Notes 99(4) : Proof of the Conventional Second Bianchi Identity from Jacobi Identity.

The proof starts from:

$$[D_\mu, D_\nu] V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma - T^\lambda{}_{\mu\nu} D_\lambda V^\rho \quad -(1)$$

Assume that:

$$T^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu} = 0, \quad -(2)$$

then:

$$[D_\mu, D_\nu] V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma \quad -(3)$$

Consider:

$$([D_\kappa, [D_\mu, D_\nu]]) V^\rho$$
$$= (D_\kappa [D_\mu, D_\nu] - [D_\mu, D_\nu] D_\kappa) V^\rho \quad -(4)$$
$$= D_\kappa (D_\mu (D_\nu V^\rho) - D_\nu (D_\mu V^\rho)) - D_\mu D_\nu (D_\kappa V^\rho) + D_\nu D_\mu (D_\kappa V^\rho) \quad -(5)$$
$$= D_\kappa ([D_\mu, D_\nu] V^\rho) - [D_\mu, D_\nu](D_\kappa V^\rho)$$
$$= (D_\kappa [D_\mu, D_\nu]) V^\rho + [D_\mu, D_\nu] D_\kappa V^\rho - [D_\mu, D_\nu] D_\kappa V^\rho \quad -(6)$$

(using ~~the~~ Leibnitz Theorem).

$$= (D_\kappa [D_\mu, D_\nu]) V^\rho \quad -(7)$$

The Jacobi identity is:

2)

$$[D_\kappa, [D_\mu, D_\nu]] + [D_\nu, [D_\kappa, D_\mu]] + [D_\mu, [D_\nu, D_\kappa]] := 0 \quad -(8)$$

so

$$D_\kappa [D_\mu, D_\nu] + D_\nu [D_\kappa, D_\mu] + D_\mu [D_\nu, D_\kappa] = 0 \quad -(9)$$

i.e. $$\left(D_\kappa R^\rho{}_{\sigma\mu\nu} + D_\nu R^\rho{}_{\sigma\kappa\mu} + D_\mu R^\rho{}_{\sigma\nu\kappa}\right) V^\sigma = 0 \quad -(10)$$

A particular solution is

$$D_\kappa R^\rho{}_{\sigma\mu\nu} + D_\nu R^\rho{}_{\sigma\kappa\mu} + D_\mu R^\rho{}_{\sigma\nu\kappa} = 0 \quad -(11)$$

which is ~~the~~ second Bianchi identity if Torsion is neglected.   Q.E.D.

In differential geometry eq. (11) is

$$D \wedge R^a{}_b = 0 \quad -(12)$$

which is therefore a statement of eq. (3)

Q.E.D.

It is known from paper 88 that eq. (12) is incomplete. There is only one Bianchi identity:

3)

$$D \wedge T^a := R^a{}_b \wedge q^b \qquad (13)$$

and "the second Bianchi identity" it should be:

$$D \wedge (R^a{}_b \wedge q^b) := D \wedge (D \wedge T^a) \neq 0 \qquad (14)$$

This must be derived from eqs. (1) and (9),

giving:

$$\left(D_\kappa R^\rho{}_{\sigma\mu\nu} + D_\nu R^\rho{}_{\sigma\kappa\mu} + D_\mu R^\rho{}_{\sigma\nu\kappa}\right) V^\sigma \neq 0$$

$$= D_\kappa\left(T^\lambda{}_{\mu\nu} D_\lambda V^\rho\right) + D_\nu\left(T^\lambda{}_{\kappa\mu} D_\lambda V^\rho\right) + D_\mu\left(T^\lambda{}_{\nu\kappa} D_\lambda V^\rho\right) \qquad (15)$$

It is clear that eq. (14) is a more elegant expression of eq. (15).

More generally, eq. (9) is:

$$\left(D_\kappa[D_\mu, D_\nu] + D_\nu[D_\kappa, D_\mu] + D_\mu[D_\nu, D_\kappa]\right) V^\rho = 0 \qquad (16)$$

$$\boxed{\begin{aligned} \text{i.e. } & D_\kappa\left(R^\rho{}_{\sigma\mu\nu} V^\sigma - T^\lambda{}_{\mu\nu} D_\lambda V^\rho\right) \\ & + D_\nu\left(R^\rho{}_{\sigma\kappa\mu} V^\sigma - T^\lambda{}_{\kappa\mu} D_\lambda V^\rho\right) \\ & + D_\mu\left(R^\rho{}_{\sigma\nu\kappa} V^\sigma - T^\lambda{}_{\nu\kappa} D_\lambda V^\rho\right) = 0 \end{aligned}} \qquad (17)$$

the conventional "second Bianchi identity",

4) eq. (11), is a special case of eq. (17) when:

$$T^{\lambda}{}_{\mu\nu} = 0 \quad -(18)$$

and

$$R^{\rho}{}_{\sigma\mu\nu} D_{\kappa} V^{\sigma} + R^{\rho}{}_{\sigma\kappa\mu} D_{\nu} V^{\sigma} + R^{\rho}{}_{\sigma\nu\kappa} D_{\mu} V^{\sigma} = 0 \quad -(19)$$

<u>Eq. (19) comes from the Leibniz Theorem</u>. The condition (19) is necessary for eq. (11) to be true, even in the absence of torsion. It can be written as:

$$\left(R^{\rho}{}_{\sigma\mu\nu} D_{\kappa} + R^{\rho}{}_{\sigma\kappa\mu} D_{\nu} + R^{\rho}{}_{\sigma\nu\kappa} D_{\mu} \right) V^{\sigma} = 0 \quad -(20)$$

or in operator notation:

$$\boxed{R^{\rho}{}_{\sigma\mu\nu} D_{\kappa} + R^{\rho}{}_{\sigma\kappa\mu} D_{\nu} + R^{\rho}{}_{\sigma\nu\kappa} D_{\mu} = 0} \quad -(21)$$

i.e.

$$\boxed{R^{a}{}_{b} \wedge D = 0} \quad -(22)$$

<u>This appears to be a new result</u>.