^b_\sigma - \omega^a_{\sigma b} \epsilon^b_\rho \quad — (30)$$

and derives from a structure:

$$T^\mu_{\rho\sigma} = \partial^\mu \epsilon_{\rho\sigma} + \cdots \quad — (31)$$

that reflects the fundamental geometry of 4-D spacetime, another expression of which is eqn. (19).
