

**Two-Dimensional Finite Element Scheme for Conductive Material Using  
ECE Quasi-static Electromagnetic Theory – Part 1 Distributed Driving Potential**

By

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**Abstract:**

A stable method of calculating the ECE electromagnetic field in the quasi-static purely magnetic approximation using the finite element approach for a two-dimensional flat conductive surface is presented. In its present form, one component of the magnetic vector potential is specified over the entire surface of the conductor, then the scalar spin connection and the second magnetic vector potential component are calculated. The vector spin connection cannot be calculated in its entirety, but  $\omega \times A$  can. This allows the calculation of the magnetic field. There is a limitation in that the specified driving vector potential component can never be zero since this presents a singularity in the dependent variables.

**Introduction:**

The solution of partial differential equations where domain geometries are not simple straight lines has been numerically solved using a technique known as finite element analysis. The mathematical technique is well developed and has numerous commercial software implementations available in the marketplace. The analysis contained in this paper will focus on a package known as FlexPde from pdesolutions [www.pdesolutions.com]. It is extremely simple to use, quite powerful, and has a size limited version (one thousand nodes) available to students free of charge.

This preliminary analysis will be limited to a two-dimensional thin flat sheet of conductive material. This simplifies an introductory analysis, but unfortunately limits the implementation of the ECE theory to an incomplete solution.

The use of thin flat sheets in field analysis is not new. A concept from mechanical engineering known as plane stress analysis uses the elasticity equations simplified to a thin sheet where stresses are assumed zero or constant through its thickness. This has been the basis of engineering design for more than a century, and has been quite successful in its implementation in computer and non-computer applications. Commercial finite element software packages that support elasticity analysis have two-dimensional “plane stress” analysis routines built in.

**The Finite Element Method:**

*(Italicized text in quotations throughout this document are copied directly from [1])*

FlexPde, the software tool chosen for finding numerical solutions to systems of non-linear partial differential equations of ECE electromagnetic theory, uses the methods of finite element analysis. *“Rather than addressing the solution of specific equations related to a given area of application, FlexPde provides a framework for treating partial differential equation systems in general. It gives users a straightforward method of defining the equations, domains and boundary conditions appropriate to their application. From this description it creates a finite element solution process tailored to the problem.”*

It is not the intent here to provide an in depth discussion of finite element methods. The literature is rich with details on the method, and the interested person can find a wealth of material.

*“Many approaches have been devised for using computers to approximate the behavior of real systems. The finite element method is one of them. It has achieved considerable success in its few decades of existence, first in structural mechanics, and later in other fields. Part of its success lies in the fact that it approaches the analysis in the framework of integrals over small patches of the total domain, thus enforcing aggregate correctness even in the presence of microscopic error. The techniques applied are little dependent on shapes of objects, and are therefore applicable in real problems of complex configuration...The fundamental assumption is that no matter what the shape of a solution might be over the entire domain of a problem, at some scale each local patch of the solution*

*can be well approximated by a low-order polynomial. This is closely related to the well-known Taylor series expansion, which expresses the local behavior of a function in a few polynomial terms.*“

In a two-dimensional problem, for example, we divide the domain up into a large number of triangular “elements”, then in each element the dependent variable(s) can be represented by a parabolic surface for example. Stitching the elements together we get a surface that obeys the assumption of continuity for the solution value—but perhaps not for its derivatives. The patchwork of triangles is referred to as the computation “mesh”, and the sample points at vertices or elsewhere on the element are referred to as the “nodes” of the mesh.

The solution proceeds (in simplistic terms) as follows:

1. *Assign a sample value to each vertex of the triangular or tetrahedral subdivision of the domain. Then each vertex value is shared by several triangles (tetrahedra).*
2. *Substitute the approximating functions into the partial differential equation.*
3. *Multiply the result by an importance-weighting function and integrate over the triangles surrounding each vertex.*
4. *Solve for the vertex values which minimize the error in each integral.*

*“This process, known as a “weighted residual” method, effectively converts the continuous PDE problem into a discrete minimization problem on the vertex values. This is usually known as a “weak form” of the equation, because it does not strictly enforce the PDE at all points of the domain, but is instead correct in an integral sense relative to the triangular subdivision of the domain. “*

FlexPde uses either quadratic interpolation (with sample values at vertices and midsides of the triangular cells), or cubic interpolation (with values at vertices and two points along each side).

A fundamental component of any partial differential equation system is the set of boundary conditions, which alone make the solution unique. In two or three dimensions, a value or derivative condition applied over the entire bounding curve or surface provides one condition at each end of any coordinate integration path (as for example, the side of an element).

Traditional boundary conditions take on one of three forms:

1. Dirichlet condition where a value for the dependent variable is specified at a geometric boundary
2. Neuman condition where a spatial derivative of the dependent variable is specified on a boundary or the
3. Robin condition which is a combination of the other two.

FlexPde uses a slightly different approach- namely “integration by parts”, to treat the system of equations, which has the benefit of reducing the order of a derivative integrand, and also leads to a formulation of derivative boundary conditions for the PDE system.

*“Applied to the weighted residual method, this process dictates the flux conservation characteristics of the finite element approximation at boundaries between the triangular approximation cells, and also provides a method for defining the interaction of the system with the outside world, by specifying the value of the surface integrand.*

*The values of the surface integrands are the “Natural” boundary conditions of the PDE system, a term which also arises in a similar context in variational calculus.*

*FlexPde uses the term “Natural” boundary condition to specify the boundary flux terms arising from the integration by parts of all second-order terms in the PDE system. “*

### **Summary of ECE Electromagnetic Theory:**

The ECE Theory of Electromagnetism has been developed at great length elsewhere [[www.aias.us](http://www.aias.us)] and will not be reviewed here. The equations of the theory will be presented here for the purposes of document completeness.

The field equations are identical in form to Maxwell’s equations

$$(1) \quad \nabla \cdot \mathbf{B} = 0$$

$$(2) \quad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$(3) \quad \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon}$$

$$(4) \quad \nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} = \mu_o \mathbf{J}$$

where  $\mathbf{E}$  and  $\mathbf{B}$  are the electric and magnetic fields respectively and  $c$  is the speed of light in the medium. ( In this analysis  $\rho_{eh}'$ , and  $\mathbf{j}_{eh}'$  are assumed to be zero, [3]).  $\mathbf{J}$  is the current density,  $\rho$  is the charge density, and  $\varepsilon$  is the permittivity of the medium.

In ECE theory however, the electric and magnetic field have new definitions from Maxwell’s theory that incorporates a scalar ( $\omega_o$ ) and vector spin connection ( $\boldsymbol{\omega}$ ), ie.

$$(5) \quad \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \omega_o \mathbf{A}$$

$$(6) \quad \mathbf{B} = \nabla \times \mathbf{A} - \boldsymbol{\omega} \times \mathbf{A}$$

where  $\mathbf{A}$  is the magnetic vector potential and the scalar potential is zero.

This analysis is limited to the purely magnetic case where the equations reduce to [2]

$$(7) \quad \nabla \cdot (\boldsymbol{\omega} \times \mathbf{A}) = 0$$

$$(8) \quad -\nabla \times \omega_o \mathbf{A} - \frac{\partial(\boldsymbol{\omega} \times \mathbf{A})}{\partial t} = 0$$

$$(9) \quad \nabla \cdot \left( -\frac{\partial \mathbf{A}}{\partial t} - \omega_o \mathbf{A} \right) = 0$$

$$(10) \quad \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} - \nabla \times (\boldsymbol{\omega} \times \mathbf{A}) + \frac{1}{c^2} \left( \frac{\partial^2 \mathbf{A}}{\partial t^2} + \omega_o \frac{\partial \mathbf{A}}{\partial t} + \frac{\partial \omega_o}{\partial t} \mathbf{A} \right) = \mu_o \mathbf{J}$$

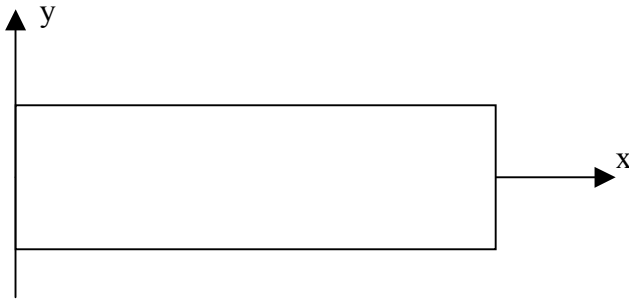
We will assume that the conductive material has a linear constitutive relationship, that is, Ohm's Law is valid.

$$(11) \quad \mathbf{J} = \sigma \mathbf{E}$$

where  $\sigma$  is the conductivity of the material and is homogeneous and isotropic.

### The Two-Dimensional Field Equations:

Consider a thin flat conductive sheet of material that is very thin in comparison to its other two geometric dimensions. It is spanned by a standard two-dimensional Cartesian frame of reference as illustrated in Figure 1. The sheet can have arbitrary spatial dimensions within this limitation, but for the purposes of this discussion, the geometry of the sheet is limited to that of a simple rectangle.



**Figure 1. Flat sheet of conducting material**

The sheet is assumed to be thin enough that there is no spatial variation in any of the system variables or parameters in the “z”, through the thickness, direction.

We will also specifically assume that

1. The conductivity of the material is high enough that there is no charge accumulation anywhere
2. No external electric potentials are applied and that the electric potential can be taken as constant which we shall set to zero without loss of generality
3. The permeability of the material is constant and is equal to  $\mu_0$ , the permeability of free space
4. The magnetic vector potential is two dimensional and is given as

$$(12) \quad \mathbf{A} = (A_x(x,y,t), A_y(x,y,t), 0)^T$$

This two dimensional limitation requires that, according to equation 5 that  $\mathbf{E}$  be two-dimensional and of the form

$$(13) \quad \mathbf{E} = (E_x(x,y,t), E_y(x,y,t), 0)^T$$

The Curl of this expression can only have a “z” component. Then by Faraday’s Law in Induction (equation 2) that the magnetic field only has a “z” component. That is

$$(14) \quad \mathbf{B} = (0, 0, B_z(x,y,t))^T$$

Given the Ampere Maxwell Law, equation 4, the form of the spin connection has to be

$$(15) \quad \boldsymbol{\omega} = (\omega_x, \omega_y, 0)^T$$

With this limitation, the divergence equation 7 is trivially zero and contributes no information.

If we substitute equation 11 into the Ampere Maxwell Equation, equation 10, noting the comment above, we get the entire set of equations governing the two dimensional problem.

$$(16) \quad -\nabla \times \omega_o \mathbf{A} - \frac{\partial(\boldsymbol{\omega} \times \mathbf{A})}{\partial t} = 0$$

$$(17) \quad \nabla \cdot \left( -\frac{\partial \mathbf{A}}{\partial t} - \omega_o \mathbf{A} \right) = 0$$

$$(18) \quad \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} - \nabla \times (\boldsymbol{\omega} \times \mathbf{A}) + \frac{1}{c^2} \left( \frac{\partial^2 \mathbf{A}}{\partial t^2} + \omega_o \frac{\partial \mathbf{A}}{\partial t} + \frac{\partial \omega_o}{\partial t} \mathbf{A} \right) - \mu_o \sigma \left( -\frac{\partial \mathbf{A}}{\partial t} - \omega_o \mathbf{A} \right) = 0$$

Simplification is possible if we assume for the purposes of discussion only that the electric field vector  $\mathbf{E} = -\mathbf{A}' - \omega_o \mathbf{A}$  is harmonic and of the form  $e^{i\beta t}$ . It is easy to show that the ratio of the wave-like term

$$\frac{1}{c^2} \left( \frac{\partial^2 \mathbf{A}}{\partial t^2} + \omega_o \frac{\partial \mathbf{A}}{\partial t} + \frac{\partial \omega_o}{\partial t} \mathbf{A} \right) \text{ to the dispersive term } \mu_o \sigma \left( -\frac{\partial \mathbf{A}}{\partial t} - \omega_o \mathbf{A} \right) \text{ in}$$

equation 18 is given by

$$(19) \quad \left| \frac{i\beta}{\mu_o \sigma c^2} \right| \ll 1$$

For frequencies in the lower rf range (hundreds of kilohertz), the wave-like component is insignificant in comparison to the dispersive component causing a simplification of equation 18 to

$$(20) \quad \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} - \nabla \times (\boldsymbol{\omega} \times \mathbf{A}) - \mu_o \sigma \left( -\frac{\partial \mathbf{A}}{\partial t} - \omega_o \mathbf{A} \right) = 0$$

This is the typical quasi-static assumption for conductive materials [4] and has the effect of allowing the change of Maxwell's equation from hyperbolic to elliptic. As will be discussed later, this form of equation is solved numerically with much less difficulty. The ECE equations are slightly different. A wave-like term still remains within the dispersive term, causing the equation set to be hyperbolic. In a subsequent section, the equations will be recast in an elliptic format to get around some of the numerical instability issues present when seeking solutions for the hyperbolic form.

Equations 16, 17, and 20 do not form a well posed problem- the number of unknowns exceed the number of equation, five to four. This is not an unusual problem in the simplification of three-dimensional field theory to a two dimensional planar problem. For example, in elasticity theory, a set of equations known as the compatibility equations, which essentially guarantee that mechanical loads not create any voids in the material, are violated.

One can arrive at a method of solution however. If one takes the Curl of equation 16 and the time derivative of equation 20, the term  $\boldsymbol{\omega} \times \mathbf{A}$  can be eliminated to give

$$(21) \quad \nabla(\nabla \cdot \mathbf{A}') - \nabla^2 \mathbf{A}' + \nabla \times \nabla \times \omega_o \mathbf{A} - \mu_o \sigma \frac{\partial(-\mathbf{A}' - \omega_o \mathbf{A})}{\partial t} = 0$$

Using the identity

$$\nabla \times \nabla \times \mathbf{A}' = \nabla(\nabla \cdot \mathbf{A}') - \nabla^2 \mathbf{A}'$$

equation 21 simplifies to

$$(22) \quad \nabla^2 \left( -\frac{\partial \mathbf{A}}{\partial t} - \omega_o \mathbf{A} \right) - \mu_o \sigma \frac{\partial \left( -\frac{\partial \mathbf{A}}{\partial t} - \omega_o \mathbf{A} \right)}{\partial t} = 0$$

Equation 17 has been used to get rid of the divergence term so is in fact linear dependent on equations 22. This shows up in FEM as a matrix with zero elements on the diagonal.

We note immediately that given the definition of the electric field (equation 5), equations 22 and 17 are simply

$$(23) \quad \nabla^2 \mathbf{E} - \mu_o \sigma \frac{\partial \mathbf{E}}{\partial t} = 0$$

$$(24) \quad \nabla \cdot \mathbf{E} = 0$$

which are familiar from standard electromagnetic theory.

It is noted that even with this approach we are short one equation (two equations in unknowns  $A_x$ ,  $A_y$  and  $\omega_o$ ). As an initial attempt to get around this problem,  $A_x$  will be specified over the entire surface. This limits the solution to those cases where  $A_y$  is small and that any change in  $A_x$  due to  $\omega_o$  can be neglected.

### Calculating $\omega$ :

In this formulation it is not possible to calculate  $\omega$  explicitly. We can however calculate  $\omega \times \mathbf{A}$  directly from equation 16, ie.

$$(25) \quad \omega \times \mathbf{A} = -\int (\nabla \times \omega_o \mathbf{A}) dt$$

This does not allow the solution of  $\omega$  and  $\mathbf{A}$  independently but can be used to calculate  $\omega \times \mathbf{A}$  which in turn enables the calculation of the magnetic field.



**Boundary Conditions:**

Equations 22, and 25 are the equations that specify the field for the two dimensional flat conductive sheet of arbitrary geometry. Boundary conditions are required at the periphery of the sheet to in essence specify the constants of integration generated in solving the system of equations.

Consider the divergence equation 17.

The divergence theorem in two dimensions requires that this be

$$(26) \quad \int \nabla \cdot \left( -\frac{\partial \mathbf{A}}{\partial t} - \omega_o \mathbf{A} \right) \cdot d\mathbf{s} = \oint \left( -\frac{\partial \mathbf{A}}{\partial t} - \omega_o \mathbf{A} \right) \cdot \mathbf{n} \cdot dl = 0$$

where  $\mathbf{n}$  is the unit normal vector at the boundary of the domain,  $d\mathbf{s}$  is a surface element and  $dl$  is a line element

The boundary conditions for the system of equations then take the form value  $(-\mathbf{A}' - \omega_o \mathbf{A})$  specified or value of  $(-\mathbf{A}' - \omega_o \mathbf{A}) \cdot \mathbf{n}$  specified.

In FlexPde, the terminology for these conditions is

$$\begin{array}{ll} \text{Value}(-\mathbf{A}' - \omega_o \mathbf{A}) = 0 & \text{and} \\ \text{Natural}(-\mathbf{A}' - \omega_o \mathbf{A}) = 0 & \end{array}$$

respectively, depending on whether the value of a variable or its flux needs specification.

In a three dimensional problem, equation 7 is not trivial and provides for further boundary condition specification. Following the integration as above we would have the boundary specification

1. value  $(\omega \times \mathbf{A})$  specified or
2. value of  $(\omega \times \mathbf{A}) \cdot \mathbf{n}$  specified

Of course in this two dimensional case,  $\omega \times \mathbf{A}$  is trivial under all conditions.

**Initial Conditions:**

It will be generally assumed that  $\omega_o$  and  $\omega$  are zero at the start of the analysis. Initial values of  $\mathbf{A}$ ,  $\mathbf{E}$ , and/or  $\mathbf{B}$  must then be specified for the problem, which must be converted to the field variable being solved.

**Load:**

To complete the description of a problem for a finite element solution, the “loads” to be put on the field equations need to be specified. Traditionally this could be done by specifying some form of  $\mathbf{A}$ ,  $\mathbf{E}$ , and/or  $\mathbf{B}$  or their derivatives. Since we don’t know the form of  $\omega_o$  and  $\boldsymbol{\omega}$  at the boundaries, the only load type that can be specified is for the magnetic vector potential  $\mathbf{A}$ . This is done either at a periphery, or over some area (perhaps all) of the conductive sheet. This analysis will be limited to the application of a load over the entire conductive sheet.

**Problem 1 Uniformly Loaded Rectangular Sheet**

The simple time dependent problem that will be analyzed is that of a rectangular sheet where one component of the vector potential is completely specified in space and time. This is similar in nature to the problem analyzed previously for a two-dimensional cylindrical geometry [3]. Coupling though  $\omega_o$  will be seen to generate a perpendicular component to the specified vector potential, something not possible in traditional Maxwell electromagnetism.

The geometry of the area to be modeled is given in Table 1.

**Table 1 Model Dimensions**

Parameter	Value
Length in “x” direction	2
Length in “y” direction	2
Number of nodes	1133
Number of elements	536
Number of unknowns	4532

Following the lead of a previous publication [3] we assume that the driving or specified vector potential is given by

$$(27) \quad A_x = \sin^6(\beta t) + \delta$$

where  $\delta$  is a small numerical factor that prevents a numerical “divide by zero” situation.

The flux of  $(-\mathbf{A}' - \omega_o \mathbf{A})$  normal to the surface is taken to be zero at all boundaries or else specified. All field variables are assumed to be zero at the start of the calculation. For the purposes of this simple problem,

$$\mu_o \sigma = k = 1 \quad \text{and} \quad \beta = 1 \quad (\text{For copper } \mu_o \sigma = 75 \text{ and for nichrome } \mu_o \sigma = 1.25 \text{ in SI units}).$$

**The Solution:**

Equations 22 were recast in three different forms to see the impact of the nature of the field equations on the solution. This is illustrated in Table 2.

Form 1 consists of equations 5 and 23.

Form 2 is equations 22.

Form 3 is the expansion of form 2 using the relation

$$(28) \quad \nabla^2 \omega_o A = \omega_o \nabla^2 A + A \nabla^2 \omega_o + 2(\nabla \omega_o \bullet \nabla)A$$

which is easily verified using a symbolic algebraic processor such as Mathematica.

Using this identity on equation 22 gives for the  $A_x$  component

$$(29) \quad \nabla^2 \omega_o - k \left( \frac{A_x''}{A_x} + \omega_o \frac{A_x'}{A_x} + \frac{\partial \omega_o}{\partial t} \right) = 0$$

and for the  $A_y$  component

$$(30) \quad \nabla^2 A_y' + \omega_o \nabla^2 A_y + A_y \nabla^2 \omega_o + 2 \left( \frac{\partial \omega_o}{\partial x} \frac{\partial A_y}{\partial x} + \frac{\partial \omega_o}{\partial y} \frac{\partial A_y}{\partial y} \right) - k (A_y'' + \omega_o A_y' + \omega_o' A_y) = 0$$

**Table 2 Solution Time and Stability for Various Forms of Equations**

Form	Form of Equations	CPU run time(hr:min.)*	Comment
1	ex: $\text{Del2}(ex)-k*\text{Dt}(ex)=0$ wo: $ex=-\text{Dt}(Ax)-wo*Ax$ ey: $\text{Del2}(ey)-k*\text{Dt}(ey)=0$ Ay: $ey=-\text{Dt}(Ay)-wo*Ay$	27	Stable, noise free calculation
2	wo: $\text{Del2}(Apx+wo*Ax)-k*\text{Dt}(Apx+wo*Ax)=0$ Ay: $Apy=\text{Dt}(Ay)$ Apy: $\text{Del2}(Apy+wo*Ay)-k*\text{Dt}(Apy+wo*Ay)=0$ ey: $ey=-\text{Dt}(Ay)-wo*Ay$	1:43	Tended towards instability. Collapsed to zero after two cycles
3	wo: $\text{Del2}(wo)-k*(Axi*\text{Dt}(Apx)+wo*Axi*Apx+\text{Dt}(wo))=0$ Apy: $Apy=\text{Dt}(Ay)$ Ay: $\text{Del2}(Apy)+wo*\text{Del2}(Ay)+Ay*\text{Del2}(wo)+2*(Dx(wo)*Dx(Ay)+Dy(wo)*Dy(Ay))-k*(\text{Dt}(Apy)+wo*Apy+\text{Dt}(wo)*Ay)=0$ ey: $e ey=-\text{Dt}(Ay)-wo*Ay$	incomplete	Unstable after two cycles.

\*all calculations were performed on a 1.75 GHz 64bit CPU with 2 GBytes of memory running a Windows XP64 operating system.

Note that  $Axi=I/A_x$

The script for a typical analysis presented is included in Appendix I.

Form 1 is on the surface, an elliptic system with non-linear dependent variable combinations. Forms 2 and 3 tend to be increasingly hyperbolic, that is, the hyperbolic nature of the equations is increasingly visible to the finite element solution algorithm. This shows up clearly in the ease at which a solution is found and the “noisiness” of the solution.

These forms all generate a trivial solution for  $A_y$  unless one or both of the following is met

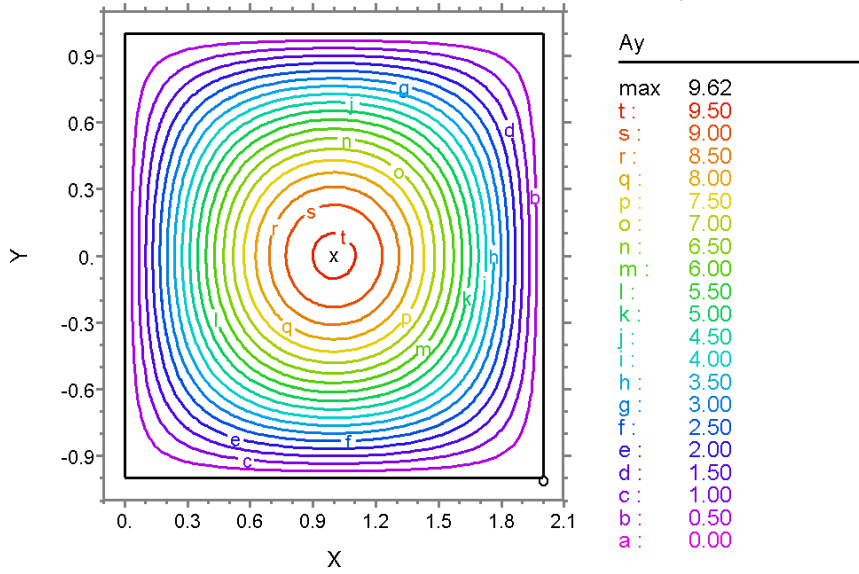
- $A_y$  is given a non-trivial initial condition
- $E_y$  is given a slight non-zero off-set at one of the boundaries

The values calculated for the dependent variables depend significantly on the values given to either the initial conditions or a non-zero boundary condition.

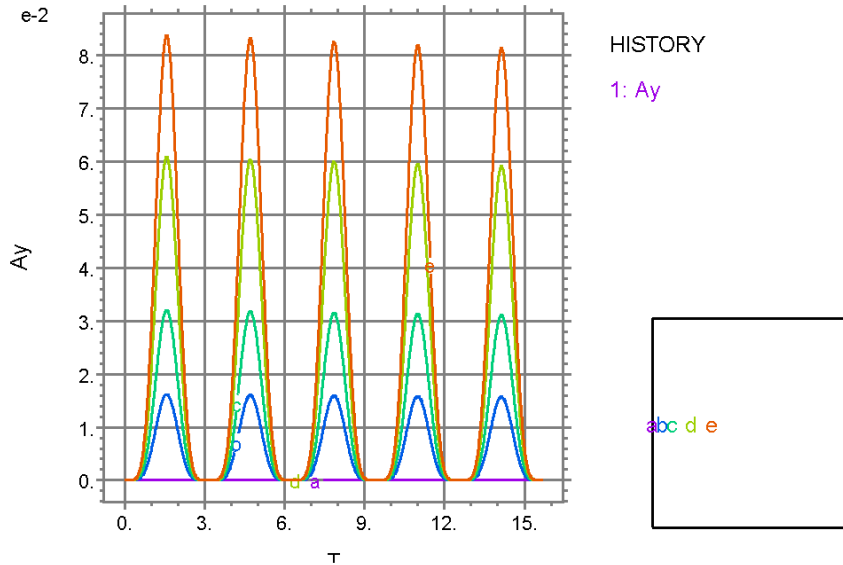
The solution for Form 1 is presented in Figures 2 where initially

$$A_y = 0.00001 \cdot \cos\left(\frac{\pi y}{2L_y}\right) \sin\left(\frac{\pi x}{L_x}\right)$$

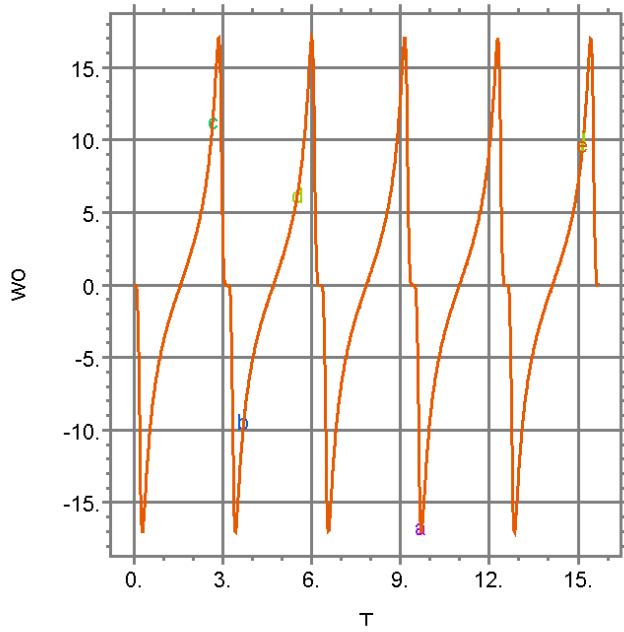
Problem 1 k=1 beta=1



Problem 1 k=1 beta=1

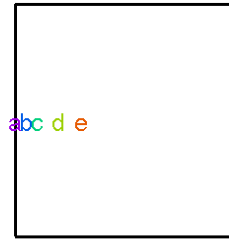


Problem 1 k=1 beta=1

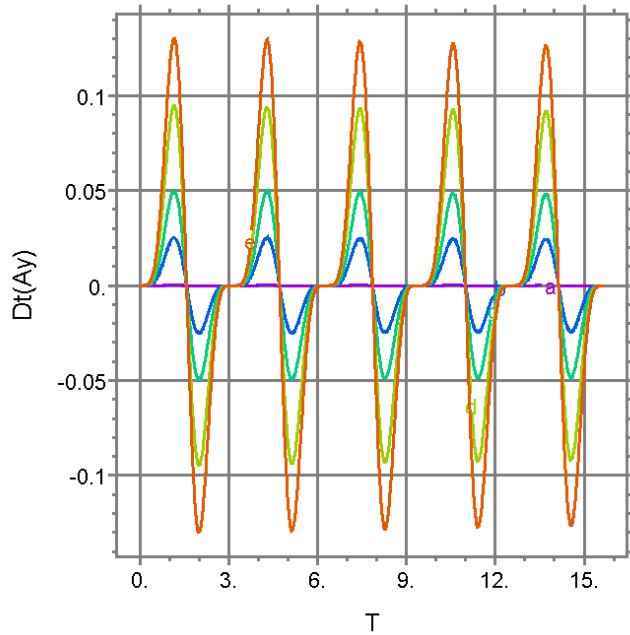


HISTORY

1: w0

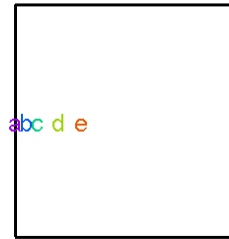


Problem 1 k=1 beta=1

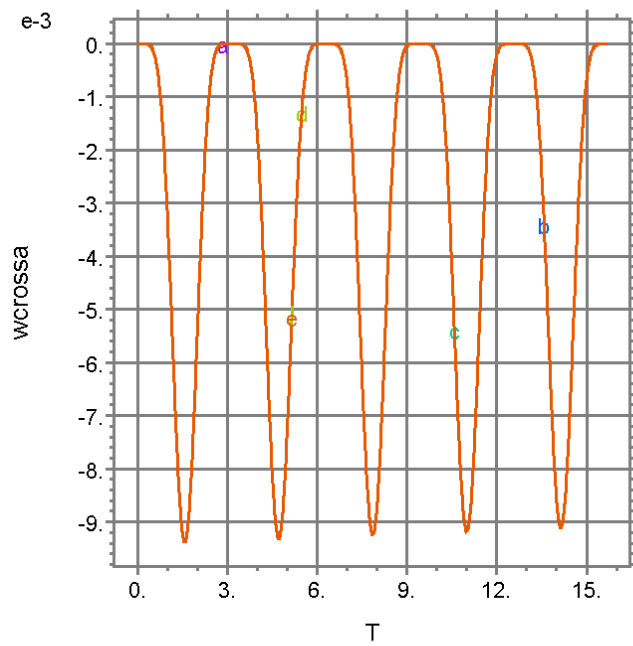


HISTORY

1: Dt(Ay)

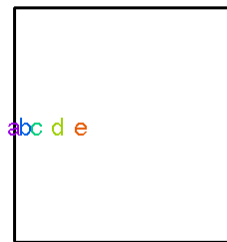


Problem 1 k=1 beta=1

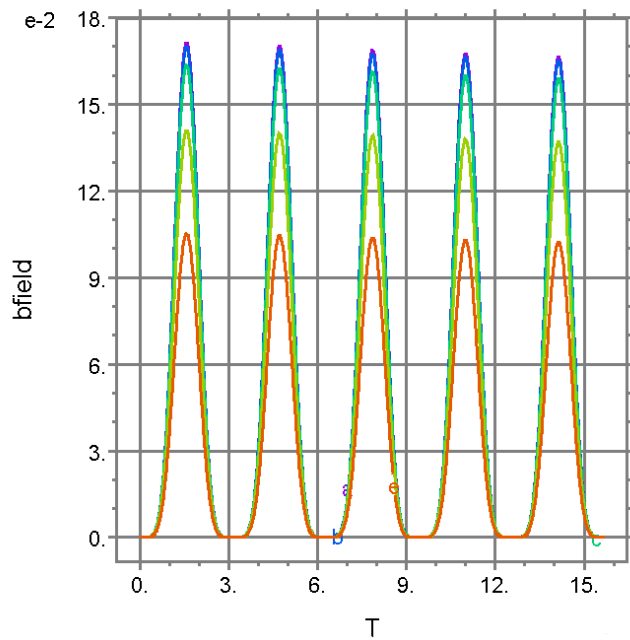


HISTORY

1: wcrossa



Problem 1 k=1 beta=1



HISTORY

1: bfield

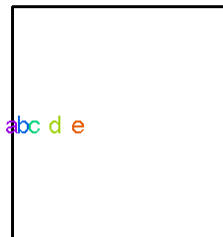


Figure 2 Form 1 equations with non=zero initial conditions on  $A_y$

A stable solution presents itself which is essentially a periodic function overlaid on a very slightly time dependent substrate. The contours of  $A_y$  are uniform and symmetric indicating a stable solution. Unstable solutions are marked by solution “hot spots” which migrate in spatial position on the domain. The boundary conditions are that there is no flux of the electric field at the boundary.

As the equation set gets “more hyperbolic” going from Form 1 to Form 3, the stability of the solution degrades. The smooth symmetric aspect of the diffusion process is lost in the hyperbolic formulation due to localized solution instabilities. Form 1 was the only equation set that generated a “long term” solution.

The following were immediately observed

- The solution is stable for a non-zero initial condition on  $A_y$  consistent with the boundary conditions
- The value of  $\omega_o$  is spatially independent and stationary periodic
- $A_y$ ,  $\omega \times \mathbf{A}$ , and  $\mathbf{B}$  all decrease slightly with time. Whether there is a convergence to a fixed zero or non-zero value is not known. It is postulated but not verified that the clipping observed on  $B_z$  is due to not allowing  $A_x$  to become zero.
- Even though this model contains spatially dependent variables, the similarity in form to that of a previous calculation [3] should be noted.
- $A_y$  follows  $A_x$  and is an “amplified version of the initial value of  $A_y$ ”. The peaks of  $\omega_o$  correspond to areas where  $A_y$  and hence  $A_x$  is nearly zero.

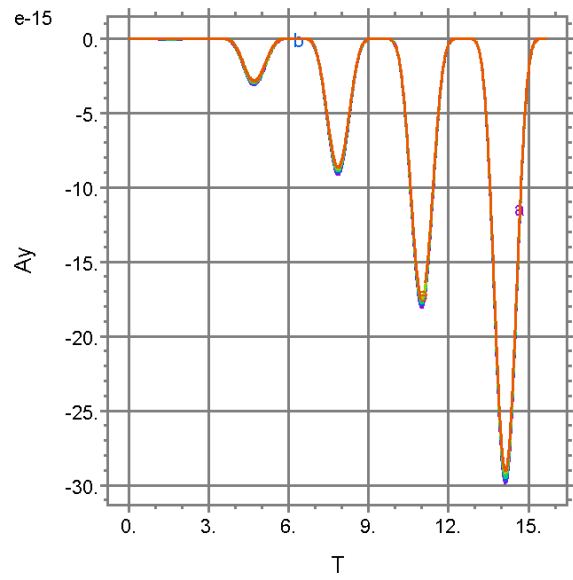
A quasi-analytic comparison for the solution given in Figure 2 is presented in Appendix III. It is seen that even though the limit of  $A_y \ll A_x$  required by the quasi-analytic analysis is not strictly adhered to, the solution agreement for the dependent variables are very good. This lends credibility to the method for loading conditions and geometries that cannot be modeled analytically.

Also presented, in Appendix II, is an analytic solution for this problem with constant initial  $A_y$  along with an FEA solution for the same conditions. Agreement is excellent.

Figure 3 is the solution for equation form 1 with zero initial conditions on  $A_y$  but with a slight non-zero flux on the left-most vertical boundary. It is noted that  $A_y$  increases as a flux of  $E_y$  continually enters from the left-most boundary. The magnetic field component also increases with time.

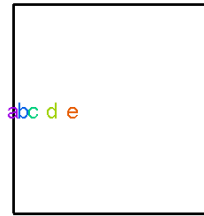


Problem 1  $k=1$   $\beta=1$   $BC=1e-19$

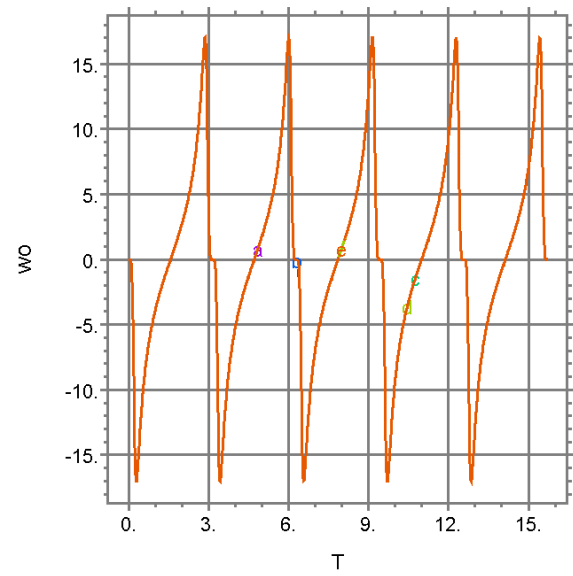


HISTORY

1: Ay

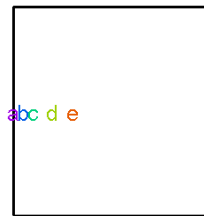


Problem 1  $k=1$   $\beta=1$   $BC=1e-19$

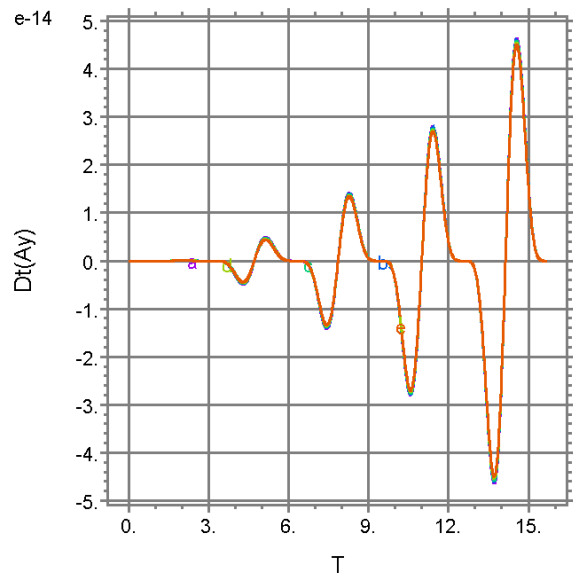


HISTORY

1: w0

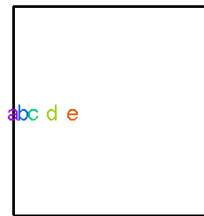


Problem 1 k=1 beta=1 BC=1e-19

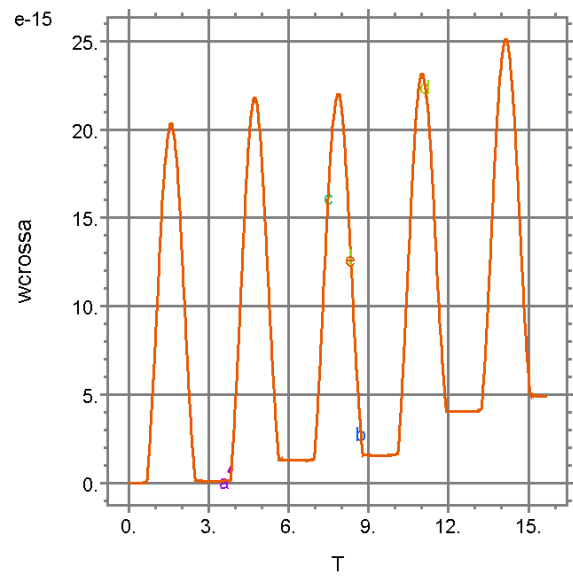


HISTORY

1: Dt(Ay)

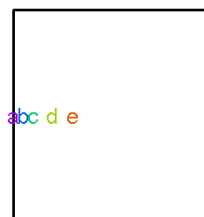


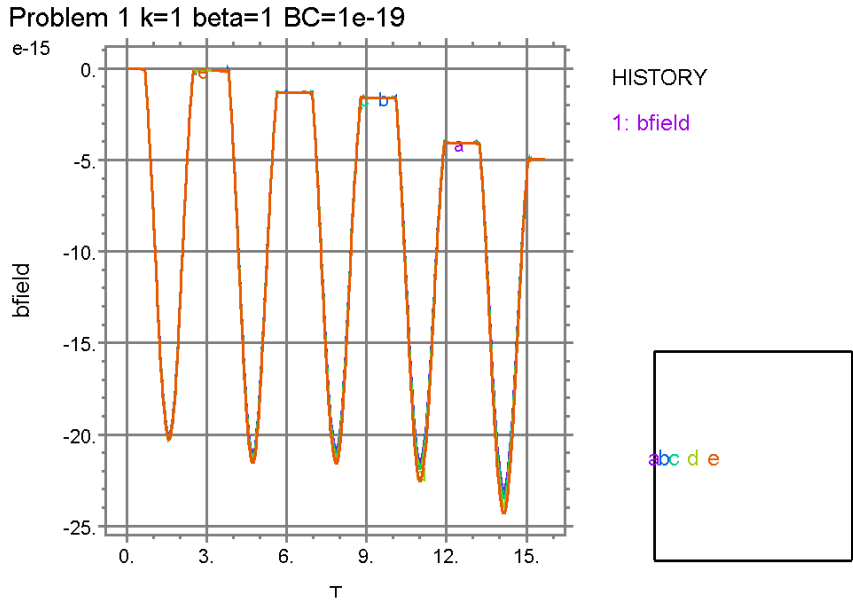
Problem 1 k=1 beta=1 BC=1e-19



HISTORY

1: wcrossa





**Figure 3** Form 1 equations for zero initial condition on  $A_y$  and slight boundary flux of  $10^{-19}$  on left-most vertical boundary

**Stability for Variations in the Conditions on  $A_y$**

To see the restrictions imposed on the size of the initial conditions, solutions were calculated when  $A_y$  was given an initial condition of

$$(31) \quad A_{y(initial)} = \varepsilon \sin\left(\frac{\pi x}{L_x}\right) \cos\left(\frac{\pi y}{2L_y}\right) + \delta$$

Some stability restrictions were discovered on the solution for some values chosen for  $\varepsilon$  and  $\delta$  as shown in Table 3.

**Table 3** Stability for Variations in the Initial Conditions on  $A_y$

$\varepsilon$	$\delta$	Boundary Condition	Stability
.00001	0	0	stable
.00001	1	0	stable
.0001	.0001	$10^{-19}$	stable
.001	0	0	marginally stable
.001	.0001	$10^{-19}$	marginally stable
.01	0	0	unstable
1	0	0 or $10^{-19}$	unstable
1	.0001	0 or $10^{-19}$	unstable
1	1	0 or $10^{-19}$	unstable

Only a small spatially variable initial condition or an off-set boundary flux lets the solution proceed. Whether the instability is due to the equation structure or in a numerical artifact is not known. The forms and values of the dependent variables are very similar to that presented earlier.

**Convergence:**

Stationary finite element solutions as a rule converge to a stable value. This is not necessarily so for time dependent problems where the existence of a solution is not even guaranteed.

Three parameters in the solution scheme implemented in FlexPde were varied to see their effect on the solution.

*Errlim – “FlexPde applies a consistency check to integrals of the PDE's over the mesh cells. From this it estimates the relative uncertainty in the solution variables and compares this to an accuracy tolerance. If any mesh cell exceeds the tolerance, that cell is split, and the solution is recomputed. The default value of ERRLIM is 0.001, which means that FlexPde will refine the mesh until the estimated error in any variable (relative to the variable range) is less than 0.1% over every cell of the mesh. This does not mean that FlexPde can guarantee that the solution is accurate to 0.1% over the domain. Individual cell errors may cancel or accumulate in ways that are hard to predict.”* This does not modify the grid structure for the next time-step calculation.

*Fixdt –* Fixing the time step disables the automatic time step adjustment provided by the software.

*Mesh\_Density-* The mesh density option allows user control over the density of elements in the solution domain. Normally the software algorithms automatically determine this, but the effect of increasing density beyond that normally suggested was observed by forcing a specified density of elements.

To investigate the effects of the above solution parameters on solution stability, the time step was varied between 0.1 seconds and 0.001 seconds. The error limit was varied from  $10^{-2}$  to  $10^{-4}$  and the mesh density varied from a coarse mesh to a very fine mesh.

**Table 4 Convergence of Solution** (536 elements, 1133 nodes)

<b>Errlim</b>	<b>Dt</b>	<b>bfield(max)</b>	<b><math>\omega_o(min)</math></b>	<b><math>A_y(max)</math></b>	<b><math>\omega \times A (min)</math></b>
$10^{-2}$	.01	$17 \times 10^{-2}$	-17	$8.4 \times 10^{-2}$	$-9.4 \times 10^{-3}$
$10^{-3}$	.01	$17 \times 10^{-2}$	-17	$8.4 \times 10^{-2}$	$-9.4 \times 10^{-3}$
$10^{-4}$	.01	$17 \times 10^{-2}$	-17	$8.4 \times 10^{-2}$	$-9.4 \times 10^{-3}$
$10^{-3}$	.05	$32 \times 10^{-2}$	-17	$25 \times 10^{-2}$	$-17 \times 10^{-3}$
$10^{-3}$	.1	$17 \times 10^{-2}$	-17	$8.4 \times 10^{-2}$	$-9.4 \times 10^{-3}$
$10^{-3}$	.001	$17 \times 10^{-2}$	-17	$8.3 \times 10^{-2}$	$-9.2 \times 10^{-3}$

(Errlim= $10^{-3}$ ; Dt=.01)

<b>Elements</b>	<b>Nodes</b>	<b>bfield(max)</b>	<b><math>\omega_o(min)</math></b>	<b><math>A_y(max)</math></b>	<b><math>\omega \times A (min)</math></b>
536	1133	$17 \times 10^{-2}$	-17	$8.4 \times 10^{-2}$	$-9.4 \times 10^{-3}$
764	1601	$20 \times 10^{-2}$	-17	$8.4 \times 10^{-2}$	$-4.3 \times 10^{-3}$
944	1969	$13 \times 10^{-2}$	-17	$8.4 \times 10^{-2}$	$-4.3 \times 10^{-3}$
1352	2801	$20 \times 10^{-2}$	-17	$8.4 \times 10^{-2}$	$-3.8 \times 10^{-3}$
2104	4329	$17 \times 10^{-2}$	-17	$8.4 \times 10^{-2}$	$-8.8 \times 10^{-3}$
3730	7621	$15 \times 10^{-2}$	-17	$8.4 \times 10^{-2}$	$-26 \times 10^{-3}$

(Automatic Mode: Fixed Dt=Off; Regrid= on)

<b>Errlim</b>	<b>Elements</b>	<b>Nodes</b>	<b>bfield(max)</b>	<b><math>\omega_o(min)</math></b>	<b><math>A_y(max)</math></b>	<b><math>\omega \times A (min)</math></b>
$10^{-3}$	536	1133	$17 \times 10^{-2}$	-17	$8.2 \times 10^{-2}$	$-9.1 \times 10^{-3}$
$10^{-4}$	4132	n/a	$16 \times 10^{-2}$	-17	$8.2 \times 10^{-2}$	$-4.3 \times 10^{-3}$
$10^{-5}$	18808	9345	$5.6 \times 10^{-2}$	-17	$8.2 \times 10^{-2}$	$-15 \times 10^{-3}$

Convergence is noted as long as the mesh density doesn't get too high, the time step too small, or the error limit too small. It is suspected that numerical errors compound when these values are too far from the default values chosen by FlexPde. The default values for FlexPde are an error limit of  $10^{-3}$  and automatic time step adjustment. If derivatives of functions are important, it has been suggested that an error limit of  $10^{-4}$  be used. It is seen that these default values generates a solution consistent with the converged values when parameters are not allowed to vary under software control. Solution times increased approximately linearly with both the error limit and the time-step for the range of variation explored.

#### **Methods of Improving Stability:**

Several methods avail themselves internal and external to the calculating engine in FlexPde for improving the stability of non-linear time dependent systems of partial differential equations [5]. (Some of this section is quoted directly from [5] and is in quotations)

**Putting highest derivative on left hand side** – It has been suggested that putting the highest time derivative as the only term on the left had side of a partial differential equation may help stability. This was tried, but no noticeable difference in the solution or its execution time was observed.

**Upwinding** –“Standard Galerkin Finite Element techniques generate oscillatory solutions to hyperbolic systems. One way of rectifying this problem is the use of “upwinding”, which takes its name from an analogous Finite Difference technique, forming a one-sided difference approximation to the first derivative. FlexPde automatically implements upwinding, but it also allows the user to select an arbitrary scaling multiplier on the standard form.” The upwinding factor was varied from 1 to 1000, with only incidental improvement in the stability of the unstable system.

**Vanderberg method** – “FlexPde implements an optional Conjugate-Gradient solver due to Van den Berg. This technique uses the matrix product  $A'A$  to symmetrize the system, and is sometimes effective in solving hyperbolic systems.” It has been used extensively in this analysis and provides the best stabilizing influence of any of the parameters available.

**Laplacian damping factor** – Adding a small Laplacian term in the unstable variable has often been used to filter unwanted numerically induced oscillations. This was tried with little success.

**Automatic regridding** –FlexPde uses a method of “adaptive mesh refinement“ to improve solution accuracy in areas of high solution gradients. *“The problem domain presented by the user is divided into a triangular mesh dictated by the feature sizes of the domain and the input controls provided by the user. The problem is then constructed and solved, and the cell integrals of the weighted residual method are crosschecked to estimate their accuracy. In locations where the integrals are deemed to be of questionable accuracy, the triangles are subdivided to give a new denser mesh, and the problem is solved again. This process continues until FlexPde is satisfied that the approximation is locally accurate to the tolerance assigned by the user.”* Turning off the automatic regridding proved very effective in minimizing numerically induced instability.

**Automatic time stepping** –“FlexPde uses a variable-order implicit backward difference method (BDM) as introduced by C.W. Gear. In most cases, second order gives the best tradeoff between stability, smoothness and speed, and this is the default configuration for FlexPde. This method fits a quadratic in time to each nodal value, using two known values and one future (unknown) value. It then solves the coupled equations for the array of nodal values at the new time. By looking backward one additional step, it is possible to infer the size of the cubic term in a four-point expansion of the time behaviour of each nodal value. If these cubic contributions are large, the time-step is reduced, and if extreme, the current step repeated. “

**Effect of Reducing Zero Off-Set on  $A_x$ :**

$A_x$  was off set by a specified amount that prevented  $A_x$  from being zero. This offset directly affected the peak values that  $\omega_0$ ,  $A_y$  and their time derivatives. This is illustrated in Table 5 for several values of the off set.

**Table 5 Effect of Reducing Off-set on  $A_x$** 

Off Set	$A_y$	$A_y'$	$\omega_o$	$\omega_o'$
$10^{-1}$	$6.5 \times 10^{-19}$	$9.5 \times 10^{-19}$	-3.4	-10
$10^{-2}$	$2.9 \times 10^{-18}$	$4.5 \times 10^{-18}$	-6.8	-28
$10^{-3}$	$1.5 \times 10^{-18}$	$2.3 \times 10^{-17}$	-11	-70
$10^{-4}$	$8.4 \times 10^{-17}$	$1.6 \times 10^{-16}$	-17	-160
$10^{-5}$	$4.2 \times 10^{-16}$	$4.5 \times 10^{-16}$	-26	-360
$10^{-6}$	$2.8 \times 10^{-15}$	$4.5 \times 10^{-15}$	-38	-800
$10^{-7}$	$2.0 \times 10^{-14}$	$3.1 \times 10^{-14}$	-56	-1800
$10^{-8}$	$3.3 \times 10^{-13}$	$4.8 \times 10^{-13}$	-85	-3900

The solution degrades without further reduction of the error limit and time step for off-sets smaller than  $10^{-8}$ . There does not appear to be any form of stabilization of the dependent variables to a stationary value as the off set is decreased. In fact it appears that  $\omega_o$  may be singular. This drives  $A_y$  to an increasingly higher value, likely also singular. For offsets greater than  $10^{-1}$ , the character of  $\omega_o$ ,  $A_y$  and their time derivatives changes dramatically.

#### Limits on Complexity of Applied Vector Potential:

To determine how stable equation form 1 is to changes in the complexity of the driving magnetic vector potential component, the applied potential  $A_x$  was allowed to vary in the two in-plane directions in a prescribed manner. A variety of forms for  $A_x$  were analyzed, all which satisfied the diffusion equation for  $A_x$ , as shown in Table 6. Stability is challenged for several spatial variations in  $A_x$ ; if at any point in the plane  $A_x$  becomes zero, the solution becomes unstable.

**Table 6 Stability with Various forms for  $A_x$** 

Form of $A_x$	Stable	CPU time (minutes)
$\text{Sin}(\pi x/Lx) \text{Sin}^6(t) + \delta^*$	yes	5:16
$\text{Sin}(2\pi x/Lx) \text{Sin}^6(t) + \delta$	no	Unstable at t=0.25 sec
$\text{Sin}^2(2\pi x/Lx) \text{Sin}^6(t) + \delta$	yes	7:40
$\text{Sin}(\pi y/Ly) \text{Sin}^6(t) + \delta$	yes	7:55
$\text{Sin}(2\pi y/Ly) \text{Sin}^6(t) + \delta$	no	Unstable at t=0.25 sec
$\text{Sin}^2(2\pi y/Ly) \text{Sin}^6(t) + \delta$	yes	8:28
$\text{Sin}(\pi y/Ly) \text{Sin}(2\pi x/Lx) \text{Sin}^6(t) + \delta$	no	Unstable at t=0.25 sec
$\text{Sin}(2\pi x/Lx) \text{Sin}^6(t) + \varepsilon^{**}$	yes	3:05
$\text{Sign}(\text{Sin}(2\pi x/Lx))(\text{Abs}(\text{Sin}(2\pi x/Lx) + \delta) \text{Sin}^6(t) + \delta)$	no	Unstable at t=.025 sec
* $\delta=0.0001$		
** $\varepsilon=1.0001$		

What is noted is that as long as  $A_x$  is positive, the calculation is stable. An attempt was made to run

$$A_x = \text{Sign}(\text{Sin}(2\pi x/Lx))(\text{Abs}(\text{Sin}(2\pi x/Lx) + \delta) \text{Sin}^6(t) + \delta)$$

as a functional form for  $A_x$  that would be always non-zero. This also became unstable even though  $A_x$  was never actually allowed to become zero. It is thought that the instabil-

ity in this case is caused by the fact that  $A_x$  is discontinuous at its zero. Instability of the “zero crossing” problem was “fixed” by the addition a large shifting term so that  $A_x$  was always greater than zero however, the solution may be of little practical value.

### **Conclusions:**

A method of calculating the ECE electromagnetic field in the quasi-static purely magnetic approximation using the finite element approach for two-dimensional flat conductive surfaces has been presented. In its present form, one component of the magnetic vector potential is specified and the scalar spin connection and the second magnetic vector potential component are calculated. There is a limitation in that the specified driving vector potential component can never be zero as this causes a singularity in  $\omega_0$  which in turn drives  $A_y$  to incalculable values.

It was found that it was not possible in principle to calculate the vector spin connection, however  $\omega \times A$  could be calculated. This then allowed the calculation of the magnetic field.

### **REFERENCES**

(all web addresses valid on date of writing)

- [1] “Applications of FlexPde”; Vol. 1 Rev. 1; [www.pdesolutions.com](http://www.pdesolutions.com)
- [2] Eckardt H.;“ECE Engineering Model- The Basic for Electromagnetic and Mechanical Applications”; [www.aias.us](http://www.aias.us)
- [3] Evans M., Eckardt H., Hubbard C.,Shelburne J.; “Spin Connection Resonance in the Bedini Machine”; [www.aias.us](http://www.aias.us)
- [4] Lorrain P., Larson D.,and Lorrain F.; “Electrodynamic Fields and Waves”; W.H. Freeman & Company, New York, 1987
- [5] Nelson R. G.; “Solving Hyperbolic Systems with FlexPde”; [www.flexpde5.com/download/hyperbolic.doc](http://www.flexpde5.com/download/hyperbolic.doc)



## APPENDIX I

### TYPICAL SCRIPT

#### TITLE

'Problem 1 k=1 beta=1 '

#### SELECT

errlim= 1e-4 { Limit of relative error }  
vandenberg=on  
nonlinear=on  
prefer\_stability=on  
smoothinit=on  
fixdt=on

#### COORDINATES cartesian2

#### VARIABLES

Ay(threshold=100)  
ey(threshold=.01)  
wo(threshold=.1)  
ex(threshold=.01)  
enorm(threshold=.01)

#### DEFINITIONS

Lx= 2            Ly=1 { SI units throughout }  
                          { Domain size }  
beta=1  
Ax=(Sin(beta\*t))^6 +.0001 { Input field }  
k=1 {  $\mu_0 \cdot \sigma$  }  
a=Vector(ax,ay)  
e=Vector(ex,ey)  
am=Vector(ax,0)  
Q=Dy(wo\*ax)-Dx(wo\*ay) { Curl(wo\*a) }  
wcrossa=-TIME\_INTEGRAL(-Q) { Cross(wo,a) }  
bfield=Curl(a)-wcrossa

#### INITIAL VALUES

Ay=.00001\*cos(pi\*y/2\*Ly)\*Sin(pi\*x/Lx)

#### EQUATIONS

ex:            Del2(ex)-k\*Dt(ex)=0  
wo:            ex=-Dt(Ax)-wo\*Ax  
ey:            Del2(ey)-k\*Dt(ey)=0  
Ay:            ey=-Dt(Ay)-wo\*Ay  
enorm:        enorm=Normal(e)

#### BOUNDARIES

##### region 1 'strip'

start (0,-Ly)            Natural(enorm)=0  
line to (0,Ly)            Natural(enorm)=0  
line to (Lx,Ly)            Natural(enorm)=0  
line to (Lx,-Ly)            Natural(enorm)=0  
line to close

TIME 0 TO 5\*pi by .01

## MONITORS

for cycle=1

Contour(Ax)  
Elevation(Ax) from (0,0) to (Lx,0)  
Elevation(Ax) from (0,-Ly) to (0,Ly)  
Elevation(Ax) from (.1\*Lx,-Ly) to (.1\*Lx,Ly)

Contour(Ay)  
Elevation(Ay) from (0,0) to (Lx,0)  
Elevation(Ay) from (0,-Ly) to (0,Ly)  
Elevation(Ay) from (.1\*Lx,-Ly) to (.1\*Lx,Ly)

## HISTORIES

history(Ax) at (0,0)(.05\*Lx,0)(.10\*Lx,0)(.20\*Lx,0)(.30\*Lx,0)  
history(Ay) at (0,0)(.05\*Lx,0)(.10\*Lx,0)(.20\*Lx,0)(.30\*Lx,0)  
history(Dt(Ax)) at (0,0)(.05\*Lx,0)(.10\*Lx,0)(.20\*Lx,0)(.30\*Lx,0)  
history(Dt(Ay)) at (0,0)(.05\*Lx,0)(.10\*Lx,0)(.20\*Lx,0)(.30\*Lx,0)  
history(wo) at (0,0)(.05\*Lx,0)(.10\*Lx,0)(.20\*Lx,0)(.30\*Lx,0)  
history(wo\*Ax) at (0,0)(.05\*Lx,0)(.10\*Lx,0)(.20\*Lx,0)(.30\*Lx,0)  
history(wo\*Ay) at (0,0)(.05\*Lx,0)(.10\*Lx,0)(.20\*Lx,0)(.30\*Lx,0)  
history(Dt(wo)) at (0,0)(.05\*Lx,0)(.10\*Lx,0)(.20\*Lx,0)(.30\*Lx,0)  
history(ex) at (0,0)(.05\*Lx,0)(.10\*Lx,0)(.20\*Lx,0)(.30\*Lx,0)  
history(ey) at (0,0)(.05\*Lx,0)(.10\*Lx,0)(.20\*Lx,0)(.30\*Lx,0)  
history(wcrossa) at (0,0)(.05\*Lx,0)(.10\*Lx,0)(.20\*Lx,0)(.30\*Lx,0)  
history(Curl(a)) at (0,0)(.05\*Lx,0)(.10\*Lx,0)(.20\*Lx,0)(.30\*Lx,0)  
history(bfield) at (0,0)(.05\*Lx,0)(.10\*Lx,0)(.20\*Lx,0)(.30\*Lx,0)

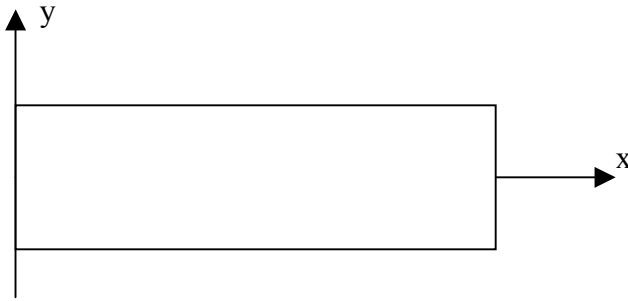
## PLOTS

END

## Appendix II Analytic Solution- Justification for Use of Finite Element Method

An analytic solution is available for the quasi-static magnetic solution of the ECE equations for a very simple geometry and constant loading condition.

Consider a thin flat conductive sheet of material that is very thin in comparison to its other two geometric dimensions. It is spanned by a standard two-dimensional Cartesian frame of reference as illustrated in Figure AII-1. The sheet can have arbitrary spatial dimensions within this limitation, but for the purposes of this discussion, the geometry of the sheet is limited to that of a simple rectangle.



**Figure AII-1. Flat sheet of conducting material**

The sheet is assumed to be thin enough that there is no spatial variation in any of the system variables or parameters in the “z”, through the thickness, direction.

We will also specifically assume that

5. The conductivity of the material is high enough that there is no charge accumulation anywhere
6. No external electric potentials are applied and that the electric potential can be taken as constant which we shall set to zero without loss of generality
7. The permeability of the material is constant and is equal to  $\mu_0$ , the permeability of free space
8. The magnetic vector potential is two dimensional and is given as

$$(1) \quad \mathbf{A} = (A_x(x,y,t), A_y(x,y,t), 0)^T$$

This two dimensional limitation requires that, as before in this paper that  $\mathbf{E}$  be two-dimensional and of the form

$$(2) \quad \mathbf{E} = (E_x(x,y,t), E_y(x,y,t), 0)^T$$

The Curl of this expression can only have a “z” component. Then by

Faraday's Law in Induction (equation 2) that the magnetic field only has a "z" component. That is

$$(3) \quad \mathbf{B}=(0,0,B_z(x,y,t))^T$$

Given the Ampere Maxwell Law, equation 4, the form of the spin connection has to be

$$(4) \quad \boldsymbol{\omega}=(\omega_x, \omega_y, 0)^T$$

Earlier in this paper, the following equation was developed as a means of implementing this problem in a finite element scheme. It was

$$(5) \quad \nabla^2\left(-\frac{\partial \mathbf{A}}{\partial t}-\omega_o \mathbf{A}\right)-\mu_o \sigma \frac{\partial\left(-\frac{\partial \mathbf{A}}{\partial t}-\omega_o \mathbf{A}\right)}{\partial t}=0$$

Based on observations from FEA runs, it is postulated that a very simple analytic solution exists for which

- $A_x$  does not vary spatially
- $\omega_o$  does not vary spatially
- $\boldsymbol{\omega} \times \mathbf{A}$  does not vary spatially
- $A_y$  does not vary spatially if initially  $A_y$  is independent of position.

Substituting these assumptions into the x component of equation (5), and noting that both  $A_x$  and  $\omega_o$  are zero at the start of the analysis, we have that

$$(6) \quad \omega_o = -\frac{1}{A_x} \frac{\partial A_x}{\partial t} \quad \text{and}$$

$$(7) \quad \frac{\partial A_y}{\partial t} = -\omega_o A_y$$

From Paper 1, we have the expression for  $\boldsymbol{\omega} \times \mathbf{A}$  given by

$$(8) \quad \boldsymbol{\omega} \times \mathbf{A} = -\int (\nabla \times \omega_o \mathbf{A}) dt$$

From the assumptions above, this is identically zero.

The geometry of the area modeled is given in Table AII-1.

**Table AII-1**

Parameter	Value
Length in “x” direction	2
Length in “y” direction	2
Number of nodes	1133
Number of elements	536
Number of unknowns	4532

In accordance with this paper we assume that the driving or specified vector potential is given by

$$(9) \quad A_x = \sin^6(\beta t) + \delta$$

where  $\delta$  is a small numerical factor that prevents a numerical “divide by zero” situation. In this example it was chosen to be 0.0001.

The flux of  $(-\mathbf{A}' - \omega_0 \mathbf{A})$  normal to the surface is taken to be zero at all. All field variables are assumed to be zero at the start of the calculation. For the purposes of this simple problem,

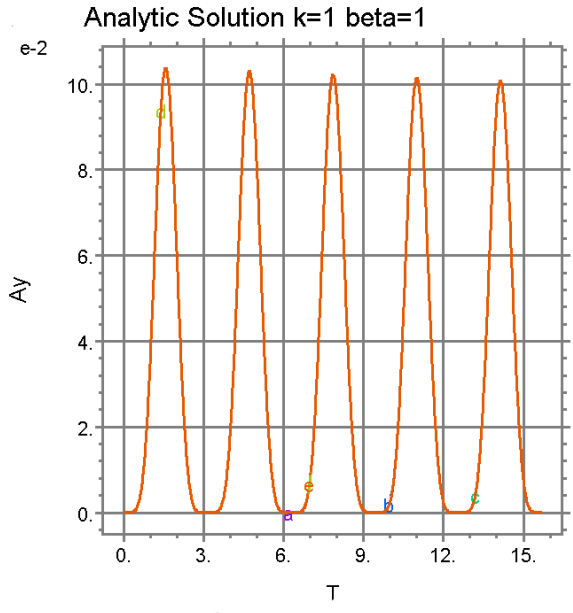
$\mu_0 \sigma = k = 1$  and  $\beta = 1$  (For copper  $\mu_0 \sigma = 75$  and for nichrome  $\mu_0 \sigma = 1.25$  in SI units).

**The Solution:**

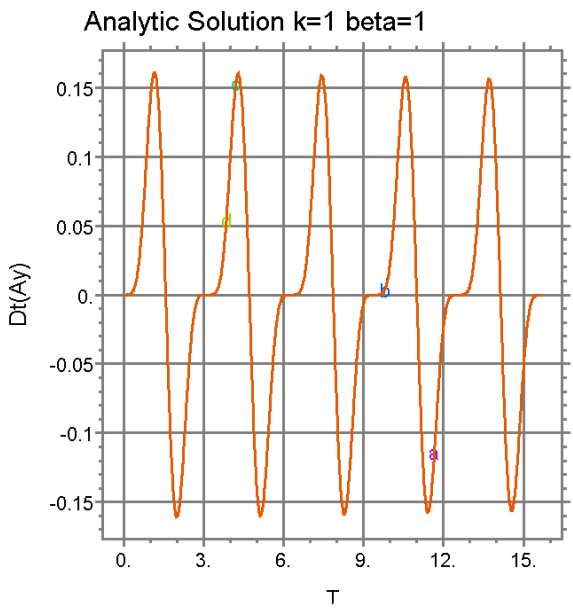
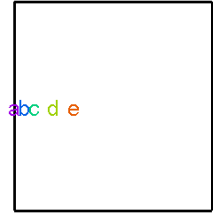
The solution for Form 1 is presented in Figures AII- 1 where initially

$$A_y = 1$$

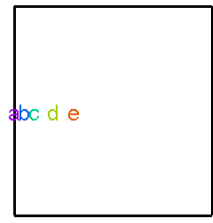
The script for the problem on Flexpde is presented at the end of this appendix



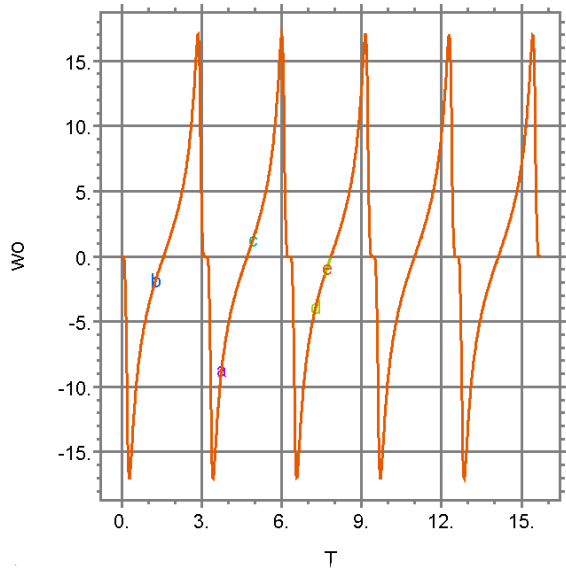
HISTORY  
1: Ay



HISTORY  
1: Dt(Ay)

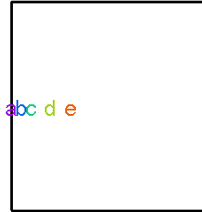


Analytic Solution k=1 beta=1

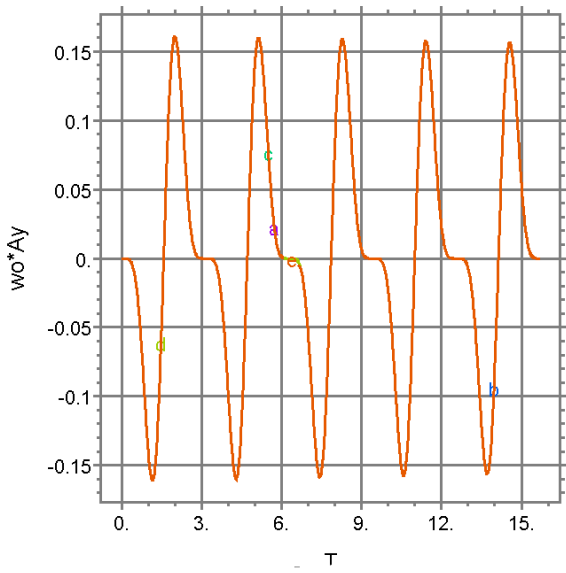


HISTORY

1: w(t)

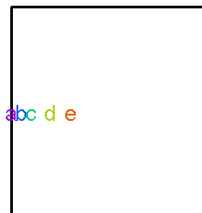


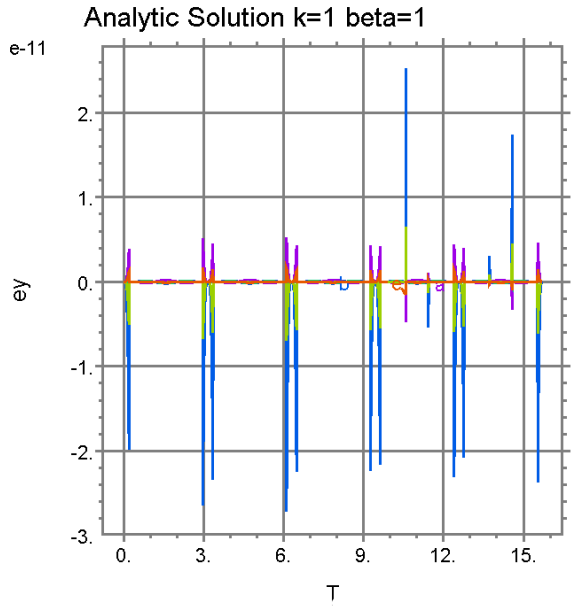
Analytic Solution k=1 beta=1



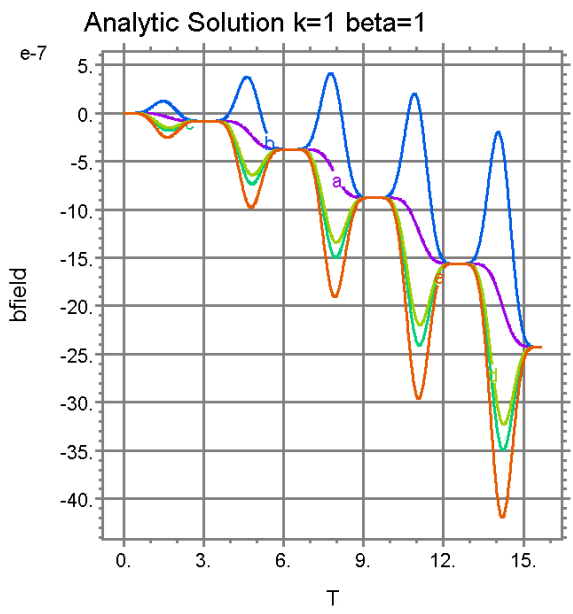
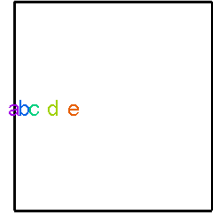
HISTORY

1: w(t)\*Ay

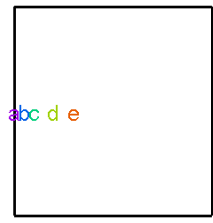




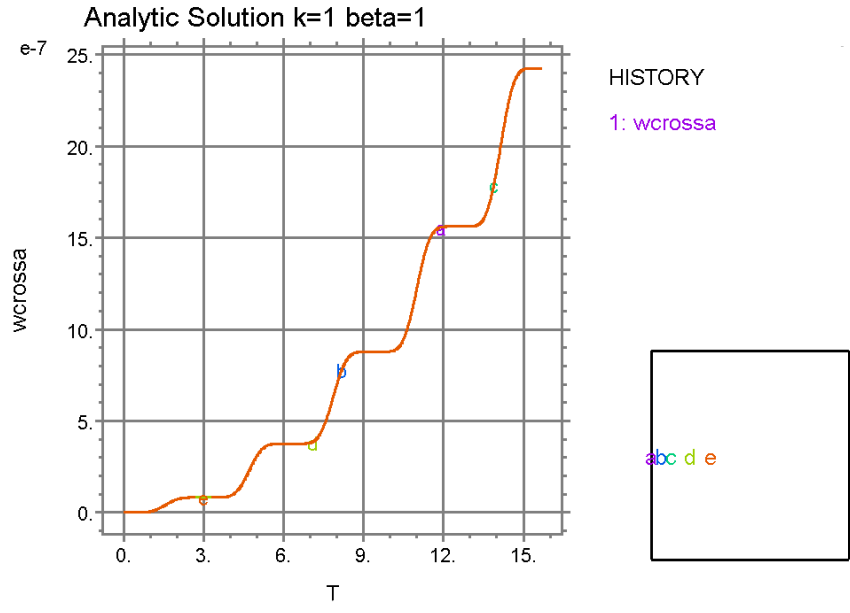
HISTORY  
1: ey



HISTORY  
1: bfield



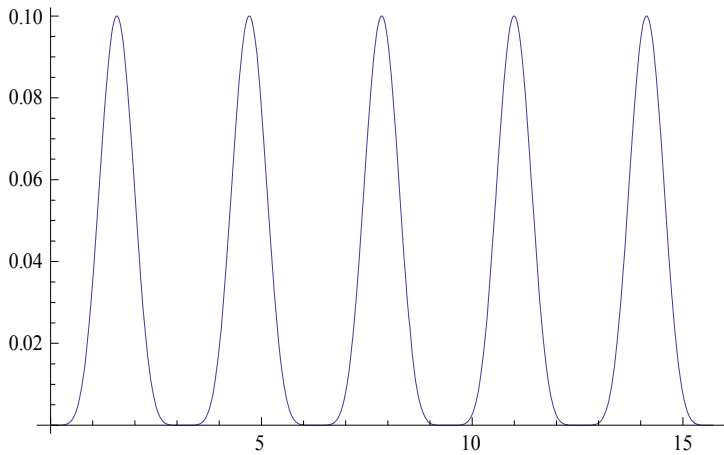




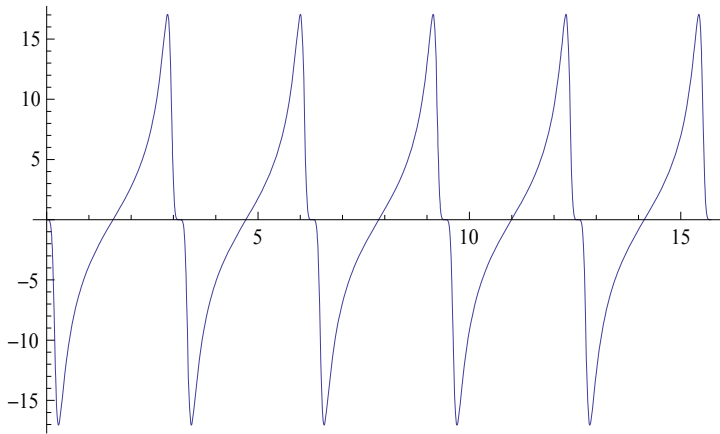
**Figure AII-1 Flexpde Solution**

An analytical solution was developed using Mathematica

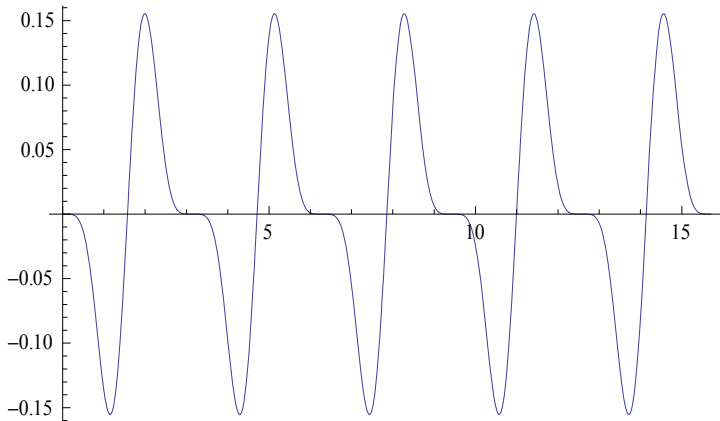
$$(10) \quad A_y = const \cdot \left( \frac{1. + 3. \tan[t]^2 + 3. \tan[t]^4 + 1001. \tan[t]^6}{\sec[t]^2} \right)^3$$



**Figure AII-2a Ay vs t**



**Figure AII-2b**  $\omega_o$  vs  $t$



**Figure AII-2c**  $\omega_o A_y$  vs  $t$

For the calculation of  $\omega_o$ ,  $A_y$ , and  $\omega_o A_y$ , the finite element solution is in very good agreement with the analytical solution. It should be noted that the y component of the electric field is not exactly zero in the finite element solution nor is  $\text{Curl}(\mathcal{A})$  nor the  $\mathbf{B}$  field as would be expected from the analytic solution. The values are very small; many orders of magnitude below that of the other calculated field variables, and appear to be due to the manner in which the field variables are calculated (ie. a numerical artifact).

The magnification of  $A_y$  is very much governed by the size of  $\delta$ . No interpretation of the amplification is given here; it is just reported as part of the mathematical analysis.

As a conclusion, it seems that FEA can be used to solve ECE electromagnetic problems with sufficient accuracy.



TIME 0 TO 5\*pi by .01

## MONITORS

```
for cycle=1
  Contour(ey)
  Vector(e)
  Contour(Ay)
  Contour(Ax)
  Elevation(Ay) from (0,0) to (Lx,0)
  Elevation(Ay) from (0,-Ly) to (0,Ly)
  Elevation(Ay) from (.5*Lx,-Ly) to (.5*Lx,Ly)
```

## HISTORIES

```
history(Ax) at (0,0)(.05*Lx,0)(.10*Lx,0)(.20*Lx,0)(.30*Lx,0)
history(Ay) at (0,0)(.05*Lx,0)(.10*Lx,0)(.20*Lx,0)(.30*Lx,0)
history(Dt(Ax)) at (0,0)(.05*Lx,0)(.10*Lx,0)(.20*Lx,0)(.30*Lx,0)
history(Dt(Ay)) at (0,0)(.05*Lx,0)(.10*Lx,0)(.20*Lx,0)(.30*Lx,0)
history(wo) at (0,0)(.05*Lx,0)(.10*Lx,0)(.20*Lx,0)(.30*Lx,0)
history(wo*Ax) at (0,0)(.05*Lx,0)(.10*Lx,0)(.20*Lx,0)(.30*Lx,0)
history(wo*Ay) at (0,0)(.05*Lx,0)(.10*Lx,0)(.20*Lx,0)(.30*Lx,0)
history(Dt(wo)) at (0,0)(.05*Lx,0)(.10*Lx,0)(.20*Lx,0)(.30*Lx,0)
history(ex) at (0,0)(.05*Lx,0)(.10*Lx,0)(.20*Lx,0)(.30*Lx,0)
history(ey) at (0,0)(.05*Lx,0)(.10*Lx,0)(.20*Lx,0)(.30*Lx,0)
history(-Dt(Ay)-wo*ay) at (0,0)(.05*Lx,0)(.10*Lx,0)(.20*Lx,0)(.30*Lx,0)
history(wcrossa) at (0,0)(.05*Lx,0)(.10*Lx,0)(.20*Lx,0)(.30*Lx,0)
history(Q) at (0,0)(.05*Lx,0)(.10*Lx,0)(.20*Lx,0)(.30*Lx,0)
history(Curl(a)) at (0,0)(.05*Lx,0)(.10*Lx,0)(.20*Lx,0)(.30*Lx,0)
history(bfield) at (0,0)(.05*Lx,0)(.10*Lx,0)(.20*Lx,0)(.30*Lx,0)
history(ay/ax) at (0,0)(.05*Lx,0)(.10*Lx,0)(.20*Lx,0)(.30*Lx,0)
```

## PLOTS

END

## Mathematica Script for Analytical Solution

Simplest Analytical Solution for Problem 1

Specify Fields

Note that wo is only a function of time.  $w \times A$  is also only time dependent.

```
<< VectorAnalysis`
```

```
SetCoordinates[Cartesian[x, y, z]];
```

```
a = {ax[t], ay[t], 0};
```

```
ax[t] = Sin[t]^6 + .0001;
```

```
wo[t] = -(1/ax[t])*D[ax[t], t];
```

```
eqn1 = D[ay[t], t] + ay[t]*wo[t] == 0
```

```
s = DSolve[eqn1, ay, t]
```

```
Plot[Evaluate[ay[t] /. s /. C[1] -> .00001], {t, 0, 5*Pi}]
```

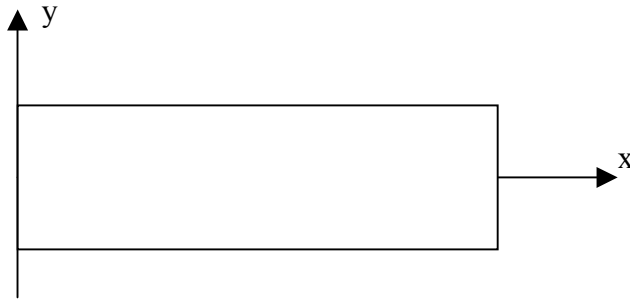
```
Plot[Evaluate[wo[t]], {t, 0, 5*Pi}]
```

```
Plot[Evaluate[wo[t]*ay[t] /. s /. C[1] -> .00001], {t, 0, 5*Pi}]
```

### Appendix III Quasi-Analytic Solution for Non-Uniform $A_y$

A quasi-analytical solution is developed to verify the FEA solution presented in this paper.

Consider a thin flat conductive sheet of material that is very thin in comparison to its other two geometric dimensions. It is spanned by a standard two-dimensional Cartesian frame of reference as illustrated in Figure AIII-1. The sheet can have arbitrary spatial dimensions within this limitation, but for the purposes of this discussion, the geometry of the sheet is limited to that of a simple rectangle.



**Figure AIII-1. Flat sheet of conducting material**

As in the body of this paper, the sheet is assumed to be thin enough that there is no spatial variation in any of the system variables or parameters in the “z”, through the thickness, direction.

From the body of this paper, equation (22), the relevant equations requiring solution are

$$(A-1) \quad \nabla^2 \left( -\frac{\partial A}{\partial t} - \omega_o A \right) - \mu_o \sigma \frac{\partial \left( -\frac{\partial A}{\partial t} - \omega_o A \right)}{\partial t} = 0$$

We will assume for this quasi-analytic problem that  $A_x$  does not vary spatially and is specified over the entirety of the sheet. We will assume that  $A_y$  is so small that it does not influence the value of  $\omega_o$ . Although not specifically necessary from equation (5), we will assume that  $\omega_o$  is also spatially invariant.

Substituting these assumptions into the x component of equation (A-1), and assuming that both  $A_x$  and  $\omega_o$  are zero at the start of the analysis, we have that

$$(A-2) \quad \omega_o = -\frac{1}{A_x} \frac{\partial A_x}{\partial t}$$

The geometry of the area to be modeled is given in Table AIII-1.

**Table AIII-1 Model Geometry**

Parameter	Value
Length in “x” direction	2
Length in “y” direction	2
Number of nodes	1133
Number of elements	536
Number of unknowns	4532

In accordance with the body of this paper we assume that the driving or specified vector potential is given by

$$(27) \quad A_x = \sin^6(\beta t) + \delta$$

where  $\delta$  is a small numerical factor that prevents a numerical “divide by zero” situation.

The flux of  $(-A' - \omega_0 A)$  normal to the surface is taken to be zero. All field variables are assumed to be zero at the start of the calculation. For the purposes of this simple problem,

$\mu_0 \sigma = k = 1$  and  $\beta = 1$  (For copper  $\mu_0 \sigma = 75$  and for nichrome  $\mu_0 \sigma = 1.25$  in SI units).

**The Solution:**

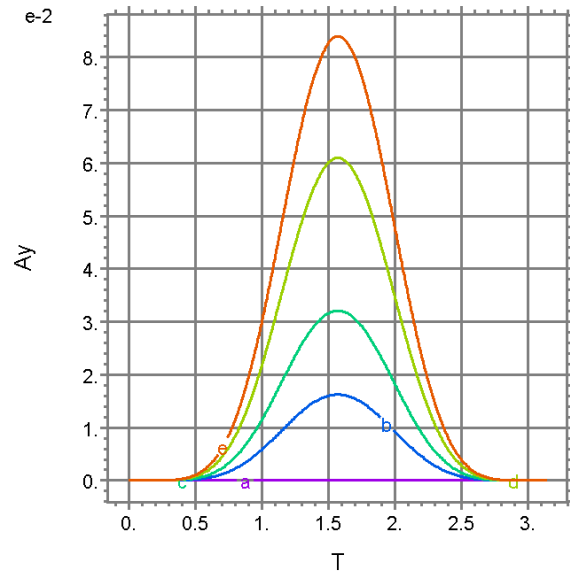
The solution is presented in Figures AIII-2 where initially

$$A_y = 0.00001 \cdot \cos\left(\frac{\pi y}{2L_y}\right) \sin\left(\frac{\pi x}{L_x}\right)$$

Agreement with the solution presented in the body of the paper is very good indicating that the method of solution is acceptable.

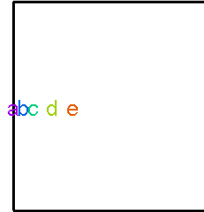
The script for the problem on Flexpde is presented at the end of this appendix.

Quasi-Analytic Solution k=1 beta=1

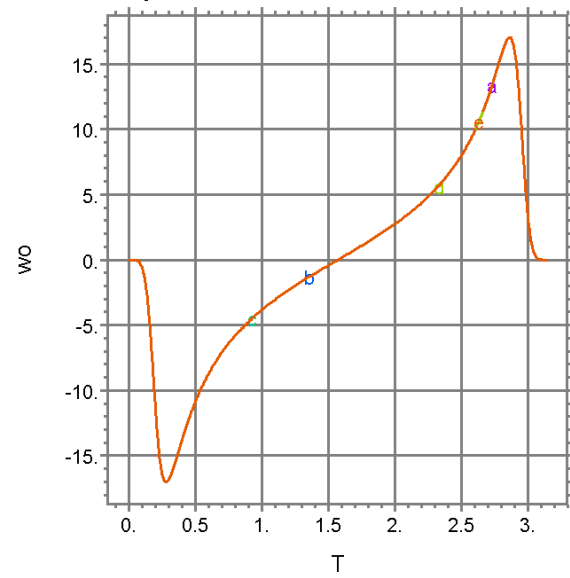


HISTORY

1: Ay



Quasi-Analytic Solution k=1 beta=1

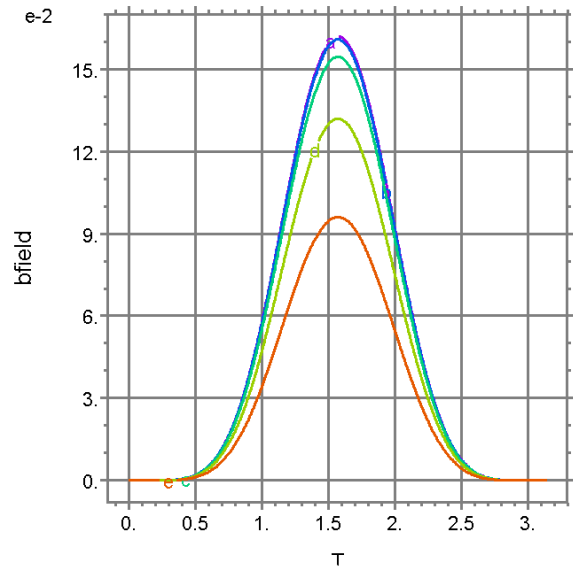


HISTORY

1: w0

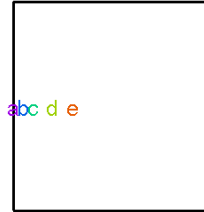


Quasi-Analytic Solution  $k=1$   $\beta=1$



HISTORY

1: bfield



**Figure AIII-2 Quasi-analytic solution**



## Script

```

TITLE
'Quasi-Analytic Solution k=1 beta=1 '

SELECT
    errlim= 1e-4                { Limit of relative error }
    vandenberg=on
    nonlinear=on
    prefer_stability=on
    smoothinit=on
    fixdt=on
    regrid=off

COORDINATES cartesian2
VARIABLES
    Ay(threshold=1)
    apy(threshold=1)
    {ey(threshold=1)}
    enorm(threshold=1)

DEFINITIONS
    Lx= 2                Ly=1                { SI units throughout }
    beta=1                { Domain size }
    k=1                { frequency }
    Ax=(Sin(beta*t))^6 +.0001                {mu0*sigma}
    Apx=Dt(Ax)                { Input field }
    wo=-apx/ax
    ey=-Apy-wo*Ay
    ex=-apx-wo*Ax
    a=Vector(ax,ay)
    e=Vector(ex,ey)
    Q=Dy(wo*ax)-Dx(wo*ay)                { Curl(wo*a) }
    wcrossa=-TIME_INTEGRAL(-Q)                { Cross(wo,a) }
    bfield=Curl(a)-wcrossa

INITIAL VALUES
    Ay=(.00001)*cos(pi*y/2*Ly)*Sin(pi*x/Lx)

EQUATIONS
Apy:    Apy=Dt(Ay)
Ay:    Del2(-Apy-wo*Ay)-k*Dt(-Apy-wo*Ay)=0
enorm:    enorm=Normal(e)

BOUNDARIES
    region 1 'strip'
        start (0,-Ly)
            Natural(enorm)=0                line to (0,Ly)
            Natural(enorm)=0                line to (Lx,Ly)
            Natural(enorm)=0                line to (Lx,-Ly)
            Natural(enorm)=0                line to close

TIME 0 TO pi by .01

MONITORS
for cycle=1
    Contour(ey)
    Vector(e)
    Contour(Ay)
    Elevation(Ay) from (0,0) to (Lx,0)
    Elevation(Ay) from (0,-Ly) to (0,Ly)
    Elevation(Ay) from (.5*Lx,-Ly) to (.5*Lx,Ly)

HISTORIES
    history(Ax) at (0,0)(.05*Lx,0)(.10*Lx,0)(.20*Lx,0)(.30*Lx,0)
    history(Ay) at (0,0)(.05*Lx,0)(.10*Lx,0)(.20*Lx,0)(.30*Lx,0)
    history(Dt(Ax)) at (0,0)(.05*Lx,0)(.10*Lx,0)(.20*Lx,0)(.30*Lx,0)

```

```
history(Dt(Ay)) at (0,0)(.05*Lx,0)(.10*Lx,0)(.20*Lx,0)(.30*Lx,0)
history(wo) at (0,0)(.05*Lx,0)(.10*Lx,0)(.20*Lx,0)(.30*Lx,0)
history(wo*Ax) at (0,0)(.05*Lx,0)(.10*Lx,0)(.20*Lx,0)(.30*Lx,0)
history(wo*Ay) at (0,0)(.05*Lx,0)(.10*Lx,0)(.20*Lx,0)(.30*Lx,0)
history(Dt(wo)) at (0,0)(.05*Lx,0)(.10*Lx,0)(.20*Lx,0)(.30*Lx,0)
history(ex) at (0,0)(.05*Lx,0)(.10*Lx,0)(.20*Lx,0)(.30*Lx,0)
history(ey) at (0,0)(.05*Lx,0)(.10*Lx,0)(.20*Lx,0)(.30*Lx,0)
history(-apy-wo*ay) at (0,0)(.05*Lx,0)(.10*Lx,0)(.20*Lx,0)(.30*Lx,0)
history(wcrossa) at (0,0)(.05*Lx,0)(.10*Lx,0)(.20*Lx,0)(.30*Lx,0)
history(Curl(a)) at (0,0)(.05*Lx,0)(.10*Lx,0)(.20*Lx,0)(.30*Lx,0)
history(bfield) at (0,0)(.05*Lx,0)(.10*Lx,0)(.20*Lx,0)(.30*Lx,0)
history(ay/ax) at (0,0)(.05*Lx,0)(.10*Lx,0)(.20*Lx,0)(.30*Lx,0)
history(ay/(1e-8)*cos(pi*y/2*Ly)*Sin(pi*x/Lx)) at (0,0)(.05*Lx,0)(.10*Lx,0)(.20*Lx,0)(.30*Lx,0)
```

PLOTS

END